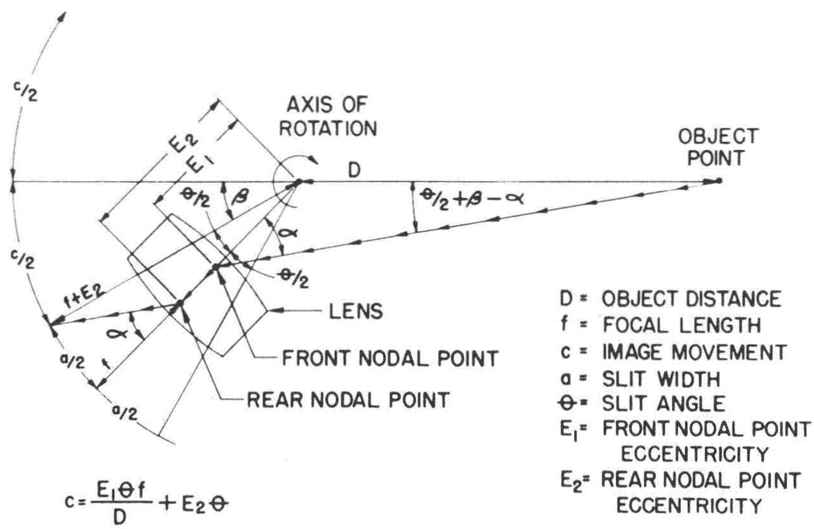

FIG. 16. Image-movement resulting from the eccentricity of nodal points.

When E_2 is small compared to f, the following is a close approximation:

$$f + E_2 = f. \tag{6}$$

Substituting equations (4) and (6) into equation (5)

$$c = \theta f\left(\frac{E_1}{D} + \frac{E_2}{f}\right)$$

$$c = \frac{\theta E_1 f}{D} + \theta E_2. \tag{7}$$

It is apparent from the above formula that there is no image-movement or blur when (1) the front and real nodal points are on the axis of rotation ($E_1 = E_2 = 0$), or (2) the rear nodal point **is** on the axis of rotation ($E_2 = 0$) and the object is located at infinity ($D = \infty$).

When the front nodal point is on the axis of rotation ($E_1 = 0$) or when the object is located at infinity ($D = \infty$), the image-movement formula reduces to the simplified form

$$c = \theta E_2.$$

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Patent No. 2,794,379 dated June 4, 1957.

*A Tilted Line Approach to Photogrammetric Determinations of Volume**

WILLIAM W. MENDENHALL, JR.‡

Introduction

THE usual scheme of computing earthwork volumes for highway, railroad, and canal projects is the cross-section method. This has been used for many decades, and must be accepted as a time-tested approach to the problem of volume determination. In brief, this "standard" method consists moving along the proposed centerline of the highway, and taking cross-sections at regular intervals so as to determine the shape of the original ground surface at these intervals. If the ground surface changes abruptly, special cross-sections are taken at the critical places.

This standard method is a logical outgrowth of the "field" approach. It represents a convenient solution to several problems encountered by the instrumentman. Since brushing is commonly required, it makes good

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‡ At the time of submitting this paper, the author was a Graduate Student at Cornell University, Ithaca, N. Y. His address is now Mendenhall Aerial Surveys, Fairbanks, Alaska.

sense to take cross-sections only at certain regular intervals. Also, a new instrument set-up is generally required at each cross-section, and thus, the fewer sections taken, the better the economy. The regular field procedure seems to be a fair compromise; more sections would increase the accuracy, but they also increase the cost.

With the advent of modern photogrammetric plotters, earthwork determination can be moved indoors, so to speak. It is very commendable that so many engineering firms have switched to photogrammetric determinations of volume. It is rather unfortunate, however, that they have stuck with the same methods of cross-sectioning that are practical in the field, to the exclusion of other approaches perhaps more ideally suited to photogrammetric techniques. One of these approaches will be called the "tilted line approach" to volume determination.

The idea behind cross-sectioning is to faithfully record the shape of the surface along the cross section. When this original surface is combined with the proposed shape of the highway, there results a cross-sectional area of cut or fill. When these areas are multiplied by the linear distance between cross-sections, the result is a determination of volume. This is the ultimate goal. It would seem proper to see if this ultimate result can be obtained in some easier way.

