The Determination of the Angle between the Fiducial Axes (The 90° Condition)

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ABSTRACT: *The deviation of the angle between the fiducial axes from rectangularity is determined without the use of a right-angle master. The difference between the supplementary angles is measured after superimposing the calibration negative plate with the calibration positive plate. The algebraic expression for the angle of deviation, ϑ, is given when the approximation $\sin\vartheta = \vartheta$ is valid for all angles measured and calculated. A precise collineation of the two calibration plates is not required. The case when a right-angle master is employed is shown to be a special case.*

ONE of the conditions which the four fiducial marks of precision mapping cameras have to satisfy and which is sometimes referred to as the 90° condition, is, as stated in the early literature, as follows: " . . . the lines joining opposite members of the two pairs of index markers shall intersect at an angle of 90° ±1 minute. . . ."[1] Thus, regarding Figure 1, we have for the angles φ and ϑ between the fiducial axes *NS* and *WE* of the fiducial cross:

$$\begin{aligned} \varphi + \vartheta &= 90^\circ, & \vartheta &\leq 1 \text{ minute of arc}, \\ &= \pi/2, & &\leq 0.000\,291 \text{ radian}. \end{aligned} \tag{1}$$

On a 9×9 inch camera frame, this is equivalent to a permissible peripheral displacement of ±0.065 mm. The two supplementary angles of the fiducial cross are then

$$\varphi_1 = \frac{\pi}{2} - \vartheta, \qquad \varphi_2 = \frac{\pi}{2} + \vartheta \tag{2}$$

and their difference is

$$\varphi_2 - \varphi_1 = 2\vartheta. \tag{3}$$

It will be noted that the angle ϑ is so small that the relation holds

$$\sin\vartheta = \vartheta. \tag{4}$$

This appropriation will simplify the subsequent calculations. It can also be applied to many different cases found in the photogrammetric literature, yet surprisingly is hardly ever used. Provided the limit of its validity is chosen properly, the relation (4) need not be a source for an increase in error.

For instance in the present case, if we consider as the limit of accuracy for the determination of the angle ϑ an error ϵ of

$$\begin{aligned} &= \pm\ 0.01 \text{ minute of arc}, \\ &= \pm\ 0.000\,003 \text{ radian}, \end{aligned} \tag{5}$$

then relation (4) will be valid for all angles

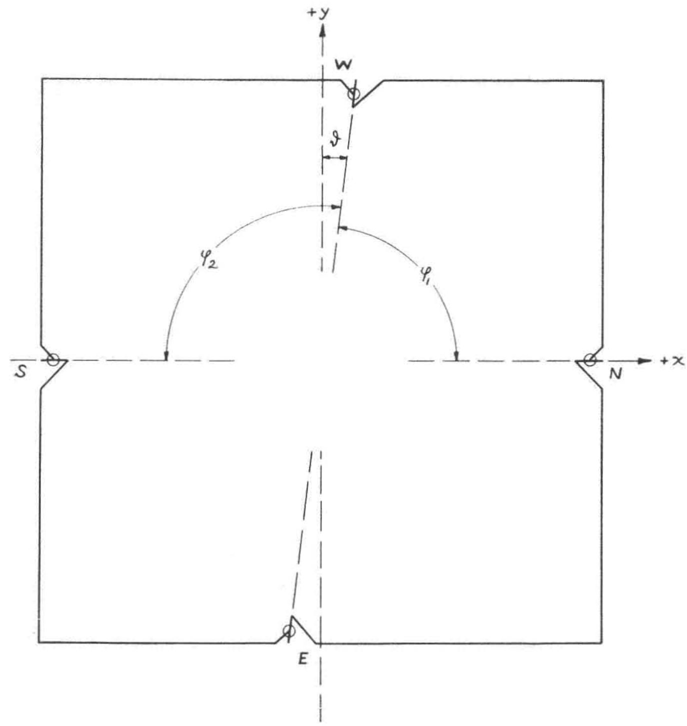

FIG. 1. The four fiducial marks, *N*, *W*, *S*, *E*, with *N* near the positive direction of the *x*-axis. The angle ϑ is the deviation from the rectangularity of the two fiducial axes *NS* and *WE*. The details measured are encircled.

$$\begin{aligned} \vartheta &< 1 \text{ degree of arc}, \\ &< 0.017453 \text{ radian}, \end{aligned} \tag{6}$$

i.e. for angles up to sixty times the tolerable deviation from rectangularity.

