Torsion Constants of Certain Cross-Sections by Non-Topographic Photogrammetry *

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ABSTRACT: *Determining the torsion constants of all but a few cross-sections is mathematically very complicated. A membrane analogy is often used to overcome this difficulty. The volume under the elevated membrane is proportional to the torque, and the slope of the surface at every point is proportional to the stress at that point.*

This paper describes a method of determining the torsion constants by mapping the surface of the membrane stretched over the cross-section. The contours are plotted photogrammetrically as opposed to the conventional method of mapping, by actually measuring the elevation of many points on the membrane.

The investigation described in this paper was carried out as part of the requirements for the Master of Science degree at Iowa State College under the direction of Professor Stephen J. Chamberlin.

HISTORICAL FOREWORD

THE general theory of torsion was first accurately presented by Saint Venant in 1854 and has since become known as Saint Venant's torsion problem. The theory mathematically is very involved for all but the simplest shapes. In 1903 Prandtl suggested the membrane analogy as a means of solving the torsion problem for irregular shapes. His suggestion was first tried by Anthes and then extended by Griffith and Taylor (1) in England and by Trayer and March (2) in the United States. Thiel (3) seems to be the first to apply the principles of photogrammetry to the solution of the membrane analogy.

For a discussion of the torsion problem the reader is referred to the three papers by Professor T. J. Higgins (4, 5, 6). The bibliographies contained in these works are by far the most complete this author has seen.

Reference to standard textbooks on advanced strength of materials (7) or the theory of elasticity (8) should be made for a complete discussion of the theories of torsion and the membrane analogy. Only a brief summary is included in this work.

TORSION THEORY

When a shaft of uniform circular cross-section is twisted, a section that was plane before twisting remains plane. Internal resistance to twisting is maintained by shearing stresses which are proportional to the distance from the center of the section. The total resisting moment may be expressed as

$$T = JG\theta \tag{1}$$

in which

T = torsional moment of torque
J = polar moment of inertia of cross-sectional area
G = modulus of rigidity of the material
θ = angle of twist per unit length in radians.

At any distance, ρ, from the center of the cross-section, the shearing stress is:

$$S = T\rho/J \tag{2}$$

These simple relationships are valid only for circular cross-sections. In a shaft of any other cross-section there is warping of plane sections during twisting, and the polar moment of inertia of the section has,

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in general, no direct relationship to the torsional problem. It is possible, however, to determine a torsion constant, K, which may be substituted for J in Equation 1, which then becomes

$$T = KG\theta \tag{3}$$

The shearing stress at any point is proportional to the ratio T/K multiplied by some function of the thickness of the material. It is imperative in designing to be able to calculate the torsion constant (9).

The general solution for the shearing stresses in a shaft subjected to a torsional moment was developed by Saint Venant. The method involves the solution of a partial differential equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta \tag{4}$$

in which ϕ is a torsional stress function in terms of the coordinate axis of the section under consideration. The stress function is defined by the following equations:

$$S_{xz} = \frac{\partial\phi}{\partial y} \tag{5}$$

and

$$S_{yz} = -\frac{\partial\phi}{\partial x} \tag{6}$$

in which S_{xz} and S_{yz} are components of the shearing stress in the plane of the cross-section in the x and y directions respectively. It may also be shown that the torsional moment can be represented by (7):

$$T = 2\int_0^A \phi dA \tag{7}$$

THE MEMBRANE ANALOGY

In 1903 Prandtl showed that there exists an analogy between the differential equations of the torsion problem and the differential equations representing the surface of a thin membrane stretched across an opening having the same shape as the section to be investigated, and slightly elevated by a small change in pressure.

By applying the equations of equilibrium to the free body of an element of the distended membrane, it can be shown that the equation of the surface of the film is:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{P}{F} \tag{8}$$

in which P is the unit pressure under the film and F is the tensile force per unit length in the film (7).

Because of the similarity between Equations 4 and 8, it is evident that ϕ is proportional to z, the elevation of the distended membrane. Applying this proportionality to Equations 5, 6, and 7 it can be shown that the shearing stress at a point is proportional to the slope of the membrane at the point and that the torsional moment is proportional to the volume enclosed by the membrane and the plane of the cross-section.

In practice the need to determine the proportionality constant is eliminated by including a circular cross-section in addition to the cross-section under investigation. The following relationship can then be determined (10):

$$K = \frac{V}{V^1} J \tag{9}$$

in which V is the volume for the irregular section and V^1, for the circular section.