Rather than have a ragged series of line segments connecting the selected points along a cross section, it is possible to construct one straight line which gives the best fit of the points so connected.

In Figure 1, points A, B, C, D, and E represent critical points on the ground (i.e. points of change of slope) and the dotted line connecting them represents an approximation to the true ground surface. Note that it is not the true surface, but merely an approximation, since we have selected not an infinite number of points, but only the major slope-break points. The straight line MN is drawn so as to approximate the true ground surface. Many straight lines tend to approximate the surface, but there is one true "best-fit" line which is determined by the method of least squares. If this best-fitting line really does give a true approximation of the ground surface, then this single line, in many cases, can be used to determine volumes. The one line can thus replace the many line segments which make up the "true" cross-section. It remains to be seen, however, just how well a stereoplotter operator can determine this best fitting line by his own judgment, without recourse to mathematics. It is quite apparent that this single-line method should be easier for the photogrammetrist than for the field surveyor, since the former can see the entire cross-section at a glance.

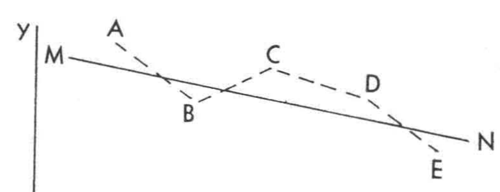

FIG. 1

To determine the practicability of the straight-line method, a test project was run. The area involved was a portion of Interstate Route No. 5, Section 7C-1 in Jefferson County, New York. A 1,000 foot long section was investigated, between sections 292+00 and 302+00. There were 21 sections spaced 50 feet apart along the centerline. Ground elevations were furnished the author by the firm of Lewis-Dickerson Associates of Watertown, New York.

The same firm took photography of the area with a camera using a 6-inch Metrogon lens. Flight height was 1,500 feet above mean terrain, giving a photo-scale of 250 feet per inch. The plotting was done on a Balplex 525 plotter at a scale of 75 feet per inch. Two models were involved, using three diapositives. Horizontal and vertical control were adequate, and there was very little warp. The terrain was gently rolling.

To conduct a test using the straight-line approach only, would not be conclusive one way or the other. Such factors as the visual acuity of the operator, quality of photography, etc. all enter into the problem. It was decided, therefore, that the author would run photogrammetric cross-sections in the usual manner (i.e. spot heights at certain points along the cross sections.) He would then take straight-line readings in a manner to be explained later. The results would then give a true comparison of the two methods, using the same operator, same photos, same plotter, etc.

Standard Cross Section Method

As stated before, the terrain was gently rolling. For this reason the ground elevations had been determined at seven places along each cross-section: at the proposed centerline, and also at 50, 100, and 150 foot offsets both to the left and to the right.

In order to compare elevations determined photogrammetrically with those obtained by spirit leveling, the photogrammetric spot elevations were taken in the same places as the ground data.

For a section 100 feet wide, three elevations were used; namely the centerline, and the 50 foot offsets both to the left and right. For a section 200 feet wide, five elevations were used; for a section 300 feet wide, all seven elevations were used.

For a section 100 feet wide, the average ground-elevation is obtained by merely taking the arithmetic mean of the three elevations involved. The comparison of the average ground-elevation obtained by spirit leveling with the average ground-elevation obtained photogrammetically gives us the indication of the accuracy obtainable. The error is expressed in terms of ground method value minus photogrammetric value (in feet).

The summary of results is given in Table 1:

TABLE 1

WIDTH OF SECTIONS

	100 Feet	*200 Feet*	*300 Feet*
Mean Error:	−0.054	−0.077	−0.084
Average Error:	0.193	0.184	0.182
Maximum Error:	−0.50	−0.48	−0.49
Standard Deviation:	0.231	0.229	0.225

For the preceding comparison, the average ground elevation was obtained by a straight average of the various elevations concerned. This method is not rigorously correct, since the end result desired is a cross-sectional area.