From relation (4) follows,

$$\vartheta^2 = 0 \quad \text{and} \quad \cos\vartheta = 1.000\,00. \tag{7}$$

Methods to determine the angle ϑ have been described previously.[2] The method described in this paper employs a minimum of equipment as is shown in Figure 2. In principle, it differs mainly in two respects:

1. All methods previously described employ a right-angle master, for instance in the form of a glass plate carrying four radial diamond lines 90° apart[3] or in the form

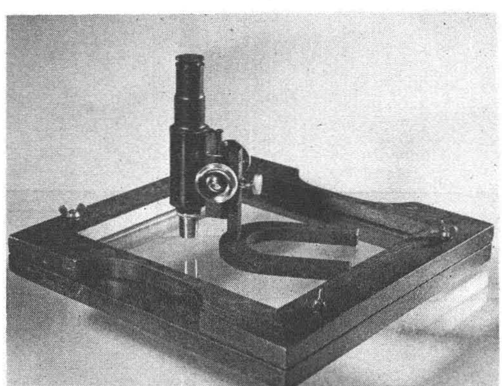

FIG. 2. The superimposed negative and positive calibration plates are held by the frame. The small distances near the edges are measured with the microscope.

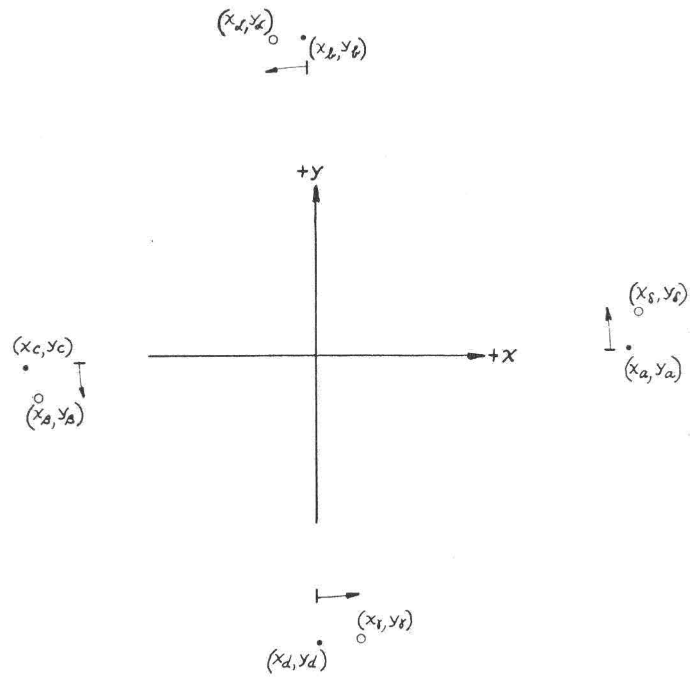

FIG. 3. The images superimposed to near registration, after a rotation of the positive plate through $\sim(\pi/2)$. Marks of the negative plate ●●●; subscripts a, b, c, d. Marks of the positive plate ○○○; subscripts α, β, γ, δ.

of a graduated circle on a rotating plate holder attached to a comparator.[4] The method to be described in this paper does not need a right-angle master, for the angle ϑ is determined by applying equation (3). As usual, a flash plate, also called a calibration plate and herein the negative plate, is made in the manner described, for instance, by Washer and Case.[5] This plate should show the images of the fiducial marks in sharp outline on a low-contrast photo. The conventional marks are thus appearing light in a background of rather low density (approximately 0.8), easily obtainable after an underdevelopment. Then, from the negative plate, a positive plate is made by contact printing, again with the high densities being kept rather low.

For the measurement, the negative and the positive plates are at first superimposed in the same manner as for contact printing. Then the positive plate is rotated through an approximate right-angle so that the first-third quadrant on the negative plate will about coincide with its supplementary angle of the second-fourth quadrant on the positive plate. See Figure 3. The difference between the two superimposed angles is then determined by measurement and calculation.