The determination of K is thereby reduced to the problem of determining the volumes for the two sections. Various techniques to calculate the volumes are available to the investigator. Hetenyi in his Handbook of Experimental Stress Analysis discusses these techniques in some detail, with the exception of the photogrammetric method. This paper presents the photogrametric method.

PHOTOGRAMMETRIC METHOD

Thiel (3) seems to be the first to apply the principles of photogrammetry to the membrane analogy. In his investigation he employed a zero pressure film and used a specially constructed stereo-camera to photograph the membrane. In order to give the membrane a surface which could be photographed, a fine dust of lycopodium powder was allowed to settle on the film after it had dried. The drying occurred as a result of the addition of a small amount of sugar to the soap solution. This is a very tedious and time-consuming process.

One year later, 1935, for his dissertation, Englemann (11), using Thiel's camera, extended the work begun by Thiel on zero-pressure films.

In 1948, Pirard (12) applied the principles of photogrammetry in mapping the surface of an elevated membrane. His method differs from the other two in that he employed a single lens camera rather than a specially constructed stereo camera. The stereo effect was produced by very carefully sliding the apparatus containing the membrane from one side of the camera to the other. Pirard intimated that pumice powder was more satisfactory than lycopodium.

The photogrammetric analysis in all three of the above investigations was accomplished with a Zeiss Stereoplanigraph.

TEST EQUIPMENT

PLEXIGLASS BOX

Since the time to stretch the film across the boundary, elevate the membrane, and take the pictures is short, the usual heavy cast-iron box was not used. In its stead a plexiglass box approximately 9 by 12 by 3 inches was constructed. The top strips, on which rested the aluminum plate, and the sides were of $\frac{1}{4}$ inch plexiglass, and the bottom was $\frac{1}{8}$ inch plexiglass. The top was milled smooth and level. Two holes were drilled and tapped as close to the bottom as possible. Into these were fitted two stopcocks; one was to be used as an inlet for the water and the other as a drain. However in the course of the investigation it was found that only one was needed.

TEST SECTIONS

The cross-sections suggested by the National Advisory Committee for Aeronautics (13), along with a circular section, were cut in $\frac{1}{16}$ inch aluminum plates. An additional plate containing a rectangle and a larger circle in addition to the test circle was used as a check on the results. The plates were blued and inscribed with a grid for proper alignment of the plates by the stereo-plotter.

SEALING AND CLAMPING DEVICES

After the plates were laid on the top of the box, strips of masking tape were placed around the edges to form a seal. The plates were clamped to the box by means of C clamps, pressure being applied to four 1 inch wide bars of steel placed along the edges.

SOAP SOLUTION

In previous investigations employing the photogrammetric method of analysis, the membrane was dried and then a powder was dusted on the surface. The amount of time required to do this is excessive. The author believes that the excessive amount of time inherent in the method is the reason for the lack of widespread acceptance of the method as a standard means of analysis of the membrane. A means of eliminating the time-consuming technique was therefore sought. Several techniques were tried before it was found that the best method, and by the way the easiest, is to add a small amount of titanium dioxide to the soap solution. This addition seems to give it the required photographic properties without detracting from any of the properties required by the membrane analogy. The particles of titanium dioxide disperse through the membrane giving it a white paint-like appearance.

Almost any soap solution can be used with this method because the amount of time required to stretch the film across the boundary, to elevate the membrane and to take the photographs is small. There is no need therefore to obtain solutions which will give long lasting soap films if these solutions are not readily available. The solution used in this investigation had the following composition by weight:

1 part triethanolamine oleate
3 parts glycerine
6 parts distilled water.

PHOTOGRAMMETRIC EQUIPMENT

The Wild A-7 Autograph was chosen for this investigation. The instrument and its operators were made available by the Aeronautical Chart and Information Center.

PHOTOGRAPHIC EQUIPMENT

Because of the great versatility of the Autograph two cameras can readily be used. It is not essential that the lenses in the two cameras be a matched pair. Any two lenses will be satisfactory as long as they will allow the principal distance to lie within the range of operation of the Autograph. A 4 by 5 Crown Graphic press camera and a 4 by 5 Speed Graphic press camera were used. Two wide-angle lenses were used, one a Schneider-Kreuznach Angulen 1:6 8/90 lens and the other a $3\frac{1}{2}$ inch (90 mm.) Graflex Optar W.A. f 6.8 lens. The membrane was illuminated by a Heiland Strobonar IV speed light and extension lamp.