Referring to Figure 2, if the cross-sectional area between the ground surface and any arbitrary base line is to be determined, the correct formula for this area is:

$$A = \frac{(h_1 + h_2)d_1}{2} + \frac{(h_2 + h_3)d_2}{2}$$

It is easily seen that the end elevations (i.e. h_1 and h_3) appear only half as frequently as the other elevation h_2. In fact, when d_1 and d_2 are equal (as they are in the problem under investigation) the formula reduces to:

$$A = d_1\left[\frac{h_1}{2} + h_2 + \frac{h_3}{2}\right]$$

Therefore, if true cross-sectional areas are desired, then the correct "average" elevation (the quantity within brackets) is obtained by weighting the end elevations only half as much as the other elevation. Similarly, for five-elevation and seven-elevation sections, the end elevations are weighted only half as much as the other elevations.

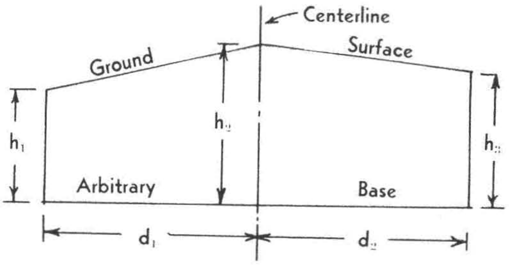

FIG. 2

If the "average" ground-method elevations are compared with the "average" photogrammetric-method elevations, with this weighting effect taken into account, the results are included in Table 2:

TABLE 2

WIDTH OF SECTIONS

	100 Feet	*200 Feet*	*300 Feet*
Mean Error:	−0.051	−0.069	−0.081
Average Error:	0.215	0.182	0.179
Maximum Error:	0.53	−0.46	−0.48
Standard Deviation:	0.261	0.218	0.220

A comparison of these "weighted" results with the previous "unweighted" results shows very little difference.

To obtain some indication of the over-all accuracy, a mean error of about −0.067 feet can be assumed. Taking a standard deviation of about 0.225 feet, it can be shown that 90% of all average elevations would lie within 1.71 standard deviations from their correct values. (If the mean error had been zero, then 90% would lie within 1.645 standard deviations.) This 90% limit is therefore 1.71×0.225 feet or 0.384 feet. A common specification for spot heights is that 90% of them shall fall within ¼ of the contour interval. This would permit a contour interval of 1.54 feet, yielding a C-factor of 975. The Balplex 525 projector is generally considered to have a *C*-factor of about 800 or 900. However, as has been stated before, it was not the intention of this project to arrive at any absolute measure of accuracy, but rather to compare this "standard" photogrammetric-method with the proposed straight-line method.

TILTED LINE METHOD

Before presenting the method and results obtained by the straight-line procedure, a brief explanation of the tilting platen tracing table is in order. The platen is constructed so as to tilt about one axis. Figure 3 shows a close-up of this device.

This type of platen has been used mainly in geological work, where it is tilted to approximate geologic strata, thus yielding strike and dip. For the specific purpose of this project, two lines were drawn on the top of the platen. One line, parallel to the axis of tilt, represented the proposed highway centerline; the second line at right-angles to the first represented the cross-section. Floating dots were not used, but rather a series of small ticks on the cross-section line, to allow the eye to focus on specific points. Seen in stereo, this was a "floating line" which not only could be made to move up and down, but also to assume any desired slope. It could be made to fit the surface of the ground, or more precisely, approximate the one straight-line which "best fits" the actual ground surface. The straight line can be determined by two factors—the slope, and the elevation of any point along the line. For this test, the centerline elevation was read off the Veeder-Root counter of the tracing table. The slope was indicated by a thin black thread supporting a lead weight. A production model could be greatly improved, but for this test the protractor and vertical thread arrangement was completely satisfactory. The protractor read in degrees, and the slopes could be estimated to tenths of degrees.

For any desired cross-section, since the true ground-elevations were known, the best fitting straight-line could be computed by the method of least squares. Then, photogrammetrically, by trying to fit the line on the tracing table to best fit the ground surface, another straight-line is determined. Ideally, of course, the two lines should be the same. The differences, or "errors" between the two lines can be studied. The results of these comparisons are given in Table 3, below:

The complete results cannot be presented due to lack of space. However, it is worth noting that the results obtained on moderate slopes (about 10%) were about the same as on very flat slopes.