2. Most methods described previously require a collineation of certain details on the master plate with corresponding details on the calibration plate[6] attainable with the use of special adjustment attachments. Such a registration, tedious and the source of a systematic error, is not required, as has been pointed out by Washer.[7] Only a near and arbitrary coincidence is necessary. "Interpretation of the data thus gained yields the amount of departure from 90°."[7] This interpretation not earlier described is given in the following.

Figures 3 and 4 illustrate the superposition of the negative and positive plates when arranged for the measurements. Both plates are held together in a fixed arbitrary position in the frame shown in Figure 2. The measurements are made through the microscope shown, equipped with a micrometer-scale in the ocular. The magnification used is approximately 50 times.

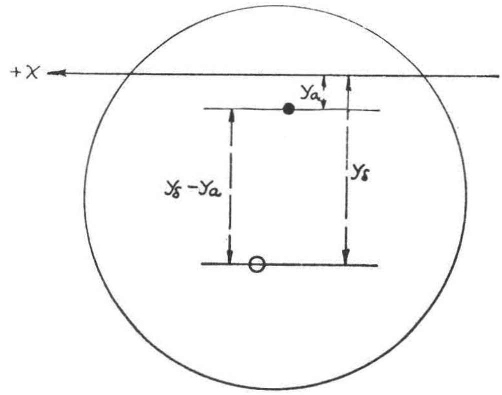

FIG. 4. The difference of the coordinates γ_δ and γ_a as seen and measured in the microscope. This is an enlarged detail from Fig. 3, shown upside down.

To the four fiducial marks are assigned the letters of the four cardinal points of the compass, N, W, S, E, in counterclockwise rotation, with N indicating the positive direction of the x-axis. Placed in a rectangular coordinate system, the coordinates of the negative and positive images of the fiducial marks are as follows (Figures 3 and 4):

mark symbol for negative	N	W	S	E
mark symbol for positive	E	N	W	S
on the negative plate	(x_a, y_a)	(x_b, y_b)	(x_c, y_c)	(x_d, y_d)
on the positive plate	(x_δ, y_δ)	(x_α, y_α)	(x_β, y_β)	(x_γ, y_γ)

The symbols used as subscripts are self-explanatory: a, b, c, d for the negative and α, β, γ, δ for the positive.

The equations, $L=0$, of the fiducial axes are now written down. The equation of the negative fiducial axis NS is:

$$L_{ac}: \quad \frac{x - x_a}{y - y_a} = \frac{x_a - x_c}{y_a - y_c}$$

or

$$L_{ac} \equiv x(y_a - y_c) - y(x_a - x_c) + c_{ac} = 0 \tag{8)a}$$

where the constant term c_{ac} has the value

$$c_{ac} = x_a y_c - x_c y_a$$

and is of no importance here.

Similarly, by cyclic interchange:

$$L_{bd} \equiv x(y_b - y_d) - y(x_b - x_d) + c_{bd} = 0, \tag{8)b}$$

and for the positive fiducial axes:

$$\begin{aligned} L_{\delta\beta} &\equiv x(y_\delta - y_\beta) - y(x_\delta - x_\beta) + c_{\delta\beta} = 0, \\ L_{\alpha\gamma} &\equiv x(y_\alpha - y_\gamma) - y(x_\alpha - x_\gamma) + c_{\alpha\gamma} = 0. \end{aligned} \tag{9}$$

The superimposed angles φ_n and φ_p between the pairs of the negative resp. positive fiducial axes are then given by the relations:

$$\cos \varphi_n = \frac{(y_a - y_c)(y_b - y_d) + (x_a - x_c)(x_b - x_d)}{R_n} \tag{10)a}$$

for the angle of the first quadrant of the negative fiducial cross; and

$$\cos \varphi_p = \frac{(y_\delta - v_\beta)(y_\alpha - y_\gamma) + (x_\delta - x_\beta)(x_\alpha - x_\gamma)}{R_p} \tag{10)b}$$

for the angle of the fourth quadrant of the positive fiducial cross. The denominators have the values:

$$R_n = \sqrt{(x_a - x_c)^2 + (y_a - y_c)^2} \cdot \sqrt{(x_b - x_d)^2 + (y_b - y_d)^2},$$

$$R_p = \sqrt{(x_\delta - x_\beta)^2 + (y_\delta - y_\beta)^2} \cdot \sqrt{(x_\alpha - x_\gamma)^2 + (y_\alpha - y_\gamma)^2}.$$

