quently the best target location capabilities, are greatest for the frame type, followed by the panoramic and the strip cameras respectively.

For the X-coordinate the percentage error in locating targets between frame and panoramic camera is 8.92%, between the frame and strip 13.87% and 4.73% between the panoramic and strip camera. For the Y-coordinate the percentage error between the frame and panoramic is 15.77%, between frame and strip 23.13% and 6.36% between the panoramic and strip.

There are a number of exterior orientation elements which have not been considered in the camera comparison; however, they are factors which affect the system error to the same extent regardless of camera type. Since one purpose of this investigation is an accuracy comparison between camera types, these added elements have not been considered.

*The Utilization of Constraints in Analytical Photogrammetry**

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(*Abstract is on next page*)

INTRODUCTION

WITH the introduction of high speed, large capacity digital computers, the possibility of solving elaborate problems in analytical photogrammetry, in particular the simultaneous adjustment of large blocks of aerial photographs in a control extension, has now become a reality. A number of techniques that adequately express the projective relations of photogrammetry have gained wide acceptance in recent years; however, most of these techniques have limited themselves to the case in which the camera and ground-point parameters are exactly known or completely unknown. It is the purpose of this paper to describe a number of constraints and their application to some of these techniques.


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Such constraints are obtained by enforcing the camera or points in the object space to conform to some functional or geometrical relationship or to lie within certain bounds as defined by weighting. The value of constraints lies in their ability (1) to utilize geometrical properties of the physical situation which may lead to a reduction in the number of unknown camera or ground-point parameters and, hence, to a reduction in the size of the normal equation matrix which must be inverted; and (2) to utilize these same geometrical properties and/or weighting fac-

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tors to reduce the magnitude of error ~~propagation~~ accumulation particularly in areas lacking in ground control.

The Projective Equations

In order to better explain the manner in which constraints may be utilized in analytical photogrammetry, a description of the projective equations of photogrammetry, the formation of the condition equations, and the solution of the resulting normal equations is given below. The description is brief since complete derivations have been given by Schmid (1959) and Brown (1958).

ABSTRACT: *The universality of the projective equations of analytical photogrammetry, as first developed by von Gruber, has been well documented in the work of Hellmut Schmid and Duane Brown. These equations have shown themselves amenable to both aerial and terrestrial photography and are completely rigorous from a statistical viewpoint.*

This rigorousness has led to the possibility of utilizing various constraints in the solution of problems in analytical photogrammetry. The utilization of both weight constraints, in which the camera and control point parameters may be weighted in accordance with how well they are known, and geometrical constraints, in which the camera and control point parameters are enforced to some geometrical pattern, are described.

The projective equations, completely stating the relations which exist between the image-space (photograph), the center of perspective (the lens), and the object-space were first developed by von Gruber (1932). They are

$$
\begin{aligned}
x_{ij} &= x_{p_i} - f_i \frac{(a_{11})_i(X_j - X_i^c) + (a_{21})_i(Y_j - Y_i^c) + (a_{31})_i(Z_j - Z_i^c)}{(a_{13})_i(X_j - X_i^c) + (a_{23})_i(Y_j - Y_i^c) + (a_{33})_i(Z_j - Z_i^c)} \\
y_{ij} &= y_{p_i} - f_i \frac{(a_{12})_i(X_j - X_i^c) + (a_{22})_i(Y_j - Y_i^c) + (a_{32})_i(Z_j - Z_i^c)}{(a_{13})_i(X_j - X_i^c) + (a_{23})_i(Y_j - Y_i^c) + (a_{33})_i(Z_j - Z_i^c)}
\end{aligned}
\tag{1}
$$

in which

the subscripts i and j indicate the ith photograph and the jth object-space point;
x_{ij} and y_{ij} are the photographic coordinates of the jth object-space point imaged on the ith photograph;
x_{p_i}, y_{p_i}, and f_i are the coordinates of the principal-point and focal-length of the ith photograph;
$(a_{11})_i$ through $(a_{33})_i$ are the elements of the orientation matrix relating the photograph coordinate system to the object-space coordinate system (these elements are functions of the rotations roll, ω, pitch, ϕ, and yaw, κ);
X_j, Y_j, and Z_j are the object-space coordinates of the jth object-space point; and
X_i^c, Y_i^c, and Z_i^c are the object-space coordinates of the center of perspective (or camera station).

