*Radial Distortion: Its Calibration, Computation in Non-Gaussian Image Planes, and Compensation**

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ABSTRACT: *The customary procedure in designing photogrammetric objectives is to correct distortion in the Gaussian image-plane for the intended magnification. Since in some instruments, such as the Multiplex, the 55/525 Projector, and similar instruments, the region of actual imagery substantially departs from the Gaussian image-plane, it is of importance to know the distortion in this region. The problem was investigated and formulas were derived for computing distortion in non-Gaussian image-planes. The significance of distortion calibration was reviewed, and a complete set of calibration equations was derived. The results of the study were illustrated with two representative types of photogrammetric objectives. An investigation was also made of the problem of distortion compensation, and analytical conditions of compensation were derived.*

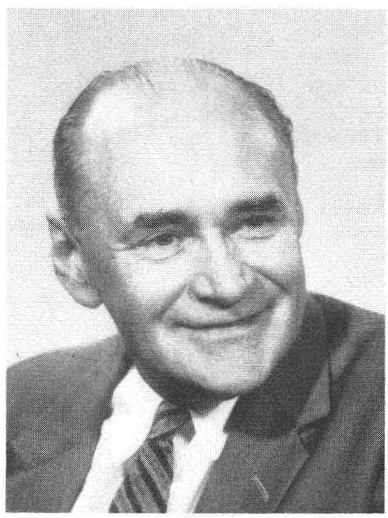
I. INTRODUCTION

THE usual procedure for computing the distortion of a plane-image produced by a centered lens system is based on two assumptions:

1. The position of the image-plane satisfies the Gaussian image equation:

$$1/s' - 1/s = 1/f, \tag{1}$$

   where s and s' are, respectively, the axial object and image distances from the corresponding principal (nodal) points of the system, and f is the equivalent focal-length determined by a paraxial computation.
2. The positions of the image-points are determined by the intersections of the principal rays with the image plane defined by the equation given above.



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If y and y' are the radial distances of the conjugate object and image points from the optical axis, and α is the half-field angle in the object space, the condition of freedom from distortion is:

$$\begin{aligned} &y' = My \text{ for finite conjugates; and} \\ &y' = F \tan\alpha \text{ for an object at infinity.} \end{aligned} \tag{2}$$

M and F in these expressions are certain constants, the significance of which will be presently explained.

For a general system of imagery (not necessarily involving optics), where their basic parameters—for example, the object distance and the equivalent focal-length—are not established, the constants may be assigned any value. In other words, an infinite number of orthoscopic images may be constructed for a given object. Thus utilizing orthogonal projection for three-dimensional objects, or orthogonal or conic projection for

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plane objects, any convenient scale factors may be used for the drawings. These factors analytically correspond to the constants of the equations given above.

When, however, the Gaussian image-equation is satisfied, the constant assumes the value of the paraxial lateral magnification (m_0) when the conjugates are finite, and it becomes equal to the equivalent focal-length (f) when the object is at infinity.

In optical imagery the condition of orthoscopy can be strictly satisfied only in some special cases. Generally the positions of the image-points deviate from their ideal position because of distortion. When the Gaussian image-equation is satisfied, the measure of linear distortion is:

$$\begin{aligned} D_g &= y' - M_0 y \text{ (finite conjugates)} \\ D_g &= y' - f \tan\alpha \text{ (object at infinity)} \end{aligned} \tag{3}$$

where D_g is the linear distortion in the Gaussian plane, and m_0 and f are the paraxial values of the constants.

Strictly speaking, the plane in which the image is received hardly ever exactly coincides with the Gaussian image-plane. This is due to the residual aberrations and uncertainty of focusing. With corrected lens systems, this discrepancy in most cases is not sufficient to cause doubt of the validity of the established procedure for computing the distortion.

In some cases, however, the focusing discrepancy may be so great that the image-equation cannot be considered even approximately satisfied. For example, with the Multiplex, the Twinplex, and other similar photogrammetric instruments, even the plane of optimum imagery (the nominal projection-plane) is far from the Gaussian image-plane. This is because of the substantial spherical aberration of the projection lenses customarily used in these instruments. Furthermore, these instruments utilize an extended region of imagery on both sides of the nominal projection plane. Although this region of imagery may be considered to be in good focus for practical purposes, it is theoretically out of focus with respect to the uniquely defined Gaussian image-plane. Therefore the distortion computed in this plane cannot be accepted without a further investigation as a representative measure of the distortion in the entire region of imagery.

Accurate knowledge of the distortion produced by a given system is of basic importance in the design and use of photogrammetric optics. Accordingly the problem of determining the condition of orthoscopy, and computing the distortion in non-Gaussian image-planes, warrants an investigation.

The preliminary results of this investigation were presented in 1948 before a meeting of the Optical Society of America.[1] The author now summarizes the derivations that previously were not published, and applies them to an analysis that may be of particular interest to those dealing with photogrammetric optics.