The following simplifications can be made:

1. With respect to the denominators R_n and R_p; the distance s_{ac} between the negative images of the fiducial marks $N(x_a, y_a)$ and $S(x_c, y_c)$ is given by the relation

$$\cos \eta_{ac} = \frac{x_a - x_c}{s_{ac}}$$

where η_{ac} is the angle between the fiducial axis NS and the abscissa. Since the placement of the coordinate system in relation to the fiducial crosses is arbitrary, we can make the angle η_{ac} so small that

$$\cos \eta_{ac} = 1. \tag{11}$$

Then we have

$$s_{ac} = x_a - x_c. \tag{12)a}$$

Similarly

$$s_{bd} = y_b - y_d. \tag{12)b}$$

Usually the two distances s_{ac} and s_{bd} between the two pairs of opposite fiducial marks are equal, but even for any difference

$$\delta < 1 \text{ mm.}$$

so that

$$s_n = s_{ac} - \delta = s_{bd} + \delta \tag{13}$$

we may write

$$s_n^2 = s_{ac} \cdot s_{bd}$$

since the terms s_{ac} and s_{bd} are quantities of the order of 9 inches or 225 mm.

The other quantities of the denominator, $(y_a - y_c)^2$ and $(x_b - x_d)^2$, are squares of quantities sufficiently small to be neglected, again following relation (7). Thus we have

$$R_n = s_n \cdot s_p = s^2 \quad \text{and similarly} \quad R_p = s_n \cdot s_p = s^2.$$

2. With respect to the numerators of equations (10), following (12) and (13), f. i. for the numerator of the right side of formula (10)a:

$$(y_a - y_c)(y_b - y_d) + (x_a - x_c)(x_b - x_d)$$
$$= (y_a - y_c) \cdot s_{bd} + s_{ac}(x_b - x_d) = s_n(y_a - y_c) + s_n(x_b - x_d) - \delta(y_a - y_c) + \delta(x_b - x_d).$$

The terms having δ as a factor are products of two small quantities and may therefore be neglected. The numerator is then simplied to:

$$s[(y_a - y_c) + (x_b - x_d)]$$

and we obtain finally the simplified expressions for (10)a and (10)b:

$$\cos\varphi_n = \frac{y_a - y_c + x_b - x_d}{s}$$
$$\cos\varphi_p = \frac{y_\delta - y_\beta + x_\alpha - x_\gamma}{s}. \tag{14}$$

3. With relation (4), we have furthermore:

$$\cos\varphi_n = \sin\vartheta = \vartheta,$$
$$\cos\varphi_p = -\sin\vartheta = -\vartheta. \tag{15}$$

Now from the difference between the two relations (14) and after the proper rearrangement in the numerator, we have

$$\vartheta = \frac{-(y_\delta - y_a) - (x_\alpha - x_b) + (y_\beta - y_c) + x_\gamma - x_d}{2s}. \tag{16}$$

In this equation, the quantity s as well as the four quantities in the parentheses are directly measurable, as is seen f.i. from Figure 4 in regard to the difference $(y_\delta - y_a)$.

In order to avoid a confusion of the signs of the coordinate differences as they are measured directly, the following procedure may be adopted:

1. The negative and the positive calibration plates are superimposed in the following manner: The positive plate is rotated through slightly more than a right-angle, so that the positive images are displaced with regard to the nearby negative images in a counterclockwise sense as indicated by the arrows in Figure 4. Then the distances between adjacent negative and positive images are always of the following signs:

$$y_\delta - y_a > 0 \qquad |y_\delta - y_a| = q_1$$
$$y_\alpha - y_b < 0 \qquad |y_\alpha - y_b| = q_2$$
$$y_\beta - y_c < 0 \qquad |y_\beta - y_c| = q_3$$
$$y_\gamma - y_d > 0 \qquad |y_\gamma - y_d| = q_4$$

2. Taking the absolute values q_i and substituting in equation (16), we have

$$\vartheta = \frac{-q_1 + q_2 - q_3 + q_4}{2s} \tag{17}$$

Thus, the distances read through the microscope are substituted directly into equation (17).