These projective equations apply equally well to terrestrial and aerial photography, to any number of cameras, and to any number of object-space points imaged on the photographs. The single condition enforced is that an object-space point, a corresponding image-space point, and the perspective center are collinear (after corrections for lens distortion, film shrinkage, atmospheric refraction, etc., have been applied). The condition equations, fulfilling this requirement, are formed by equating the projective equations to zero, that is

$$\begin{aligned} e_{ij} &= F(x_{ij}, x_{p_i}, f_i, \omega_i, \phi_i, \kappa_i, X_i^c, Y_i^c, Z_i^c, X_j, Y_j, Z_j) = 0 \\ \tilde{e}_{ij} &= F(y_{ij}, y_{p_i}, f_i, \omega_i, \phi_i, \kappa_i, X_i^c, Y_i^c, Z_i^c, X_j, Y_j, Z_j) = 0 \end{aligned} \tag{2}$$

These equations are solved for the camera inner-orientation parameters x_{p_i}, y_{p_i}, and f_i; the camera exterior-orientation parameters roll, ω, pitch, ϕ, and yaw, κ (implicit in the a's); the camera-position parameters X_i^c, Y_i^c, and Z_i^c; and the object-point coordinates X_j, Y_j, and Z_j. However, in order to obtain a solution, the condition equations must first be linearized.

Let

$$\begin{aligned} x_{p_i} &= x_{p_i}{}^0 + \delta x_{p_i}, & \omega_i &= \omega_i{}^0 + \delta\omega_i, \\ y_{p_i} &= y_{p_i}{}^0 + \delta y_{p_i}, & \phi_i &= \phi_i{}^0 + \delta\phi_i, \\ f_i &= f_i{}^0 + \delta f_i, & \kappa_i &= \kappa_i{}^0 + \delta\kappa_i, \\ X_i^c &= (X_i^c)^0 + \delta X_i^c, & X_j &= X_j{}^0 + \delta X_j, \\ Y_i^c &= (Y_i^c)^0 + \delta Y_i^c, & Y_j &= Y_j{}^0 + \delta Y_j, \\ Z_i^c &= (Z_i^c)^0 + \delta Z_i^c, & Z_j &= Z_j{}^0 + \delta Z_j, \end{aligned} \tag{3}$$

and

$$\begin{aligned} x_{ij} &= x_{ij}{}^0 + v_{ij}, \\ y_{ij} &= y_{ij}{}^0 + \bar{v}_{ij}, \end{aligned}$$

in which the superscript $(^0)$ indicates measured or estimated values, the δ's are corrections to the initial approximations, and the v's are residuals of the photographic measurements. Equations (3) are substituted into equations (2) which, after linearization by Taylor's expansion, become

$$\begin{aligned} v_{ij} + b_{ij}{}^1\delta x_{p_i} + b_{ij}{}^2\delta y_{p_i} + \cdots + b_{ij}{}^9\delta Z_i{}^c + b_{ij}{}^{10}\delta X_j + b_{ij}{}^{11}\delta Y_j + b_{ij}{}^{12}\delta Z_j + \epsilon_{ij} = 0 \\ \bar{v}_{ij} + \bar{b}_{ij}{}^1\delta x_{p_i} + \bar{b}_{ij}{}^2\delta y_{p_i} + \cdots + \bar{b}_{ij}{}^9\delta Z_i{}^c + \bar{b}_{ij}{}^{10}\delta X_j + \bar{b}_{ij}{}^{11}\delta Y_j + \bar{b}_{ij}{}^{12}\delta Z_j + \bar{\epsilon}_{ij} = 0 \end{aligned} \tag{4}$$

in which

my error

$\epsilon_{ij} = e_{ij}{}^0$, $\bar{\epsilon}_{ij} = \tilde{e}_{ij}{}^0$ (equations (2) with approximate values)

and

$$b_{ij}{}^1 = \frac{\partial \epsilon_{ij}}{\partial x_{p_i}}, \qquad b_{ij}{}^2 = \frac{\partial \epsilon_{ij}}{\partial y_{p_i}},$$

and so on.

The condition equations (4) are solved for the differential corrections (δx_{p_i}, δy_{p_i}, $\cdots$, $\delta Z_i{}^c$, δX_j, δY_j, δZ_j) which are then added to the original approximations ($x_{p_i}{}^0$, $y_{p_i}{}^0$, $\cdots$, $(Z_i{}^c)^0$, $X_j{}^0$, $Y_j{}^0$, $Z_j{}^0$) to form new approximations. The iteration is continued until such time as the δ's become negligibly small.

Because there is usually a redundancy of data, a rigorous least-squares adjustment is employed for solving the condition equations. This adjustment requires the formation and solution of a set of normal equations. Before the normal equations are formed, the matrix partitioning of Brown (1958) is employed to simplify the notation of the condition equations.