For the purposes of this analysis, the second assumption mentioned in the beginning of this paper is accepted: namely, that the intersections of the principal rays with the image-plane, whether Gaussian or non-Gaussian, determine the image-positions in this plane. A justification of this assumption for an "in focus" image is well presented in a book by Southall,[2] who supposes that with a satisfactorily corrected optical system, the energy distribution in the image patch will be such that the center of gravity of the distribution will be in practical coincidence with the image-point determined by the principal ray. This supposition is now extended to "out of focus" imagery. There may be, of course, some cases where the supposition is not valid. Then the actual energy distribution will have to be taken into consideration for an accurate determination of distortion. An investigation of such special cases is not within the scope of this paper.

II. Basic Relationships

A generalized system of imagery is represented by Figure 1. Here the object-plane is I; the conjugate Gaussian image-plane is I_g', and the actual image-plane is I'. The object-point P is imaged at P' by the principal ray PSP', whose axial intersections are E, S, and E'. The distance of the object-plane from the first nodal-point H is s, and the distance of the actual image-plane from

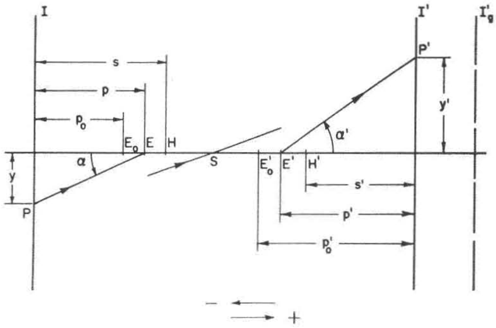

FIG. 1. A generalized system of imagery.

the second nodal-point H' is s'. The corresponding distance from the axial intersections of the principal ray are p and p'; they are p_0 and p_0' referred to the paraxial positions E_0 and E_0' of the entrance and exit pupils. The corresponding distances to the Gaussian image-plane are not indicated in the drawing to avoid its crowding; they will be designated by a subscript g in subsequent text.

The distances are negative for the object-plane, and positive for the image-plane. The slope-angle of the principal ray is α in the object space, and α' in the image space; they are positive as shown in the illustration. The object-height is y (negative); the image-height is y' (positive). The equivalent focal-length of the lens is f.

The following basic relationships of geometrical optics exist between the quantities of interest to us.

$$y = p \tan \alpha \quad \text{and} \quad y' = p' \tan \alpha' \tag{4}$$

The lateral magnification in the Gaussian image-plane is

$$m_0 = f/(f+s) \tag{5}$$

The lateral magnification at the exit pupil (E_0') is

$$m_e = \lim_{\alpha \to 0} (\tan \alpha / \tan \alpha') \tag{6}$$

The distance of the Gaussian image-plane from the paraxial exit pupil is

$$\begin{aligned} P_{0g}' &= m_e m_0 p_0 \text{ (finite conjugates)}; \\ P_{0g}' &= m_e f \text{ (object at infinity)}. \end{aligned} \tag{7}$$

III. Condition of Orthoscopy and Distortion in a Non-Gaussian Image Plane

The condition of orthoscopy in any image-plane was given in Eq. (2). Substituting there the values of y and y' from Eq. (4), and rearranging the results, we obtain:

$$\begin{aligned} M &= p' \tan \alpha' / p \tan \alpha \text{ (finite conjugates)} \\ F &= p' \tan \alpha' / \tan \alpha \text{ (object at infinity)} \end{aligned}$$

If the image is to be free from distortion, M (or F, when the object is at infinity) must be a constant for the entire image field, including the paraxial region. Consequently, the constant values (designated M_0 and F_0) may be determined using paraxial relationships.

Therefore,

$$M_0 = \lim (p' \tan \alpha' / p \tan \alpha)$$

and

$$F_0 = \lim (p' \tan \alpha' / \tan \alpha).$$

Noting that in the paraxial region $\lim p' = p_0'$, $\lim p = p_0$, and $\lim (\tan \alpha / \tan \alpha') = m_e$, we obtain the following values for the constants of orthoscopy in any image plane:

$$\begin{aligned} M_0 &= p_0'/m_e p_0 \text{ (finite conjugates)}; \\ F_0 &= p_0'/m_e \text{ (object at infinity)} \end{aligned} \tag{8}$$

While these expressions are formally the same as the generally known magnification formulas[3], their validity is not restricted by the usual condition that the actual image-plane is the Gaussian conjugate of the object-plane. The constants M_0 and F_0, given by Eq. (8) are analogous, respectively, to the paraxial lateral magnification (m_0) in the Gaussian image-plane, and to the equivalent focal-length (f), but their values differ from the Gaussian values.

Confirming the consistency of notation and derivation, these constants acquire their Gaussian values when the image is in the Gaussian image-plane. This can be easily demonstrated by substituting for p_0' in Eq. (8) the corresponding Gaussian values p_{0g}' determined by Eq. (7).

Following the usual procedure, the linear distortion in a non-Gaussian image-plane is determined by the following equations:

$$\begin{aligned} D_0 &= y' - M_0 y \text{ (finite conjugates)} \\ D_0 &= y' - F_0 \tan \alpha \text{ (object at infinity)} \end{aligned} \tag{9}$$

Contrary to the possible direct geometrical interpretation of the situation, it is of importance to note that $M_0 \neq s'/s$ and $F_0 \neq s'$; however the equal sign is valid for a Gaussian imagery. It is true that using $M = s'/s$ or $F = s'$ in a case of non-Gaussian imagery, or assigning any other arbitrary values to these constants, we would also define an orthoscopic-image. Then we could use these values in computing the distortion in accordance with Eq. (9). But an orthoscopic-image thus defined would be of a different size from the one uniquely determined by the paraxial values, and the corresponding distortion would have some values resulting from an implicit arbitrary calibration. The results so obtained would tend to be confusing, and generally they would not yield a distortion distribution that would be most favorable for the intended photogrammetric application.