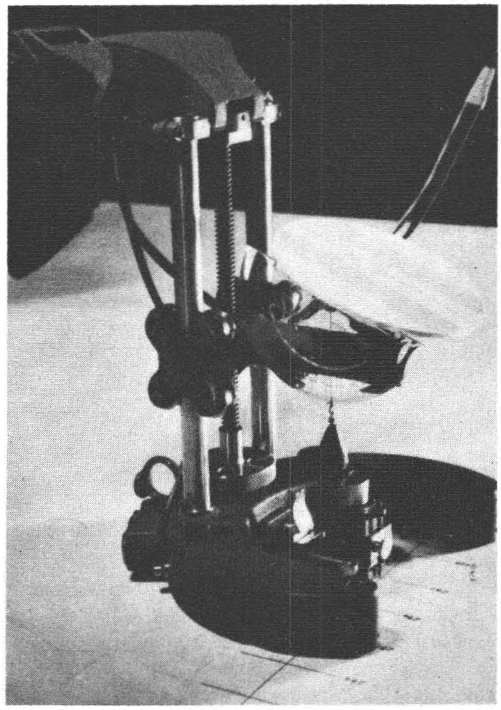

FIG. 3. Tracing table with tilting platen.

Averaging the results for 100, 200, and 300 foot widths, the following values are obtained:

Standard deviation of centerline elevation error: 0.544 feet.

Standard deviation of slope error: 0.00543.

In the tilted straight-line method, as performed in this experiment, the sides were always balanced; i.e., it was always assumed that the ground surface extended an equal distance each side of the centerline. In actual highway practice this is rarely the case, except in level ground.

Using balanced sides, it can be seen that

TABLE 3

	100 Foot Widths		*200 Foot Widths*		*300 Foot Widths*	
	Ctr. Elev. Diff.	*Slope (Tangent) Diff.*	*Ctr. Elev. Diff.*	*Slope (Tangent) Diff.*	*Ctr. Elev. Diff.*	*Slope (Tangent) Diff.*
Mean Error:	0.263	0.002803	0.046	0.000992	−0.015	0.002804
Average Error:	0.511	0.006105	0.358	0.003440	0.452	0.003818
Maximum Error:	1.58	0.01634	1.16	0.01067	−1.24	0.01039
Standard Deviation:	0.600	0.007497	0.471	0.004714	0.561	0.004093

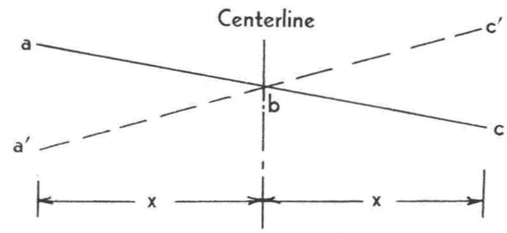

FIG. 4

the recorded slope has no effect whatsoever on the average elevation; only the centerline elevation is of any consequence.

In Figure 4 suppose that *abc* represents the best fitting straight line to the actual ground surface. If center elevation *b* remains the same, but line *a'bc'* is recorded, it is seen that elevation *a'* is lower than *a* by exactly the same amount that *c'* is higher than *c*. Therefore, the average elevation is unchanged.

Since unbalanced sides do occur in actual practice, their effect on the accuracy of the cross-sections will be investigated.

In Figure 5 let *s* be the error in slope between the true best fitting line (least square method) and the slope actually read from the tracing table. Also, *e* represents the excess of one side over the other. All of the error in area is represented by the shaded trapezoid. Since an equivalent elevational error is desired, then the error in area must be divided by the total width of cross-section. It can thus be shown that:

$$\text{Elevation error} = \frac{se}{2}$$

As a numerical example, if the limits of cut are 40 feet to the left, and 70 feet to the right, then the excess *e* is 30 feet. If the error in slope is 0.004 (or 0.4%) then the elevational error of the entire cut section is:

$$\frac{se}{2} = \frac{0.004 \times 30 \text{ ft.}}{2} = 0.06 \text{ feet.}$$

(Assuming the center line elevation to be correct.)