The linearized condition equations (4) for the ith camera station and the jth object-space point may be expressed in matrix form as

$$v_{ij} + \dot{B}_{ij}\dot{\delta}_i + \ddot{B}_{ij}\ddot{\delta}_j = \epsilon_{ij} \tag{5}$$

in which

$$v_{ij} = \begin{pmatrix} v_{ij} \\ \bar{v}_{ij} \end{pmatrix}, \qquad \dot{B}_{ij} = \begin{pmatrix} b_{ij}{}^1 & b_{ij}{}^2 & \cdots & b_{ij}{}^9 \\ \bar{b}_{ij}{}^1 & \bar{b}_{ij}{}^2 & \cdots & \bar{b}_{ij}{}^9 \end{pmatrix}, \qquad \dot{\delta}_i = \begin{bmatrix} \delta x_{p_i} \\ \delta y_{p_i} \\ \cdot \\ \cdot \\ \delta Z_i{}^c \end{bmatrix} \tag{6}$$

$$\ddot{B}_{ij} = \begin{pmatrix} b_{ij}{}^{10} & b_{ij}{}^{11} & b_{ij}{}^{12} \\ \bar{b}_{ij}{}^{10} & \bar{b}_{ij}{}^{11} & \bar{b}_{ij}{}^{12} \end{pmatrix}, \qquad \ddot{\delta}_j = \begin{bmatrix} \delta X_j \\ \delta Y_j \\ \delta Z_j \end{bmatrix}, \qquad \epsilon_{ij} = \begin{pmatrix} -\epsilon_{ij} \\ -\bar{\epsilon}_{ij} \end{pmatrix}.$$

Finally, by a continued application of the partitioning process, the condition equations for all object-space points and all camera stations are formulated as

$$v + \dot{B}\dot{\delta} + \ddot{B}\ddot{\delta} = \epsilon, \tag{7}$$

or, equivalently,

$$v + B\delta = \epsilon. \tag{8}$$

The normal equations, as formed from the condition equations (8), are

$$N\delta = c \tag{9}$$

where

$$N = B^T W B$$
$$c = B^T W \epsilon$$

and W is the weight matrix of the plate coordinates, usually a unit matrix. The solution of the normal equations (9) is

$$\delta = N^{-1}c. \tag{10}$$

Equation (10) gives, directly, the solution to the photogrammetric projective equations. With the partitioning employed in equations (7), the normal equations (9) become

$$\begin{pmatrix} \dot{N} & \overline{N} \\ \overline{N}^T & \ddot{N} \end{pmatrix} \begin{pmatrix} \dot{\delta} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} \dot{c} \\ \ddot{c} \end{pmatrix} \tag{11}$$

in which

$$\dot{N} = \dot{B}^T W \dot{B}, \qquad \overline{N} = \dot{B}^T W \ddot{B}, \qquad \dot{c} = \dot{B}^T W \epsilon$$
$$\overline{N}^T = \ddot{B}^T W \dot{B}, \qquad \ddot{N} = \ddot{B}^T W \ddot{B}, \qquad \ddot{c} = \ddot{B}^T W \epsilon$$

Constraints and Their Utilization

The methods of applying constraints in analytical photogrammetry, as described in this paper, follow the procedures developed by Brown (1959) in his extended solution for the Bermuda geodetic tie. The constraints are of two general types: the geometrical or functional constraint and the weight constraint.

Functional Constraint

The functional constraint is obtained from the functional relationship that may exist between the camera station or object-space point parameters and a new or different set of parameters. In some cases the new set of parameters may be smaller in number than the original, thus reducing the number of unknown parameters in the condition equations. As an example of a functional constraint, suppose that each of the camera-station position parameters X_i^c, Y_i^c, and Z_i^c is a function of the parameters α, β, and γ. That is

$$\begin{aligned} X_i^c &= F_1(\alpha, \beta, \gamma) \\ Y_i^c &= F_2(\alpha, \beta, \gamma) \\ Z_i^c &= F_3(\alpha, \beta, \gamma) \end{aligned} \tag{12}$$

The differentials of these equations are

$$\begin{aligned} \delta X_i^c &= \frac{\partial X_i^c}{\partial \alpha}\delta\alpha + \frac{\partial X_i^c}{\partial \beta}\delta\beta + \frac{\partial X_i^c}{\partial \gamma}\delta\gamma \\ \delta Y_i^c &= \frac{\partial Y_i^c}{\partial \alpha}\delta\alpha + \frac{\partial Y_i^c}{\partial \beta}\delta\beta + \frac{\partial Y_i^c}{\partial \gamma}\delta\gamma \\ \delta Z_i^c &= \frac{\partial Z_i^c}{\partial \alpha}\delta\alpha + \frac{\partial Z_i^c}{\partial \beta}\delta\beta + \frac{\partial Z_i^c}{\partial \gamma}\delta\gamma \end{aligned} \tag{13}$$

My errors

or

$$\dot{\delta}_i' = \begin{pmatrix} \delta X_i^c \\ \delta Y_i^c \\ \delta Z_i^c \end{pmatrix} = \begin{pmatrix} \frac{\partial X_i^c}{\partial \alpha} & \frac{\partial X_i^c}{\partial \beta} & \frac{\partial X_i^c}{\partial \gamma} \\ \frac{\partial Y_i^c}{\partial \alpha} & \frac{\partial Y_i^c}{\partial \beta} & \frac{\partial Y_i^c}{\partial \gamma} \\ \frac{\partial Z_i^c}{\partial \alpha} & \frac{\partial Z_i^c}{\partial \beta} & \frac{\partial Z_i^c}{\partial \gamma} \end{pmatrix} \begin{pmatrix} \delta\alpha \\ \delta\beta \\ \delta\gamma \end{pmatrix} = \dot{U}_i'\dot{\delta}' \tag{14}$$