Of course, these results could be recalibrated to suit the use. However, from the point-of-view of the lens designer it is more rational to determine the distortion values using the paraxial constants, and then obtain the desired distribution by calibration. Good arguments in favor of this approach are given in a paper by Dr. I. C. Gardner.[4]

IV. Calibrated Distortion

A. Theoretical Relationships

For a period some confusion existed among lens designers and photogrammetrists with regard to the procedures and significance of distortion calibration. Later, particularly after Dr. Gardner published the paper mentioned above, the situation became clarified. Recently, however, the significance of calibration appeared to be questioned again in a paper by J. G. Lewis.[5] It is desirable, therefore, to review the theory of calibration, and to present the calibration equations in a form which can be readily used in practice.

The concept of calibration appears to be generally known only in its application to systems with an object at infinity (such is nearly the case with aerial cameras), although a calibration procedure must also be used for the proper adjustment of such instruments as reduction printers and rectifiers, whose conjugates are finite. It may be also noted that even in a representative document, such as the Military Standard on photographic lenses,[6] no terminological differentiation is made between the distortion computed by using the paraxial constant of orthoscopy (the equivalent focal-length, when the object is at infinity) and the distortion computed with an adjusted (calibrated) value of the constant. Since such a differentiation appears to be desirable, and it is also needed for clarity of the subsequent text, the term *characteristic-distortion* will be used to denote the distortion computed with the appropriate paraxial constant. The term *calibrated-distortion* will denote the distortion based on the constant determined in the process of calibration.

Although the transformation relationships between the characteristic- and calibrated-distortion should have been of theoretical and practical interest to those dealing with photogrammetric optics, a complete presentation of these relationships has not been located in prior literature. Therefore they will be derived here. The following notation will be used:

D_0 = characteristic linear distortion.
D_r = characteristic relative distortion. The characteristic-distortions are determined with paraxial constants of orthoscopy M_0 or F_0.
D_c = calibrated linear distortion.
D_{rc} = calibrated relative distortion. The calibrated-distortions are determined with calibrated constants of orthoscopy M_c or F_c.
M_c = calibrated magnification.
F_c = calibrated focal-length.
$y_0' = M_0 y$ (or $y_0' = F_0 \tan\alpha$) ideal image-size.
$y_c' = M_c y$ (or $y_c' = F_c \tan\alpha$) calibrated image-size.
y = object-size.
y' = actual image-size.

The characteristic linear distortion is determined by the previously given Eq. (9). The calibrated linear distortion is determined by the following equations:

$$\begin{aligned} D_c &= y' - M_c y \text{ (finite conjugates)} \\ D_c &= y' - F_c \tan\alpha \text{(object at infinity)} \end{aligned} \tag{10}$$

If y' is now eliminated, using Eq. (9), the relationship between the characteristic linear distortion and calibrated linear distortion becomes

$$\begin{aligned} D_c &= D_0 + (M_0 - M_c)y; \\ D_c &= D_0 + (F_0 - F_c)\tan\alpha. \end{aligned} \tag{11}$$

Since in usual graphical representation, linear distortion is plotted on the ordinate axis as a function of the ideal image-size y_0', which is plotted on the abscissa axis, it is desirable to introduce y_0' into Eq. (11). This is done by substituting $y = y_0'/M_0$ and $\tan\alpha = y_0'/F_0$. Thus we obtain the following expressions connecting the characteristic and calibrated distortion:

$$\begin{aligned} D_c &= D_0 - (M_c/M_0 - 1)y_0' \\ D_c &= D_0 - (F_c/F_0 - 1)y_0' \end{aligned} \tag{12}$$

In accordance with relationships of analytic geometry, these equations show that the calibrated-distortion is obtained by a transformation of the original rectangular system of coordinates into an oblique one. In this transformation the new and old ordinate axes remain in coincidence, and the abscissa axis is at an angle ρ relative to the original abscissa axis. This angle is determined as

$$\begin{aligned} \tan\rho &= M_c/M_0 - 1 \text{ (finite conjugates)} \\ \tan\rho &= F_c/F_0 - 1 \text{ (object at infinity)} \end{aligned} \tag{13}$$

Distortion, being much smaller than the image-size, is usually plotted on a considerably exaggerated scale. If the relative scale-factor for the distortion plot is k, then the angle of rotation ρ_k, as would be actually measured on the graph, is given by:

$$\begin{aligned} \tan\rho_k &= k(M_c/M_0 - 1) \\ \tan\rho_k &= k(F_c/F_0 - 1) \end{aligned} \tag{14}$$

Another way of interpreting the transformation from characteristic linear to calibrated linear distortion is to visualize the distortion curve as having been rotated through an angle minus ρ_k with respect to the abscissa axis.

