The Application of Analytical Photogrammetry to Missile Trajectory Measurement[*]

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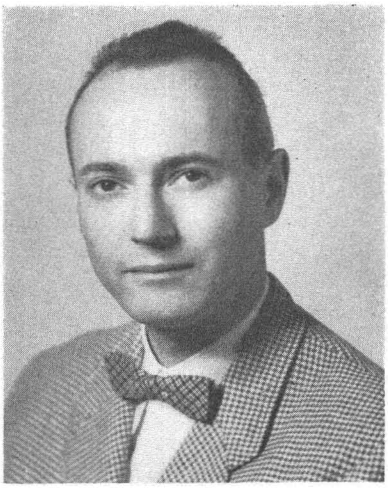
ABSTRACT: *This paper concerns the philosophy of data reduction of optical systems data. A discussion is given of the mathematical techniques used to produce the final optical information; and the peculiarities of the data from the various instrumental systems are presented.*

PREFACE

THIS paper presents a discussion of the mathematical techniques used in reducing missile test photogrammetric data. It does not enter into a presentation of the mathematical formularization, since that is adequately covered in pertinent technical reports. The fundamental concepts of Analytical Photogrammetry, both Intersection and Stereo Photogrammetry, as they apply to missile testing are explained.

The data derived from the various types of photogrammetric instrumentation retain the peculiarities of the random and systematic errors associated with the particular recording instrument and the particular data reduction instruments used in the preparation of the missile flight data. Because of this, the description of the peculiarities of the different types of optical instruments is presented as a description of the random and systematic errors affecting a photogrammetric reduction. A chart outlining many of the sources of error associated with a photogrammetric instrumentation system is included with this paper. The chart was prepared by D. C. Brown and published in reference (3). The paper also presents a discussion of the mathematical propagation of both the random and systematic errors through the data reduction process, and indicates a method of determining the error influence on the reduced data. Since a familiarization with the photogrammetric instrumentation used in missile testing is needed as background for this presentation, the following brief description of the general types of cameras used is given.



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Photogrammetric missile trajectory data are obtained from several types of instrumentation cameras, which are generally categorized as being either fixed cameras or tracking cameras. Fixed cameras are those operated in a given fixed orientation, the missile trajectory passing through the field of view of the camera. Tracking cameras are those which track the missile in flight, the camera orientation changing for each photographic frame. The high-precision fixed camera is the Ballistic Camera, which utilizes a single glass-plate negative upon which is recorded the entire event of interest. Normally, this is photographed against a background of stars, from which an accurate orientation can be calibrated. All other cameras of somewhat lower precision used in a fixed orientation are categorized as fixed cameras. These cameras normally use film as the photographic base. Either ground surveyed target boards or the

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mount dials are used to determine the camera orientation.

Tracking cameras can be categorized as metric or non-metric. The metric tracking cameras are of the Cine-theodolite type which record Azimuth and Elevation angles together with the event photographed for each frame. The non-metric tracking telescopes do not record camera orientation, but can be used for certain metric applications such as missile-attitude determination, where the telescope orientation is computed from the known missile-position data.

INTRODUCTION

Analytical Photogrammetry is the art or science of applying mathematical methods to obtaining reliable measurements by means of photographs. For many years, the application of analytical photogrammetry remained a laboratory curiosity and a classroom exercise. The increased utilization of the electronic computer in recent years has led to sophisticated developments of numerical techniques in analytical photogrammetry that were not possible in the past.

The basic information of all photogrammetry is the image-space measurement of the object-space phenomena. The measurements are normally in the form of linear or polar coordinates on the image-space medium, and may be made and utilized either directly or indirectly in the resulting photogrammetric reduction. A major advantage of analytical photogrammetry is that the basic measurements can be considered independently, and can be adjusted for all perturbations of the photogrammetric model, to the fullest desired extent of the state of human knowledge. Since it is impossible to make a perfect measurement, a mature instrumental system includes knowledge of the nature of the measuring error. Proper computational techniques allow propagation of the measuring error through the adjustment to obtain the variances of the reduced data. Proper techniques also allow an indication of the residual, uncorrected, systematic error remaining in the system. The fundamental mathematical concept applicable to analytical photogrammetry is the Method of Least Squares, in which the numerical adjustment aims to minimize the weighted sum of the squares of the measuring residuals.