In order to utilize the constraints, α, β, and γ in the condition equations, the functional relationships of equations (12) and (14) are merely substituted, where applicable, into equations (5). Thus, equation (5) becomes

$$\cancel{v_{ij} + \dot{B}_{ij}(\dot{\delta}_i'' + \dot{U}_i'\dot{\delta}') + \ddot{B}_{ij}\ddot{\delta}_j = \epsilon_{ij}} \tag{15}$$

$$v_{ij} + \dot{B}_{ij}\begin{pmatrix} \dot{\delta}_i'' \\ \dot{U}_i'\dot{\delta}' \end{pmatrix} + \ddot{B}_{ij}\ddot{\delta}_j = \epsilon_{ij}$$

where $\dot{\delta}_i''$ is given by

$$\dot{\delta}_i'' = \begin{pmatrix} \delta x_{p_i} \\ \delta y_{pi} \\ \cdot \\ \cdot \\ \delta \kappa_i \end{pmatrix} \tag{16}$$

and ϵ_{ij} is a function of α^0, β^0, and γ^0 instead of $(X_i^c)^0$, $(Y_i^c)^0$, and $(Z_i^c)^0$. As a result of the solution of equations (15) the most likely corrections $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ to the original assumed values α^0, β^0, and γ^0 are determined. If desired, the final values of X_i^c, Y_i^c, and Z_i^c may be found by substituting the adjusted values of α, β, and γ into equations (12).

The constraint just described is employed to reduce the number of unknown parameters in the condition equations. Another application of functional constraints is to perform the transformation of correlated parameters when one or more of the original parameters is unknown. Taking the previous example, suppose again that each of the camera-station position parameters X_i^c, Y_i^c, and Z_i^c is a function of the parameters α_i, β_i, and γ_i where this time α, β, and γ are different for each station. Further, suppose that one of the parameters, e.g., β_i, is unknown. That is

$$\begin{aligned} X_i^c &= F_1(\alpha_i, \beta_i^0, \gamma_i) \\ Y_i^c &= F_2(\alpha_i, \beta_i^0, \gamma_i) \\ Z_i^c &= F_3(\alpha_i, \beta_i^0, \gamma_i) \end{aligned} \tag{17}$$

In order to be able to use the functional constraint it is necessary that a further functional relationship exist between the two systems of parameters. For example, suppose that

$$\begin{aligned} (X_i^c)^0 &= G_1(\alpha_i, \beta_i^0, \gamma_i) \\ (Y_i^c)^0 &= G_2((X_i^c)^0, \alpha_i, \gamma_i) \\ (Z_i^c)^0 &= G_3((X_i^c)^0, \alpha_i, \gamma_i) \end{aligned} \tag{18}$$

Thus, while X_i^c is a function of the unknown β_i^0, Y_i^c and Z_i^c are functions of only the known parameters α_i and γ_i and the unknown $(X_i^c)^0$. The differentials become

$$\begin{aligned} \delta X &= \delta X \\ \delta Y &= \frac{\partial G_2}{\partial X}\,\delta X \\ \delta Z &= \frac{\partial G_3}{\partial X}\,\delta X \end{aligned} \tag{19}$$

with δX being the only unknown differential to be determined.

Functional constraints may be applied to the camera inner-orientation parameters, to the camera exterior-orientation parameters, and to the object-point coordinates in exactly the same manner.

WEIGHT CONSTRAINTS

The weight constraint is enforced by utilizing in the condition equations the standard deviations of the various parameters. For example, suppose that the standard deviations $\sigma_{X_i^c}$, $\sigma_{Y_i^c}$, and $\sigma_{Z_i^c}$ of the camera station parameters X_i^c, Y_i^c, and Z_i^c are known or can be closely approximated. The covariance matrix Σ_i', of the camera station parameters is

$$\dot{\Sigma}_i' = \begin{bmatrix} \sigma_{(X_i^c)}^2 & \sigma_{X_i^c Y_i^c} & \sigma_{X_i^c Z_i^c} \\ \sigma_{X_i^c Y_i^c} & \sigma_{(Y_i^c)}^2 & \sigma_{Y_i^c Z_i^c} \\ \sigma_{X_i^c Z_i^c} & \sigma_{Y_i^c Z_i^c} & \sigma_{(Z_i^c)}^2 \end{bmatrix} \tag{20}$$

in which, normally, the off-diagonal elements are zeroes. Finally, the weight matrix, W_i', of the camera-position parameters is the inverse of the covariance matrix. That is

$$\dot{W}_i' = \dot{\Sigma}_i'^{-1}. \tag{21}$$

In a similar manner the weight matrices of the camera inner-orientation parameters, the camera exterior-orientation parameters, and the object-space point coordinates may be determined.