There are two sources of error using this straight-line method. First, assuming the slope to be correct, the centerline elevation of the platen may be higher or lower than the centerline elevation of the least square line. Second, assuming centerline elevation to be correct, the platen slope may differ from that of the least square line. In general, both errors combine. A statistical analysis of the combined errors yields the results in Table 4:

TABLE 4

Excess (e) of one side over the other:	*90% of the elevational errors are less than:*
0 feet	0.895 feet
50 feet	0.923 feet
100 feet	1.000 feet

Comparison Between Standard Cross Section Method and Tilted Line Method

Referring back to the results of the standard cross-section method, it is seen that the 90% limit for the average elevation of a cross-section was 0.384 feet. The tilted-line method gives 90% limits of slightly less than 1.00 feet, depending, of course, on the excess, or unbalance of one side over the other. Thus, the standard method is about two-and-a-half times as accurate as the straight-line method. However, the latter method is about three times as fast as the regular method. The author strongly urges its use, not for pay quantities, but rather as a very rapid method in preliminary road design to compare several alternate routes, and to determine approximate earthwork quantities.

Since the whole idea of the straight-line method is to approximate the ground surface with a straight line, it is fairly obvious that the best results will occur where the ground surface is in fact very nearly a straight line. The rougher the terrain, then, the less accurate will be the results using the straight line method.

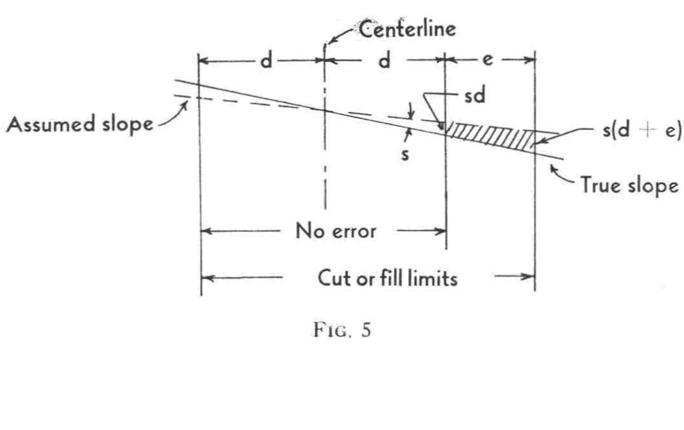

FIG. 5

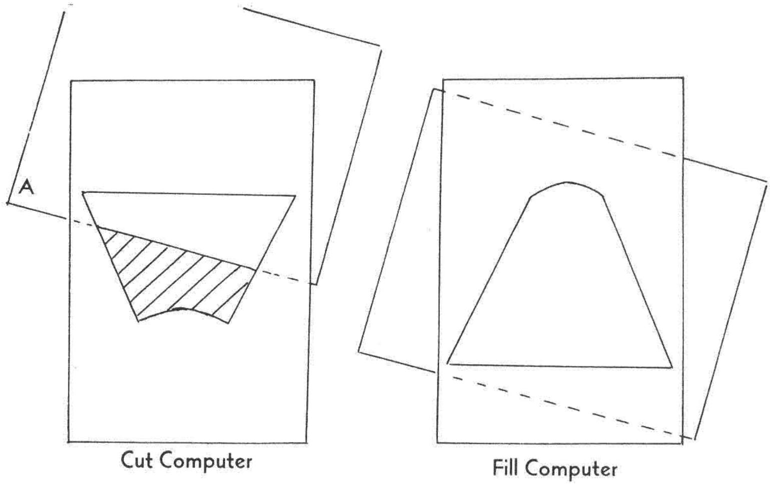

FIG. 6. Analog Computers showing positions for a section in cut.

USE OF DIGITAL AND ANALOG COMPUTERS WITH STRAIGHT LINE METHOD

There is no reason why a digital computer cannot be used in connection with the straight-line method. There are only two pieces of information which need to be read in: the centerline elevation, and the slope. Programming should be easier using the straight-line method, as opposed to the regular cross-section method.

The straight-line method brings about the intriguing possibility of using an analog computer for earthwork quantities. The basic fundamentals of such a computer will be presented. Actually, there should be two computers, one for cut, and one for fill. Each computer consists of two parts. One of the parts is basically a template for the desired shape of the cut (or fill) section. It is raised or lowered to correspond to the correct centerline elevation of the road surface. The second part is merely a piece of thin metal or hardboard, connected directly or via servo-mechanism to the tracing table platen, so that one edge of the metal assumes the exact attitude of the platen. Refer to Figures 6, 7, and 8.