If now the original abscissas are denoted X, the transformed abscissas are denoted X', and the original ordinates are denoted $Y(Y=kD_0)$, the transformation equations for abscissas are:

$$X' = X/\cos \rho_k$$

when the abscissa axis is rotated; and

$$X' = X/\cos \rho_k - Y \sin \rho_k$$

when the distortion curve is rotated.

In either case the transformed abscissas do not represent a quantity that has a photogrammetric meaning. Therefore, for a conventional graphical representation the calibrated-distortion curve must be replotted, in a rectangular system of coordinates, as a function of the calibrated image-size y_c'. As a result, the shape of the calibrated-distortion curve will differ somewhat from the original shape of the characteristic curve. As will be shown later, there may be cases in which this change cannot be disregarded. Still rotation of the abscissa axis is a useful means for rapid orientation as to a possible gain by a calibration.

As is the case with characteristic linear distortion, calibrated linear distortion is always equal to zero on the optical axis because D_0, y_0', and $\tan \alpha$ are equal to zero. It will be shown later that the calibrated relative distortion never becomes zero on the axis.

The characteristic and calibrated relative distortions are defined by the following expressions:

$$D_r = y'/M_0 y - 1 \quad \text{and} \quad D_{rc} = y'/M_c y - 1$$
$$D_r = y'/F_0 \tan \alpha - 1 \text{ and } D_{rc} = y'/F_c \tan \alpha - 1.$$

By elimination of y' we obtain:

$$\begin{aligned} D_{rc} &= D_r M_0/M_c + (M_0/M_c - 1) \text{ (finite conjugates)} \\ D_{rc} &= D_r F_0/F_c + (F_0/F_c - 1) \text{ (object at infinity).} \end{aligned} \tag{15}$$

These equations show that the calibrated relative distortion is obtained by multiplying the characteristic relative distortion by a factor, and adding a constant term. Since this factor usually does not differ significantly from one, the effect of the multiplication may be disregarded, and the calibrated-distortion obtained by a parallel translation of the abscissa (image-size) axis through a distance equal to the respective constant term of Eq. (15) taken with the opposite sign. The most interesting fact, not generally known, is that calibrated relative distortion never becomes zero on the axis, although characteristic relative distortion does. The value of the calibrated relative distortion on the optical axis is given by the constant term of Eq. (15).

It should be re-emphasized that the transformation equations for linear and relative distortions do not involve any physical changes of the optical system. Its focusing, all its physical parameters, and the actual image-positions (y') are the same both after and before the calibration. This fact was used by some optical designers as a basis for objecting to the whole concept of distortion calibration. They reasoned that the constant of orthoscopy and the corresponding orthoscopic-image are uniquely defined for a given system by the paraxial parameters (p_0', p_0, and m_e), and, consequently, the distortion curve is also uniquely determined with respect to the orthoscopic-image. Indeed, the correct physical interpretation of the situation is that, so long as the paraxial parameters remain fixed, it would be impossible to design a lens system, even of a most bizarre form, that would produce an orthoscopic (distortion-free) image different from the one defined by the paraxial constants given in Eq. (8). Therefore, the process of calibration appeared to be an attempt to use an extraneous procedure in order to change arbitrarily the ideal image-size that was uniquely determined in the course of optical design.

What had been overlooked in these arguments was that this ideal image-size is not of interest to the photogrammetrist when the image is distorted. He is interested in the actual positions of the image points and their transfer to the map with least deviations from the positions determined by the scale. The calibration procedure is equivalent to a search for a most favorable scale (in other words, a most favorable photogrammetric grid) that would minimize the position errors in the final map. This search does not require knowledge of the lens construction, its paraxial parameters, or of the ideal image-size, because a reduction of the position errors may be achieved by an empirical fitting of a grid to accommodate best the image positions of the control points. The calibration transformations, derived in the preceding text, provide a mathematical means for determining a favorable fitting on the basis of distortion data.

It is hoped that a misunderstanding about distortion calibration will not arise again, and that its significance is now clear not only to photogrammetrists but to optical designers as well.

TABLE I

154-MM. F/6.3 TOPOGON. CHARACTERISTICS DISTORTION DATA
$F_0 = EFL = 153.706$ mm.

α (degrees)	0	10	20	25	30	35	37.5	40.0	42.5	45.0
Y_0' (mm).	0.000	27.103	55.944	71.674	88.742	107.626	117.943	128.975	140.846	153.706
D_0 (mm.)	0.000	+0.017	+0.088	+0.148	+0.227	+0.321	+0.352	+0.355	+0.330	+0.272
D_r (%)	0.000	+0.063	+0.157	+0.206	+0.256	+0.298	+0.298	+0.275	+0.234	+0.177

B. ILLUSTRATION OF THE CALIBRATION PROCEDURE

To show what can be gained by calibration, a lens with a considerable distortion was purposely chosen. This is a 154 mm. f/6.3 Topogon, whose formula was derived from one given in the U. S. Patent 2,031,792.[7] The characteristic distortion data for this lens, with an object at infinity, are listed in Table I.