In order to utilize the weight matrices in the solution of problems of analytic photogrammetry (see Brown, 1959, Appendix), the partitioned normal equations (11) become

$$\begin{bmatrix} \dot{N} + [\dot{W}] & \overline{N} \\ \overline{N}^T & \ddot{N} + [\ddot{W}] \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{c} + [\dot{W}][\dot{\xi}] \\ \ddot{c} + [\ddot{W}][\ddot{\xi}] \end{bmatrix} \tag{22}$$

The augmented weight matrices, $[\dot{W}]$ and $[\ddot{W}]$, are diagonal matrices consisting of the individual weight matrices $\dot{W}_i$ and $\ddot{W}_j$ as given by the inverses of the covariance matrices $\dot{\Sigma}_i$ and $\ddot{\Sigma}_j$. For those parameters which are treated as completely unknown, the appropriate positions of the augmented weight matrix are filled with zero elements. The augmented supplementary discrepancy vectors, $[\dot{\xi}]$ and $[\ddot{\xi}]$, consist of the supplementary discrepancy vectors $\dot{\xi}$ and $\ddot{\xi}$ with zero elements in the appropriate locations for unknown quantities. The elements of the vectors $\dot{\xi}$ and $\ddot{\xi}$ are the difference between the observed values and the approximation values of the camera-station and object-space-point parameters. For the first iteration of the solution of the normal equations (22), the supplementary discrepancy vectors would normally be zero since the observed values would be taken as the first approximations. For each succeeding iteration the supplementary discrepancy vector would be the difference between the new approximations obtained from the preceding iteration and the original observed values.

As a by-product of the solution of the normal equations, new covariance matrices, $\dot{\Sigma}$ and $\ddot{\Sigma}$, of the computed or adjusted camera-station and object-space-point parameters are obtained. These covariance matrices are given by the inverse of the coefficient matrix of the normal equations. Thus, the accuracy of the various parameters, as determined from the solution of the normal equations, can be determined as a function of the accuracy of the various input data.

As with the functional constraints, the weight constraints may also be employed to perform the transformation of correlated parameters when one or more of the original parameters is unknown. In fact, the utilization of weight constraints is to be preferred since the method is completely general, and does not require that an additional functional relationship exist between the two systems of parameters as does the method of functional constraints. Suppose, as before, that each of the camera-station position parameters X_i^c, Y_i^c, and Z_i^c is a function of the parameters α_i, β_i, and γ_i and that one of the parameters, e.g., β_i, is unknown. In this event, values for the standard deviations of α_i and γ_i, σ_{α_i} and σ_{γ_i}, are chosen such that they are as realistic as possible. Then, an approximate value, β_i^0, is chosen for β_i, and the corresponding standard deviation, σ_{β_i}, is chosen to be sufficiently large that the value of β_i^0 is weighted out of the solution. The covariance matrix, $\dot{\Lambda}_i'$, of the parameters α_i, β_i, and γ_i is

$$\dot{\Lambda}_i' = \begin{pmatrix} \sigma_{\alpha_i}{}^2 & \sigma_{\alpha_i\beta_i} & \sigma_{\alpha_i\gamma_i} \\ \sigma_{\alpha_i\beta_i} & \sigma_{\beta_i}{}^2 & \sigma_{\beta_i\gamma_i} \\ \sigma_{\alpha_i\gamma_i} & \sigma_{\beta_i\gamma_i} & \sigma_{\gamma_i}{}^2 \end{pmatrix} \tag{23}$$

Then, the covariance matrix, $\dot{\Sigma}_i'$, of the camera position parameters X_i^c, Y_i^c, and Z_i^c is given by

$$\dot{\Sigma}_i' = \dot{U}_i'\dot{\Lambda}_i'\dot{U}_i'^T \tag{24}$$

where $\dot{U}_i'$ is the same as in equation (14). From this point the covariance matrix is transformed to the weight matrix and utilized in the normal equations as outlined in equations (21) and (22). Finally, the covariance matrix, $\dot{\Lambda}_i'$, of the adjusted camera station parameters is given by

$$\dot{\Lambda}_i' = \dot{U}_i'^{-1}\dot{\Sigma}_i'\dot{U}_i'^{T-1} \tag{25}$$

where, again, $\dot{\Sigma}_i'$ is the inverse of the coefficient matrix of the normal equations. A similar application of the weight constraint can be made to the camera inner-orientation parameters, to the camera exterior-orientation parameters, and to the object point coordinates.

Some Examples of Constraints

To best illustrate the manner in which functional constraints and weight constraints may be utilized in analytical photogrammetry, several examples of these constraints will be examined in detail.