VERTICAL- AND HORIZONTAL-CONTROL

The larger the number of horizontal- and vertical-control points that are provided, the greater the accuracy that the stereoplotter can locate the contours. Six aluminum blocks, $\frac{1}{2}$ by $1\frac{1}{4}$ inches, with five steps approximately 50 mils each, provided the vertical-control while the dimensions of the cross-section provided the horizontal-control.

TEST PROCEDURE

ARRANGEMENT OF THE EQUIPMENT

The equipment was arranged as shown in Figure 1. The following are the approximate pertinent dimensions:

Principal distance	4.5 inches
Distance between lenses	8 inches
Lens-membrane distance	16 inches

The plexiglass box was centered between the two cameras. Leveling of the cameras and the box was done with two ordinary spirit levels.

The principal point—the point directly beneath the center of the lens—was found by removing the lens from the camera, opening the diaphragm as wide as possible, dropping a plumb bob through the opening, then closing the diaphragm as much as possible without restricting the movement of the cord. The location of these two points, one for each camera, was referenced to the edges of the box. During the remainder of the test, care was taken not to move either the cameras or the box.

The two lamps were positioned on opposite sides of the apparatus in order to eliminate shadows as much as possible. The Strobonar unit was connected to the shutter of one of the cameras. The timing was so arranged that the shutters would open, followed a short time later by a flash of light, and then the closing of the shutters. The duration of the illumination was approximately 1/1,000 of a second.

MEMBRANE TECHNIQUE

The author first attempted to stretch the membrane across the boundary with a thin strip of plastic, such as the edge of a triangle or ruler. This did not prove to be entirely satisfactory. It was later found that excellent results could be obtained by using a strip of plastic approximately $\frac{1}{4}$ inch wide.

The strip is dipped in the soap solution, then it is drawn across the boundary. By tilting the strip, allowing only the edge to come in contact with the plate, the membrane will adhere to the strip. If the strip is drawn past the boundary a short distance, the membrane will run off the strip onto the plate and then onto the boundary. This technique tends to give a uniform boundary and to eliminate any bubbles along the boundary. If bubbles do appear they can be removed by carefully placing the strip in contact with the membrane, allowing the membrane to adhere to the strip, and stretching the membrane past the boundary once more. Another technique that is effective in removing bubbles is to carefully place the strip in contact with the bubble and slide the bubble off the membrane. By using these techniques, one is able to obtain a membrane that is free of bubbles.

Excess solution can be removed from the membrane by a process similar to the one used for removing bubbles. Before employing this technique the excess solution around the boundary should be removed from the plate. This can be done best by depressing the membrane and then wiping it with any absorbent material.

STEP-BY-STEP PROCEDURE

The following is a step-by-step procedure:

1. Arrange the equipment as shown in Figure 1.

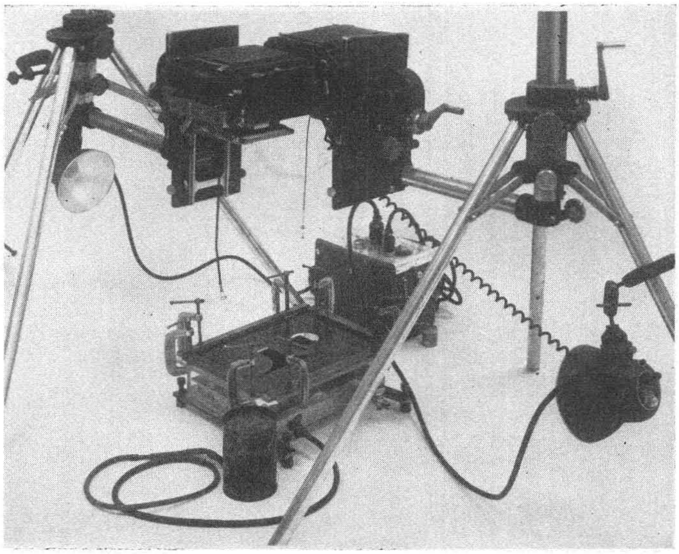

FIG. 1. Arrangement of test equipment.

2. Stretch the film across the boundary.
3. Depress the membrane.
4. Remove as much excess solution as possible.
5. Position the vertical control blocks.
6. Elevate the membrane.
7. Remove the slide from the film plate.
8. Open the shutters. (This automatically triggers the Strobonar unit.)
9. Close the shutters.
10. Replace the slide in the film plate.
11. Remove the film plate from the camera.
12. Develop the negative.
13. Make a photogrammetric analysis of the negatives.