The distortion values D_0 and D_r are plotted as functions of the ideal image-size y_0' in Figure 2. Inspection of the linear-distortion curve shows that the distortion distribution can be considerably improved by such a calibration that the linear-distortion at 37.5 degrees ($y_0' = 117.943$ mm.) would have the same absolute value as the linear-distortion at 45 degrees ($y_0' = 153.706$ mm.); but their signs would be different. To obtain this calibration, we use Eq. (12) for an object at infinity. The equation to be solved is:

$$0.352 - (F_c/153.706 - 1)117.943 = -0.272 + (F_c/153.706 - 1)153.706.$$

From this we obtain the following value for the calibrated focal length

$$F_c = 154.069 \text{ mm.}$$

Since the plotting scale for the linear distortion in Figure 2 has been exaggerated 200, the scale factor $k = 200$. Then the angle of rotation of the abscissa axis for the transformation of the characteristic linear into the calibrated distortion is, in accordance with Eq. (14):

$$\tan\rho = 200(154.059/153.706 - 1) = 0.4594;$$
$$\rho = 24.67 \text{ degrees.}$$

Confirming the validity of the theoretical formulas, the distortion values that may be read against the rotated abscissa axis agree with the directly computed values of the calibrated linear distortion listed in Table II. The value of per cent distortion on the axis ($\alpha = 0$) was computed using the expression for the constant term in Eq. (15). Thus on the axis:

$$D_{rc} = 100(153.706/154.059 - 1)\%;$$
$$D_{rc} = -0.230\%.$$

The curve of the calibrated distortion is given in Figure 3. A comparison of the data of Tables I and II and of Figures 2 and 3 shows that, as a result of calibration, the maximum value of linear distortion has been reduced from the original 0.355 mm. to 0.081 mm., and the total spread has been reduced from 0.355 mm. to 0.162 mm. This is a considerable improvement, which may be advantageous in some actual applications. As predicted theoretically, the total spread of the calibrated per cent distortion 0.299 per cent is, within the computational accuracy, the same as 0.298 per cent spread for the characteristic distortion. The abscissa axis for the per cent distortion only has shifted through an interval of +0.230 per cent.

To compare the shapes of the characteristic linear distortion and of the calibrated distortion curves, a plot is presented in Figure 4. It shows the characteristic linear distortion curve of Figure 2 rotated through an angle of 24.67 degrees toward the abscissa axis, and the calibrated linear distortion curve the same as plotted in Figure 3. The two curves are in

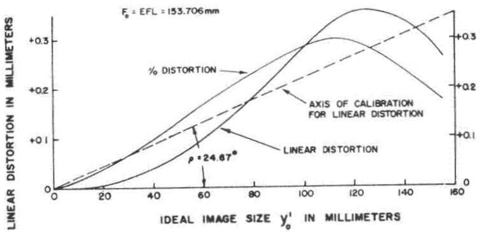

FIG. 2. Characteristic linear and % distortion 154 mm. f/6.3 Topogon.

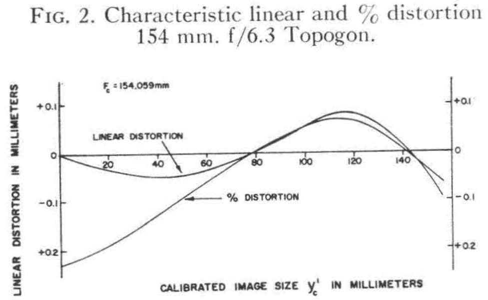

FIG. 3. Calibrated linear and % distortion for 154 mm. f/6.3 Topogon.

TABLE II

154-MM. F/6.3 TOPOGON. CALIBRATED DISTORTION DATA
$F_c = 154.059$ mm.

α (degrees)	0	10	20	25	30	35	37.5	40	42.5	45
Y_c' (mm.)	0.000	27.165	56.073	71.839	88.946	107.873	118.214	129.271	141.169	154.059
D_c (mm.)	0.000	−0.045	−0.041	−0.017	+0.023	+0.074	+0.081	+0.059	+0.007	−0.081
D_{rc} (%)	−0.230	−0.166	−0.073	−0.024	+0.026	+0.068	+0.069	+0.046	+0.005	−0.053

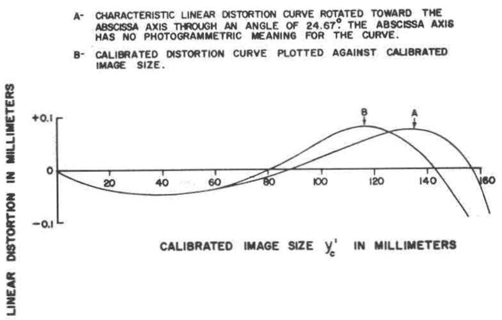

FIG. 4. Comparison of shapes of characteristic and calibrated distortion curves for 154 mm. f/6.3 Topogon.

practical coincidence up to the field angle of 25 degrees ($y_c' = 71.839$ mm.), and then depart significantly, As was mentioned before, the abscissa values have no photogrammetric meaning for the rotated characteristic linear distortion curve. Therefore, starting in this case from $y_c' = 71.839$ mm., wrong distortion values would be obtained by reading the points on the rotated curve against the abscissas which represent the calibrated image size y_c' and are in the proper relation with respect to the ordinates of the calibrated distortion curve. However, the rotated curve and the calibrated distortion curve indicate nearly the same total distortion spread.