Geodetic Constraints

One of the major problems of analytical photogrammetry is the control extension by means of aerial triangulation. Since the projective equations are based on a Cartesian coordinate system in the object space, the coordinate system used in the control extension must also be Cartesian. If the control extension is to cover only a small area, a local plane-coordinate system or a map-projection system, such as the Universal Transverse Mercator projection, may be used. However, for large areas a geocentric-coordinate system is to be preferred. In such a system (see Church, 1948) the origin is at the center of the earth, the positive Z axis is directed along the polar axis through the North Pole, the positive X axis is directed along the equatorial plane through the Greenwich meridian, and the positive Y axis is directed so as to form a right-hand system. If the latitude, ϕ, longitude, λ, and elevation, h, of a point are given, then its coordinates in the geocentric system are

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (N+h)\cos\phi\cos\lambda \\ (N+h)\cos\phi\sin\lambda \\ [N(1-e^2)+h]\sin\phi \end{bmatrix} = \begin{bmatrix} F_1(\phi,\lambda,h) \\ F_2(\phi,\lambda,h) \\ F_3(\phi,\lambda,h) \end{bmatrix} \tag{26}$$

in which

$$N = \frac{a}{\sqrt{1 - e^2\sin^2\phi}}$$

where a is the semi-major axis of the reference ellipsoid and e is the eccentricity of the reference ellipsoid. Note that h is the sum of the topographic elevation and geoid undulation.

The above relations are adequate if all three coordinate parameters (ϕ, λ, and h) of the point to be transformed are perfectly known. However, it often happens that only the elevation of a point will be known or only the latitude and/or longitude will be known. To handle this situation either the function constraint or the weight constraint may be employed. The function constraint is represented by the "geodetic restraint" (Dodge, 1959) and the weight constraint by the concept of the "relaxed absolute" control point or geoid constraint (Brown, 1959).

In the geodetic restraint method a number of different constraints are employed depending upon which elements of the geographic coordinates of a point are known. For example, if the point is known in longitude only, it is constrained to lie in the meridian plane

$$X\sin\lambda + Y\cos\lambda = 0 \tag{27}$$

Then, following the procedures outlined in equations (17) through (19), the geocentric-coordinates of the point are

$$\begin{aligned} X^0 &= X^0 = F_1(\phi^0,\lambda,h^0) \\ Y^0 &= -X^0\tan\lambda \\ Z^0 &= Z^0 = F_3(\phi^0,\lambda,h^0) \end{aligned} \tag{28}$$

and the differentials are

$$\delta X = \delta X$$
$$\delta Y = -\tan\lambda\delta X \tag{29}$$
$$\delta Z = \delta Z$$

which may be substituted into the condition equations (5).

Should a point be known only in latitude, the point is constrained to lie on the cone

$$X^2 + Y^2 - \left(\frac{Z + e^2N\sin\phi}{\tan\phi}\right)^2 = 0. \tag{30}$$

Then the geocentric coordinates of the point are

$$X^0 = X^0$$
$$Y_0 = Y^0$$
$$Z^0 = \sqrt{(X^0)^2 + (Y^0)^2}\tan\phi - e^2N\sin\phi \tag{31}$$

and the differentials are

$$\begin{aligned} \delta X &= \delta X \\ \delta Y &= \delta Y \\ \delta Z &= \frac{X^0\delta X + Y^0\delta Y}{\sqrt{(X^0)^2 + (Y^0)^2}}\tan\phi. \end{aligned} \tag{32}$$

In the same manner, if both the latitude and longitude of a point are known, the point is constrained to lie on the intersection of the meridian plane and the prime vertical plane, while if only the elevation of a point is known, the point is constrained to lie on a spheroidal surface (or, for small areas, a spherical surface) parallel to and at a distance h from the reference ellipsoid.

A considerably different approach is taken in the concept of the "relaxed absolute" control point. Here, the element or elements of the geographic coordinates which are known will have as realistic a standard deviation as possible associated with them, while the unknown elements will have standard deviations sufficiently large to weight them out of the solution. Approximate geocentric coordinates are then computed by using both known and approximate geographic coordinates in equations (26). These values, along with the standard deviations transformed to weights (equations (23), (24) and (21)), are then entered in the normal equations (22). Finally, equation (25) is used to obtain the adjusted standard deviations of the geographic coordinates.

CAMERA STATION CONSTRAINTS

The geodetic constraint offered examples of both the functional constraint and the weight constraint as employed in transforming correlated parameters from one coordinate system to another but without reducing the number of parameters. The remaining constraints to be described in this paper offer techniques for reducing the number of unknown parameters and, usually, of increasing the strength of the solution. Since the weight constraint can be applied to any parameter, including the functional parameters, as exemplified in the geodetic constraint, no further examples of the weight constraint will be given.

Flying Height and Heading Constraints

The flying height and heading constraints are particularly applicable to the problem of aerial triangulation in areas where there is little or no ground control and it would not be feasible to obtain extremely high accuracy. Thus, assumptions can be made concerning flight characteristics which are within the expected accuracy of the triangulation. These assumptions are that the aircraft is flying at a constant, though unknown, altitude and that the aircraft maintains a constant, though unknown, heading. In order to simplify the formulation, a plane-coordinate system (X, Y, h) is employed in the projective equations where X is directed east, Y is directed north, and h is the flying height above sea level. Then, by applying the geodetic constraints, the flying height and heading constraints can be transformed into the geocentric-coordinate system.