Let us follow the action of the cut computer in detail. If the lower edge of piece *A* assumes the elevation and slope of the platen, and if there is cut, then the open space (shown shaded in Figure 6) represents the area of cut. If the section is in fill, then the lower edge of piece *A* is so low that there is no open space at all. If the section is in partial

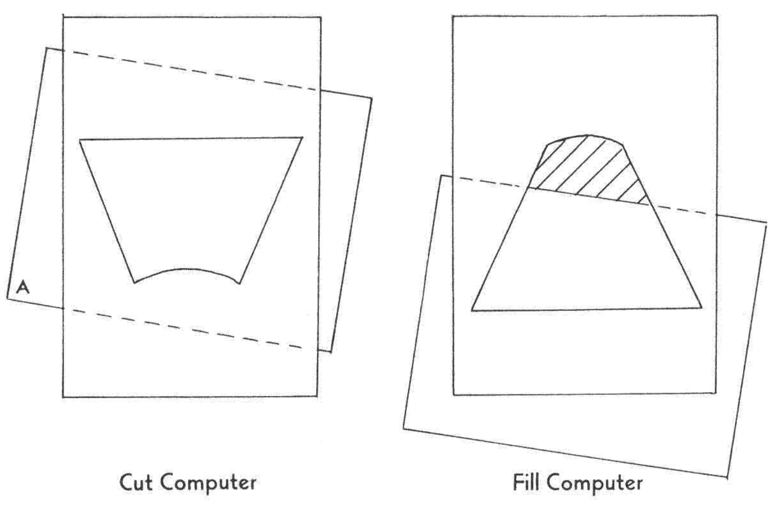

FIG. 7. Analog Computers showing positions for a section in fill.

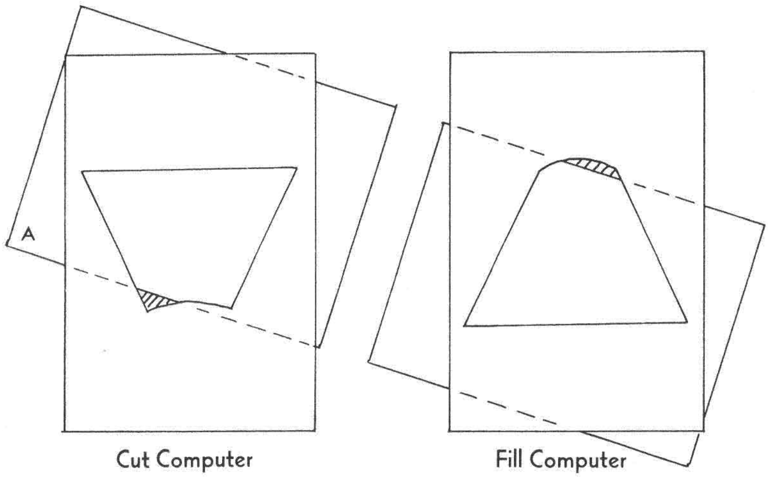

FIG. 8. Analog Computers showing positions for a section in partial cut and partial fill.

cut and partial fill, then the lower edge of piece *A* would allow a little open space to exist in the cut computer.

The actual computation is done by light. A light source is on one side of the computer, and a photo-electric cell is on the other. The amount of light passing through the computer is exactly proportional to the amount of open space. This space is of course exactly proportional to the amount of cut.

Output from the photo-electric cell can either be fed to a meter, which can be calibrated directly in square feet of cut, or it can go to some form of recording system. It might be noted that if the tracing table platen can be moved along the centerline at a constant rate, with the platen always having the proper elevation and slope, it is possible to feed the rate of movement along the route centerline as a voltage, and the cut (or fill) area as a current, and read volumes off a wattmeter-type instrument. In fact, one more integration, and the mass diagram can be plotted directly. These ideas may seem radical, but they may well find some practical application.

In summary, then, the straight-line method of approximating the ground surface of a cross-section may have some interesting possibilities. Even if this method fails to have merit on its own, perhaps its presentation may set some readers to thinking about other *truly photogrammetric* methods of volume determination, rather than mere adaptations of outdated field surveying methods.

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