ANALYTICAL PHOTOGRAMMETRY

The direct mathematical solution of a photogrammetric adjustment results in a quadratic form of the residuals, the solution of which requires complicated procedures that are somewhat impractical in nature. The popular methods for the solution of photogrammetric adjustments are based upon Newton's method of successive approximation. A set of variational equations is established from the initial photogrammetric condition equations. These equations describe the photogrammetric model in a mathematical format. The usual method of establishing the variational equations is to linearize the condition equations by a Taylor's expansion about the observations and the parameters. The solution requires approximations for the parameters which are sufficiently close to the true value to allow convergence.

The Method of Least Squares results in the computation of the most probable values for the unknown parameters, and thus reduces the effects of random errors. The least squares adjustment cannot properly allow for the influence of systematic errors, and these must be compensated in the reduction before entering into the adjustment.

The major photogrammetric reductions performed by analytical methods are those for interior and exterior orientation of the photogrammetric camera, triangulation to compute object space position information, and computations to determine the attitude of a body in object space. Interior orientation of a photogrammetric camera refers to the calibration of the optical elements of the camera body. These elements normally include the establishment of the principal-distance c and the position of the principal-point with respect in the fiducial marks of the camera x_p, y_p. The exterior orientation is that set of quantities which fixes the position of the camera station and the angular orientation of the photograph. The position is usually expressed in terms of three rectangular coordinate distances X^c, Y^c, Z^c. The elements of angular orientation are usually expressed as three angular rotations. In aerial photogrammetry the rotations may be designated ϕ, ω, κ, while in terrestrial photogrammetry they are α, ω, κ. In the usual photogrammetric case, given any three of the nine elements of orientation, it is possible to determine the remaining six, if sufficient observations and control information are available.

The solution of a photogrammetric problem may be performed using the tenets of either Intersection Photogrammetry or of Stereo Photogrammetry. Analytical Stereo Photogrammetry is based upon a mathematical formulation for the precepts of stereoscopy which attends the simultaneous consideration, in three dimensions, of an object from different perspectives. Discussion of the stereo procedures will be given later.

The concept of Intersection Photogrammetry requires first, the independent and precise orientation of each camera to be used in the photogrammetric net; and second, the determination of the most probable solution to the photogrammetric problem by means of a rigorous least squares adjustment using redundant information, considering the basic orientations to be error free. Propagation of the random error effects by co-variance analysis techniques, and determination of the systematic error effects by overdetermining the solution, indicates the accuracy of the reduced data as a by-product of the adjustment. This philosophy of computation can be applied to solution of the problems of missile-position determination (triangulation) and missile-attitude determination.

FUNDAMENTAL MEASUREMENTS

The basic data for missile-trajectory measurements consist of observations on the photographic records from the several cameras to be used in the reduction. These observations are in the form of measurements made to an image on the photographic film or plate; the form of the measurement depends upon the type of instrumentation and the data required. For example: Ballistic camera data are obtained from x, y, coordinate observations made to the symmetric images of point sources of light: star images for orientation, strobe light or flare images for triangulation. Fixed camera orientation data are obtained from x, y, coordinate observation made to the symmetric center of the images of surveyed target boards. Fixed Camera position data are obtained from x, y, coordinate observations made to the images of the images of the nose of the missile, the leading edge of the flame, or to the symmetric images of a missile-borne light source. Cine-theodolite position data are obtained from measurements made on the recorded images of the Azimuth and Elevation dials together with linear measurements in machine counts of the tracking error: ΔA_t, ΔE_t. The Tracking error measurements are made to the images of the nose of the missile, the flame, or the blob on the film, depending upon the aspect-angle and distance of the missile from the camera site.