TESTS PERFORMED

Three pairs of photographs were taken of each plate in order that the torsion constant calculated from the data would not be a result of only one test. The maximum height of the membrane was varied in each of the three stereo pairs.

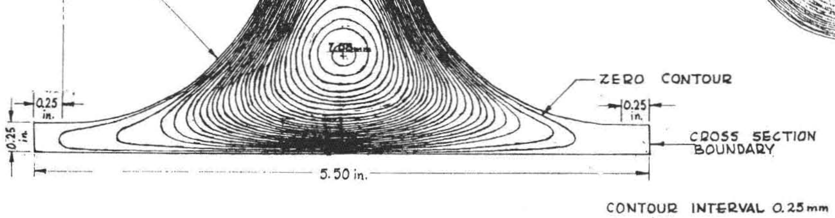

FIG. 2. Typical survey of soap film membranes for T section.

TABLE 1

COMPARISON OF THE EXPERIMENT AND THEORETICAL VALUES OF THE TORSION CONSTANTS OF THE RECTANGLE AND LARGE CIRCLE

Cross-section	Dimension, inches	Volume (mm.^{3})	Experimental K (in.^{4})	Theoretical K (in.^{4})	$\frac{K_{exp}}{K_{theo}}$
Rectangle	3.10×1.51	7,422	2.436	2.444	0.997
Circle	2.00	4,786	—	1.571	—
Circle	2.50	11,865	3.894	3.847	1.012
Rectangle	3.10×1.51	9,154	2.461	2.444	1.007
Circle	2.00	5,842	—	1.571	—
Circle	2.50	14,656	3.941	3.847	1.024
Rectangle	3.10×1.51	11,455	2.474	2.444	1.012
Circle	2.00	7,274	—	1.571	—
Circle	2.52	19,071	4.118	3.987	1.033

For circle $K=J$.

TEST RESULTS

VOLUME DETERMINATION

When the contoured plates (Figure 2) had been returned from the Aeronautical Chart and Information Center, the volume determination could be completed. The area enclosed by each contour had been determined during the plotting of the contours, by attaching a planimeter to the plotting arm. The volume was determined from these areas and the corresponding elevations by Simpson's rule.

The volume of the circular bubble obtained from these formula for the volume of a segment of a sphere of one side was in close agreement with the volume obtained by Simpson's rule. Since use of the formula required only that the height of the maximum ordinate be obtained rather than contours of the entire area, it was decided that the formula would be used for volume determination of the circular bubble.

TORSION CONSTANTS

The torsion constant of the rectangle, and the polar moment of inertia of the large circle as computed from the photogrammetric analysis, were compared with the theoretical values as a check on the method. The results of this comparison can be seen in Table 1. The theoretical value of the torsion constant of the rectangle was obtained by the formula (14):

$$K = \beta ab^3$$

in which

TABLE 2

TORSION CONSTANTS FOR THE EXPERIMENTAL CROSS-SECTION

Plate	Cross-section	Volume (mm.^{3})			K (in.^{4})			Aver. K
		Test 1	Test 2	Test 3	Test 1	Test 2	Test 3	
1	T	2,496	4,631	5,640	0.664	0.670	0.647	0.660
	O	5,845	1,047	13,555	1.555	1.555	1.555	
2	T	4,744	6,064	7,572	1.107	1.108	1.116	1.110
	O	6,666	8,513	10,549	1.555	1.555	1.555	
3	T	6,483	10,104	11,816	1.787	1.837	1.796	1.807
	O	5,791	8,778	10,498	1.596	1.596	1.596	
4	T	10,773	15,641	14,611	2.597	2.792	2.690	2.693
	O	6,515	8,800	8,532	1.571	1.571	1.571	

For circle $K=J$.

$\beta = 0.229$

a = long side of rectangle (3.10 inches)

b = short side of rectangle (1.51 inches)

The experimental value of the torsion constant was obtained from Equation 9 in the following manner:

$$K = \frac{V}{V^1} J$$

Volume of rectangle (V)	9,154 mm.^{3}
Volume of circle (V^1)	5,842 mm.^{3}
Polar moment of inertia (J)	1.571 in.^{4}
Torsion constant (K)	2.461 in.^{4}

The experimental values of the torsion constants for the cross-sections under investigation were determined by the same method used for the rectangle. The results are compiled in Table 2.

The data in Tables 1 and 2 indicate the practicality of using this method for the determination of torsion constants for irregular shaped cross-sections. The method requires little time and the accuracy of the results seems to be comparable with other methods which are more time-consuming.

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