Based on the assumptions of a constant flying height and heading and utilizing the concepts of equations (12) through (16), the coordinates of the camera stations are

$$\begin{aligned} (X_i^c)^0 &= (X_1^c)^0 + d_i^0 \sin\alpha^0 \\ (Y_i^c)^0 &= (Y_1^c)^0 + d_i^0 \cos\alpha^0 \\ (h_i^c)^0 &= (h^c)^0 \end{aligned} \tag{33}$$

and the differentials are

$$\begin{aligned} \delta X_i^c &= \delta X_1^c + d_i^0 \cos\alpha^0 \delta\alpha + \sin\alpha^0 \delta d_i \\ \delta Y_i^c &= \delta Y_1^c - d_i^0 \sin\alpha^0 \delta\alpha + \cos\alpha^0 \delta d_i \\ \delta h_i^c &= \delta h^c \end{aligned} \tag{34}$$

where X_1^c and Y_1^c are the coordinates of the first camera station, d_i is the distance from the first to the ith camera station, and α is the heading or azimuth of the flight-line measured clockwise from north.

If the coordinates of each camera station had been considered unknown, there would have been $3n$ unknowns for n camera stations. By utilizing the altitude and heading constraints, the number of unknowns is reduced to $n+4$. This represents a considerable reduction in the size of the normal equation matrix which requires inversion.

Orbital Constraints

Duane Brown (1960) has suggested that for photography taken from an orbiting satellite the camera station can be considered constrained to an orbit. If one considers that for short strips (a few thousand miles in length) the perturbations of the orbit may be ignored, the positions of the constrained camera stations will be a function of the six Keplerian orbital parameters, a, e, i, ω, Ω, τ, and of the time of exposure, t_i. The orbital parameters are defined by Kepler's equation

$$E - e \sin E = (t_i - \tau)/a^{3/2} \tag{35}$$

in which

E is the eccentric anomaly, which is solved for as a function of
e, the eccentricity of the orbit,
a, the semi-major axis of the orbit,
τ, the time of perigee passage, and
t_i, the time of exposure.

The additional elements required to transform the orbit positions to the geocentric coordinate-system are

Ω, the right ascension of the ascending node,
ω, the argument of perigee, and
i, the inclination of the orbital plane.

If the position parameters of the camera station and the partials of the position parameters are determined as functions of the orbital parameters, the geocentric coordinates of the camera station are

$$\begin{aligned} (X_i^c)^0 &= F_1(a^0, e^0, iv^0, \omega^0, \Omega^0, \tau^0; t_i) \\ (Y_i^c)^0 &= F_2(a^0, e^0, i^0, \omega^0, \Omega^0, \tau^0; t_i) \\ (Z_i^c)^0 &= F_3(a^0, e^0, i^0, \omega^0, \Omega^0, \tau^0; t_i) \end{aligned} \tag{36}$$

and the differentials are

$$\begin{aligned} \delta X_i^c &= \frac{\partial X_i^c}{\partial a}\delta a + \frac{\partial X_i^c}{\partial e}\delta e + \cdots + \frac{\partial X_i^c}{\partial \tau}\delta\tau \\ \delta Y_i^c &= \frac{\partial Y_i^c}{\partial a}\delta a + \frac{\partial Y_i^c}{\partial e}\delta e + \cdots + \frac{\partial Y_i^c}{\partial \tau}\delta\tau \\ \delta Z_i^c &= \frac{\partial Z_i^c}{\partial a}\delta a + \frac{\partial Z_i^c}{\partial e}\delta e + \cdots + \frac{\partial Z_i^c}{\partial \tau}\delta\tau \end{aligned} \tag{37}$$

Thus, if the time of each exposure is known, the determination of the $3n$ unknown coordinates of the camera stations may be replaced by the determination of the six orbital parameters.

Inner-Orientation Constraints

The projective equations (1) allow one to consider the inner orientation parameters x_{p_i}, y_{p_i}, and f_i as unknowns for each camera station. In the case of missile photogrammetry and flare triangulation, this concept is often adopted since a different camera is employed at each camera station and a star background on the photographs allows for an accurate calibration of the cameras. On the other hand, in conventional aerial photography a single, precalibrated camera is usually employed for obtaining all photography on a mission. Thus, the inner orientation elements are treated as known. Photography taken from an orbiting satellite presents still a third possibility. Here it may be impossible to maintain the camera calibration through the launching and recovery of the camera. Therefore, though the inner orientation parameters may be considered as constants during a mission, they must be treated as unknowns. Based on this assumption that the parameters are constant, the inner orientation parameters become

$$\begin{aligned} (x_{p_i})^0 &= x_p^0 \\ (y_{p_i})^0 &= y_p^0 \\ f_i^0 &= f^0 \end{aligned} \tag{38}$$

and the differentials are

$$\begin{aligned} \delta x_{p_i} &= \delta x_p \\ \delta y_{p_i} &= \delta y_p \\ \delta f_i &= \delta f \end{aligned} \tag{39}$$