V. DISTORTION IN A NON-GAUSSIAN REGION WITH A REPRESENTATIVE PROJECTION OBJECTIVE

To explore distortion variation in non-Gaussian image-planes, a formula was derived for a representative projection objective of the type that may be used in the Geological Survey Twinplex Projector 55/525. The optical layout of this lens is given in Figure 5. In the course of the design, the main emphasis was put on the practical elimination of distortion in a Gaussian projection plane, while the other aberrations were not necessarily brought into an entirely satisfactory balance. For this reason not all the design parameters of the lens are given in Figure 5. Thus the possibility is eliminated that the formula would be used for actual manufacturing.

In accordance with the Geological Survey Specifications, the lens should be set in the Twinplex projector at the principal-distance (the distance from the diapositive plane to the internal nodal-point) of 55.0 mm., have the nominal projection distance of 525.0 mm., and work satisfactorily in the region of projection distances from 425.0 mm. to 625.0 mm. Since for these projection distances, the object distance (the principal-distance) remains unchanged, the lens is required to work in an extended region of non-Gaussian image planes. Hence, the knowledge of distortion in this region is of theoretical as well as practical importance.

To satisfy the geometry of the specification, the lens should have a nominal equivalent focal-length of 49.7845 mm. and yield a magnification of -9.545455 in the nominal projection distance of 525.0 mm. from the external nodal-point. The numerical values of these quantities (as well as of the other quantities that will be used in the subsequent text) have been determined with a sufficient number of decimal places to secure the distortion values to the fourth decimal for smooth graphical representation of the curves.

Because of the substantial spherical aberra-

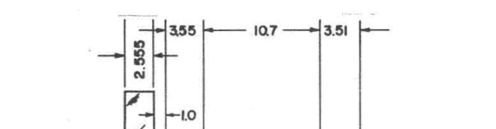

FIG. 5. Optical layout of an objective of the type that may be used in 55/525 projector

tion encountered with lenses of this type, the actual computed equivalent focal-length of the lens should be somewhat longer than the nominal focal-length as derived from the specification; otherwise the usable region of imagery will be nearer to the projector than is specified. Therefore the computed equivalent focal-length of the lens was set to be 50.2832 mm.

Then the design work proceeded until a formula was derived whose distortion was considered to be negligible at the specified magnification in the Gaussian image-plane. Following a customary procedure, no consideration was given in the course of the design to the distortion in the specified region of non-Gaussian image-planes. The distortion in these planes was computed only after the final formula had been established.

The basic paraxial parameters, used for computing the object and image distances, are listed in Table III, where EFL denotes the equivalent focal-length, and the other symbols are the same as were defined in Figure 1 and the associated text.

The distance and magnifications for the three non-Gaussian projection planes and one Gaussian plane are given in Table IV.

The characteristic linear distortion was computed in the four projection planes listed above, and the data are listed in Table V.

A graphical presentation of distortion data is given in Figure 6. The data show that the distortion in the region of actual imagery significantly departs from the distortion computed in the Gaussian image-plane. Thus, while in the Gaussian image-plane the total spread of distortion is 35 microns, with the maximum absolute values of 21 microns, the total spread is 46 microns in the far projection-plane, 58 microns in the nominal projection-plane, and 74 microns in the near projec-

TABLE III

BASIC PARAXIAL PARAMETERS

Parameter	*EFL*	m_e	HE_0	$H'E_0'$
Value	50.2832 mm.	1.001700	−0.0853 mm.	−0.0856 mm.

TABLE IV

SETTING DATA: MAGNIFICATION AND DISTANCES

Plane	*s mm.*	P_0 *mm.*	*s′ mm.*	P_0' *mm.*	M_0
Non-Gaussian	−55.0009	−54.9156	425.0000	425.0856	− 7.727571
Non-Gaussian	−55.0009	−54.9156	525.0000	525.0856	− 9.545455
Gaussian	−55.5510	−55.4657	530.2592	530.3448	− 9.545455
Non-Gaussian	−55.0009	−54.9156	625.0000	625.0856	−11.363342

TABLE V

CHARACTERISTIC LINEAR DISTORTION IN FOUR IMAGE PLANES AT DIFFERENT PROJECTION DISTANCES (s')

Nominal Field Angle in Degrees	*Projection Distances (s') in millimeters*			
	425.0	525.0	530.3	625.0
	Linear Distortion D_0 in millimeters			
10	−0.0034	−0.0023	−0.0018	−0.0011
15	−0.0069	−0.0067	−0.0053	−0.0065
20	−0.0096	−0.0112	−0.0109	−0.0129
25	−0.0112	−0.0160	−0.0179	−0.0209
30	−0.0071	−0.0156	−0.0206	−0.0241
35	+0.0053	−0.0071	−0.0174	−0.0197
40	+0.0294	+0.0133	−0.0037	−0.0030
45	+0.0628	+0.0422	+0.0146	+0.0215