3. Following the sign convention of (15), the values of the angle ϑ are
   a. negative, when the angle of the first quadrant is larger than the angle of the fourth quadrant; see Figure 5a.
   b. positive, when the angle of the first quadrant is smaller than the angle of the fourth quadrant; see Figure 5b.

It is interesting to note that, during the calculations, the location of the origin of the coordinate system remained unknown, and the location of the points of intersection of the negative resp. positive fiducial axes remained undetermined. As to the position of the coordinate axes with respect to the fiducial axes, the only restriction was imposed by the use of the approximation (11). By direct substitution it follows that the quantity ϑ of the relation (16) remains invariant

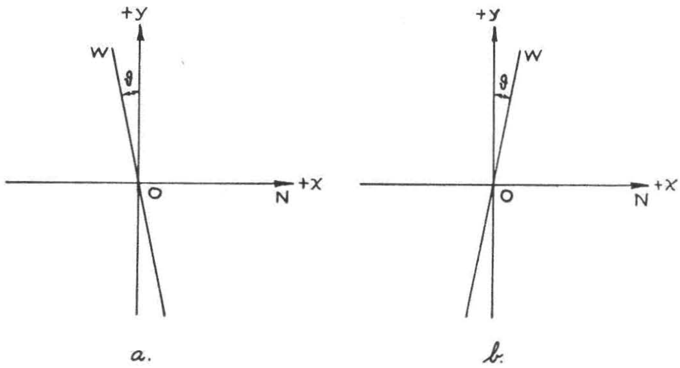

FIG. 5. (a) Angle ϑ is negative. First quadrant NOW is larger than 90°. (b) Angle ϑ is positive. First quadrant NOW is smaller than 90°.

1. through any parallel translation

$$x' = x + m,$$

$$y' = y + n.$$

The location chosen is determined alone by the necessity to read the four distances of the numerator of the relation (16) in the field of view available in the microscope.

2. through any small rotation ψ, provided it remains

$$\psi = \sin \psi.$$

Finally, it should be stated that if one of the calibration plates is substituted with a right-angle master and the procedure of measurement as described is followed, formula (15) is replaced by the new formula

$$\vartheta = \frac{-q_1 + q_2 - q_3 + q_4}{s} \tag{18}$$

with the denominator s in place of $2s$. Formula (18) is then valid for the case as described by Washer and Case.[8] This follows from equations (2) and (3), since the angle φ_1 is now equal $\pi/2$ and the difference $\varphi_2 - \varphi_1$ becomes equal ϑ.

Example. The fiducial marks were in the form of small dots:

dark on the negative, diameter 0.22 mm., and
light on the positive, diam. 0.21 mm.

From the center of the negative dot, the distances were read to the nearer and farther edge of the positive dot and were averaged.

Two sets of measurements were made:

1. fiducial marks: negative N over positive N. The result should show an angle ϑ_0 equal zero. From (17):

$q_1 = 15.15$ divisions

$q_2 = 14.35$ divisions

$q_3 = 7.1$ divisions

$q_4 = 7.85$ divisions

$$1 \text{ division equal } 0.0454 \text{ mm.}$$

$$-q_1 + q_2 - q_3 + q_4 = 0.05 \text{ division}$$
$$= 0.002\,27 \text{ mm.}$$

The distance s between opposite marks was 226 mm.

$$\vartheta_0 = \frac{0.002\,27}{452} \text{ radian} = 0.000\,005 \text{ radian}$$
$$= 1 \text{ second of arc.}$$

This result is an illustration of the accuracy attainable with the method.

2. fiducial marks: negative N over positive W.

$$q_1 = 14.4 \text{ kivisions}$$
$$q_2 = 10.35 \text{ divisions}$$
$$q_3 = 6.05 \text{ divisions}$$
$$q_4 = 10.4 \text{ divisions}$$
$$-q_1 + q_2 - q_3 + q_4 = +0.3 \text{ division}$$
$$= \frac{0.013\,6}{452} \text{ radian} = 0.000\,03 \text{ radian}$$
$$= 6 \text{ seconds of arc,}$$

and the angle of the first quadrant, between the marks N and W, is 6 seconds smaller than 90°, following (18) since ϑ is positive.

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