Attitude data, pitch and yaw, are predominantly obtained using measurements of the so-called V-angle; the angle between the image of the missile-body vector and the axes of the film coordinate system. For a tracking camera attitude reduction, only the V-angle measurements for two or more cameras are required. For a fixed camera attitude reduction, the V-angle measurements together with the x, y, plate coordinates of some point on the image of the missile are required for two or more cameras. For convenience, there is generally considered the plate coordinates for the tracking point used in the Position reduction. Ballistic Attitude reduction refers to the simultaneous determination of the three elements of attitude: the pitch, yaw, and roll, of the missile. These data are obtained from the x, y, plate coordinate measurements made to the symmetric images of either paint pattern intersections or special missile-borne light sources.

A distinction has been made between the x, y, coordinate observations and the x, y, plate coordinates. The x, y, observations are recorded pointings made in the coordinate system of the measuring comparator to the specified image on the emulsion of the photographic medium. These observations must be transformed to the coordinate system of the operational negative (the glass-plate or photographic-film) and then corrected for the perturbations introduced by the peculiarities of the particular optical instrument. The final corrected values are then commonly referred to as plate coordinates, regardless of the composition of the photographic base.

RANDOM ERRORS

The primary sources of random error in the photogrammetric reduction occur in a two-fold manner. When optimum images are being considered, such as with Ballistic Camera data or missile-borne light sources, the most important random error is due to the differences in position between the latent and the developed images. These image shifts due to random emulsion creep are caused by strain occurring in the emulsion during processing of the negatives. The other primary source of random error is the measuring-error due to:

1) setting-error, the inability of the measuring operator to repeat his measurement; and
2) identification-error, the inability of the operator to set on corresponding points of different images.

Other sources of contributing random error are caused by residual uncorrected environmental anomalies and also by random errors of the measuring comparator. The composite