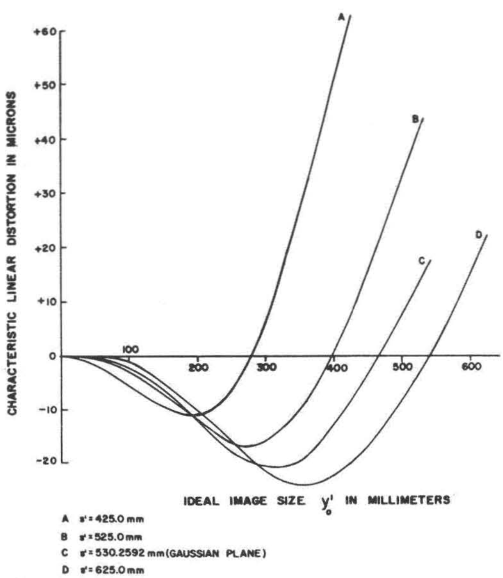

FIG. 6. An objective of the type that may be used in 55/525 projector. Characteristic linear distortion curves in image planes at different projection distances s'.

tion-plane. The respective absolute maxima are 24, 42, and 63 microns. The shapes of the distortion curves are such that not much could be gained by a calibration.

Whether or not this variation of distortion should be considered as disturbing may be evaluated by using the well-known approximate formula which determines how much a stereoscopic model of a plane will be warped in the presence of distortion. The formula is:

$$z = s'\Delta x/d$$

where:

- z is the model warpage (its departure from a true plane).
- Δx is the maximum stereoscopic parallax caused by distortion.
- s' is the projection distance.
- d is the base length (the distance between the axes of the projectors).

We may assume, for the purposes of orientation, that the maximum stereoscopic parallax is directly proportional to the total distortion spread. Then using the spreads listed above, and the respective projection-distances of 625 mm., 525 mm., and 425 mm., we determine that the warpages in these three projection-planes will be in the ratio of nearly one to one. Therefore, in this particular case, the warpage may be expected to be within the tolerance in the whole region of imagery if it is within the tolerance for the nominal projection-plane.

It is reasonable to assume that the lenses used in actual photogrammetric projectors, being of the same type as the one used in this investigation, should have the same favorable distortion characteristics. Still it may be of interest to determine what the real situation is by measuring the model warpage in the entire region of imagery of representative instruments, and not only in the nominal projection plane as is done customarily.

VI. Compensation of Distortion

If aerial negatives contain a distortion greater than can be tolerated for precision mapping, the distortion must be compensated either optically or mechanically at one of the steps of the mapping system. A representative optical compensation scheme, shown in Figure 7, utilizes a printer which produces a distortion-free positive from a distorted aerial negative. This scheme will be investigated here.

Although a solution of the compensation problem obviously requires that the distortion-curve of the aerial negative should be "matched" by the distortion-curve of the printer, the analytical expression of this condition is not obvious. The reason for this is that using calibration one may represent distortion by an infinite number of curves. The question then arises whether or not the condition of compensation imposes any inherent restriction upon the selection of distortion-curves, and whether or not a particularly favorable selection exists. The answer to this question and the compensation condition are derived from the following analysis.

In Figure 7 the aerial camera lens is at L_1; it produced a distorted image y' in the focal-plane I'. The printer lens is at L_2; it produces in the printer image-plane I'' a distortion-free image y'' of the object on the ground, which may be assumed to be at infinity. The distortion produced by the camera lens and

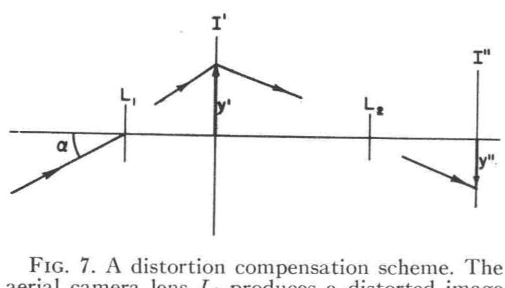

FIG. 7. A distortion compensation scheme. The aerial camera lens L_1 produces a distorted image y' in the camera focal plane I'. The printer lens L_2 produces a distortion-free image y'' of the object on the ground.

the compensating distortion produced by the printer are computed using the constants F_1 and M_2, without making an assumption whether their values are paraxial or calibrated. M_2 is the magnification of the printer in its image-space. It is more convenient, however, to use relationships in the printer object-space, where its magnification in the negative is $1/M_2$. Then the following equations may be written

$$y' = F_1 \tan \alpha + D_1; \; y' = y''/M_2 + D_2; \; y'' = MF_1 \tan \alpha$$

where D_1 is the distortion in the negative, D_2 is the compensating distortion introduced by the printer in the negative plane, and M is the factor which determines how much the actual image produced by the printer is magnified (or reduced) with respect to the assumed ideal or calibrated image-size in the negative. By eliminating y' and y'' in this set of equations, we obtain the following condition for distortion compensation:

$$D_1 - D_2 = (M/M_2 - 1)F_1 \tan \alpha \qquad (16)$$

Now we will prove that if this equation is satisfied, it will be also satisfied with any other pair of F_1' and M_2' and the corresponding distortions D_1' and D_2'.