Other Camera-Station Constraints

Normally, the use of stellar or solar photography would be considered in the same manner as missile photography or flare triangulation. That is, the stars or sun would be treated as known object-space points at an infinite distance and employed directly in the projective equations. However, in aerial triangulation where, in addition to the mapping camera, a stellar camera may have been employed to obtain orientation information, the orientation of the mapping camera with respect to the stellar camera may be unknown though it remains constant. Thus, the unknown parameters of orientation between the two cameras may logically be carried as constraints in the solution of the aerial triangulation. In the same manner, it should be possible to devise many more constraints on the various camera station parameters.

OBJECT-SPACE CONSTRAINTS

The object-space constraints are applied in the same manner as the camera-station constraints except that the object-space points are constrained instead of the camera stations. Some examples of object-space constraints follow.

Horizon-Constraints

Occasionally it may be desirable to determine the orientation parameters of individual oblique photographs, or even to carry out a complete aerial triangulation of trimetrogon or of other photography which includes photographs of the horizon. The utilization of horizon constraints is applicable in these instances, particularly when there is a lack of known ground-control points. The horizon constraint makes use of the condition that points on the horizon, as viewed from the camera station, lie on a cone which is tangent to the surface of the Earth and has its apex at the camera station and that all the horizon points are equidistant from the camera station. (This condition does not account for irregularities in the Earth's surface or that the horizon points lie above sea level. However, in lieu of sufficient ground-control, this condition will be a valuable constraint.)

As an example of the horizon constraint, the orientation parameters of a single oblique photograph of known flying height, h, will be determined. A geocentric coordinate-system is chosen such that the Z axis passes through the camera station. The coordinates of the camera station then become

$$\begin{aligned} X^c &= 0 \\ Y^c &= 0 \\ Z^c &= R + h \end{aligned} \tag{40}$$

in which R is the radius of the earth (for the expected accuracy of the solution, a mean value of the radius can probably be used). The coordinates of points on the horizon, as determined from the aforementioned conditions, are

$$\begin{aligned} (X_j^h)^0 &= (X_j^h)^0 \\ (Y_j^h)^0 &= \sqrt{R^2\left[1 - \frac{R^2}{(R+h)^2}\right] - (X_j^h)^{0^2}} \\ (Z_j^h)^0 &= \frac{R^2}{R+h} \end{aligned} \tag{41}$$

and the differentials are

$$\begin{aligned} \delta X_j^h &= \delta X_j^h \\ \delta Y_j^h &= -\frac{(X_j^h)^0}{\sqrt{R^2\left[1 - \frac{R^2}{(R+h)^2}\right]}}\,\delta X_j^h \\ \delta Z_j^h &= 0 \end{aligned} \tag{42}$$

Thus, three horizon points are sufficient to determine the three orientation parameters of the camera station. By a similar formulation, the horizon constraints may be applied to the problem of aerial triangulation.

Other Object Space Constraints

Any set of object-space points which conforms to a geometrical pattern may be constrained to that pattern. For example, if a number of points lie along a shore line (or on a very flat plane) they may all be constrained to lie at the same, though unknown, elevation. In a similar fashion, if object-space points lie along straight highways, railroads, or utility lines, they could be constrained to lie on a straight line. The formulation would be the same as that for the flying height and heading constraints as applied to the camera stations (equations (33) and (34)).

The geometrical constraints can be extremely useful in terrestrial photogrammetry, particularly in architectural and engineering work where the object photographed will almost invariably conform to a geometrical pattern. A paper by Borchers (1960) gives some idea of the constraints available in architecture which, though applied in analog fashion in a stereoplotter, are equally applicable to the analytical solution. Possibly the simplest way to utilize this type of constraint is to make the object-space coordinate system a part of the geometrical pattern. For example, the base of one corner of a rectangular building could be the origin, the Z axis could be directed vertically upward along that corner, and the X and Y axes could be directed along the base of the building on the two sides. A single known dimension on one surface of the building or the distance from the camera station to the building would be sufficient to give scale.

Conclusions

The utilization of constraints represents a powerful tool in the solution of problems in analytical photogrammetry. The weight constraint allows for every parameter in the projective equations to be assigned weights corresponding to the accuracy of the original observations or approximations, thus assuring that errors will not build up through the solution of analytical problems. In addition, use of the weight constraint makes possible the transformation from one system of correlated parameters to another when one or more of the parameters of the initial system is unknown. On the other hand, the geometric or functional constraints not only allow one to take advantage of the geometric patterns or paths to which the camera station or object space points may conform but also may lead to a considerable reduction in the number of unknown parameters which must be determined.

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