On the basis of the calibration transformations given by Eq. (11), we may write that

$$D_1 = D_1' + (F_1' - F_1) \tan \alpha;$$
$$D_2 = D_2' + (1/M_2' - 1/M_2)M'F_1' \tan \alpha$$

It should be noted that the reciprocals of M_2' and M_2 have been used because, as was before stated, the equation refers to the printer object-space. The object-size, denoted y in Eq. (11), is in this case y'' which represents the actual image-size produced by the printer. This image-size is fixed and not affected by the calibration. Therefore,

$$y'' = MF_1 \tan \alpha = M'F_1' \tan \alpha.$$

Substituting now the values of D_1 and D_2 into Eq. (16), we obtain

$$\begin{aligned} D_1' - D_2' + (F_1' - F_1) \tan \alpha \\ - (1/M_2' - 1/M_2)M'F_1' \tan \alpha \\ = (M/M_2 - 1)F_1 \tan \alpha \end{aligned}$$

This reduces to:

$$D_1' - D_2' = (M'/M_2' - 1)F_1' \tan \alpha \qquad (17)$$

Since both Eqs. (16) and (17) are satisfied, the condition of elimination of distortion will be fulfilled with any arbitrary pair of values F_1 and M_2 if it is fulfilled in the process of design for one pair of values. The following considerations introduce, however, the restriction that each pair of values must be either paraxial or calibrated, if distortion is compensated within the entire field of coverage, and that the compensating characteristic distortion-curves are necessarily identical.

Suppose we have succeeded in eliminating distortion for the entire field of coverage. This means that the condition of compensation has been satisfied with a pair of values of F_1 and M_2. Then, as was proved, the condition of compensation must be satisfied with any other values of these constants, including their paraxial values and the corresponding characteristic distortions. Hence Eq. (16) must be satisfied for a pair of characteristic distortion curves.

According to the theory of optical imagery, a characteristic distortion-curve may be represented by a series with the terms containing odd powers of the image-size, the first term being cubic. From this it follows that the difference of distortion values D_1—D_2 of two characteristic distortion-curves cannot be directly proportional to the first power of the ideal image-size $F_1 \tan \alpha$ throughout the field of coverage, as would be required by Eq. (16). The equation can be satisfied only when the value in parentheses is equal to zero, or when $M_2 = M$. The condition of compensation of characteristic distortion-curves now becomes $D_1 = D_2$, which makes the curves identical in the plane of the negative (not, as perhaps one could expect, that one curve should reproduce the other with the reversed sign of the distortion values).

Since the characteristic distortion-curves must be identical, it is obvious that a compensating calibrated-distortion-curve cannot be identical with a characteristic-distortion-curve. The impossibility of securing the identity of a characteristic-distortion-curve with a calibrated-distortion-curve may be deducted also from the following basic consideration. As before stated, a characteristic-distortion-curve may be represented by a series with the terms containing odd powers of the image-size, starting with a cubic term. A calibrated-distortion-curve according to Eq. (12) may then be represented by a similar series minus an additional term with the first power of the ideal image-size. If the curves are to be identical, all their terms containing the same powers must be equal. This condition obviously cannot be satisfied for the extra term in the calibrated-distortion-curve.

Now let Eq. (17) represent a compensation of two calibrated-distortion-curves. This equation is satisfied with any calibrated values M' and M_2' when Eq. (16), representing the compensation of characteristic-distortion-curves, is satisfied (with $M = M_2$). Therefore, we can choose $M' = M_2'$, and make the calibrated-distortion-curves also identical. Thus one may in the process of design attempt to secure the identity either of the two characteristic or of the two calibrated-distortion-curves. There is no inherent criterion for a choice between these two alternatives. The lens designer will prefer, however, to use the characteristic-distortion-curves because they are based on the paraxial constants which have to be computed anyway when the optical system is set up and when it is modified in the course of the design. Furthermore, by using the characteristic-distortion-curves, the designer will avoid the additional computations needed for calibration.

If the designer did not succeed in obtaining the identity of two characteristic-distortion-curves, Eq. (16) will not be satisfied throughout the field (it may be, however, satisfied for some image-points). Then Eq. (17) also will not be satisfied throughout the field independently of whether both distortion-curves are calibrated, or one of them is characteristic and the other calibrated. In this case the image produced by the printer will contain some residual distortion. Then by calibrating either one of the compensating curves or the residual distortion-curve, the designer may have a better evaluation of the effective residuals, and he may distribute them favorably. But, as was reasoned before, a calibration will not permit making the compensating curves identical or to eliminate distortion completely.

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*The Future for Photogrammetry and Photo Interpretation**

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ONCE Woodrow Wilson was asked if he would give a speech on a particular topic. He replied, "That depends on how long you want me to speak and how much time I'll have in which to prepare my remarks. If it's to be a 15 minute speech, I'll need nearly a week to prepare it; if it's to be a 30-minute speech, I'll need only a day or so to prepare it; but if it's to be a 60-minute speech, I'm ready to give it right now."

As I've been asked to give a 60-minute speech, I claim readiness to give it "right now," Woodrow Wilson style, without reference to notes or manuscript. I will be doing so, not because of any great ability to speak extemporaneously, but because most of my talk will be given from lantern slides and I never have acquired the ability to read a paper and point to features on the screen simultaneously.

Actually, I abandoned the idea of reading a paper this morning when I learned of the embarrassment which this habit caused a certain minister. As he faced his congregation he reached in his coat pocket for the carefully prepared manuscript of his sermon. Finding to his consternation that he had left it at home he said, "Friends in speaking to you this morning I must rely largely on the Lord for guidance; but I assure you that next Sun-

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