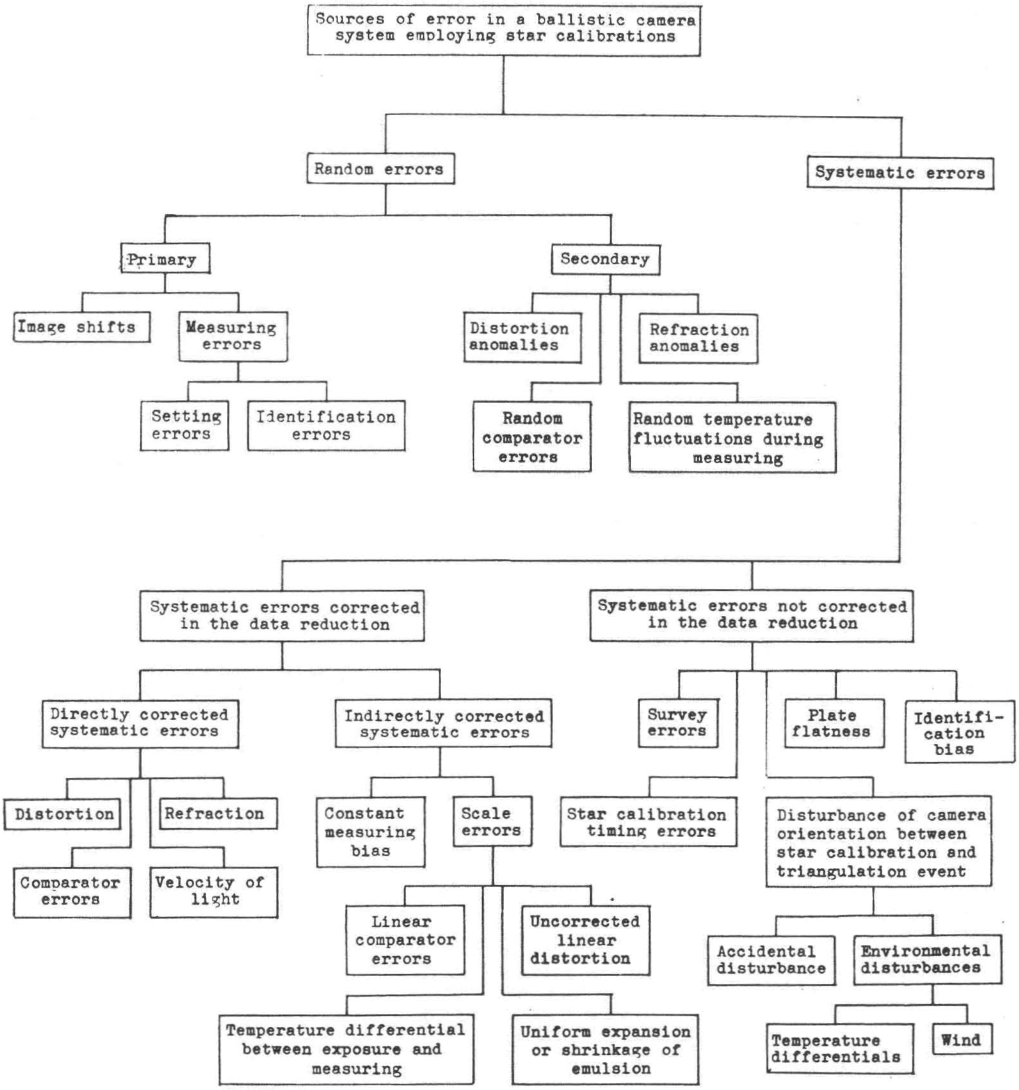

FIG. 1

of all these random errors are commonly called the reading-error.

The magnitudes of the random errors for the different types of optical instrumentation data are given below. Except for the random error of the *V*-angle attitude measurements, the values are obtained from reference (1).

For measurements made on Ballistic Camera plates to optimum-size symmetric images (approximately 50 microns diameter) using precision comparators, the average values for random errors as indicated by time series analysis of series of light-source images are typically $\sigma_{\bar{x}}, \sigma_{\bar{y}}, \approx 3$ microns.

The random error for Cine-theodolite data is considered to lie in the ability of the operator to interpolate the azimuth and elevation dial recordings together with the ability to measure the tracking-error. For measurements made with improved Theodolite measuring equipment, the combined values vary typically in the range: $\sigma_A, \sigma_E, \approx 5$ seconds to 10 seconds of arc.

The random error values for Fixed Camera data vary depending upon image quality and the measuring instrument. Typical equipment for measuring Fixed Camera film are the high-speed film reader and the precision comparator.

Table 1 summarizes the results of extensive

TABLE I

Quality of image	*Measured on film-reader* $\sigma_{\bar{x}}$ *(microns)*	$\sigma_{\bar{y}}$ *(microns)*	*Measured on comparator* $\sigma_{\bar{x}}$ *(microns)*	$\sigma_{\bar{y}}$ *(microns)*
poor (nose of missile in poor lighting)	40–60	12–25	30–45	14–20
average (nose of missile under average condition)	20–30	10–15	17–22	6–9
excellent (missile-borne light-source)	10–15	10–15	4–7	4–7

investigation by time series analysis.

In Table 1, the x coordinate is assumed to coincide with the general direction of the missile travel. For an extremely poor image such as the leading edge of the flame, the random measuring-error will be correspondingly high.

The random error associated with the V-angle measurements for determining attitude data has been investigated by Reed (2). His experience indicated that the measuring-error ranged from .1 to 5 degrees depending upon the missile-image; and that average images had a measuring-error of .5 degrees. The random error associated with the measurements on symmetric images for Ballistic Attitude data approximate 5 to 10 microns in the x and y coordinates.

The photogrammetric determination to obtain the most probable trajectory data of the object under investigation is performed in a rectangular Cartesian coordinate system. The reduction usually considers the coordinate positions of the camera stations, the orientations of the camera optical axes, and the measured plate coordinates of the images for Ballistic Camera and Fixed Camera position data; and the direction of the rays in space to the object being tracked for Cine-theodolite position data. The measured V-angles or x, y, plate coordinates are used for attitude data. The mathematics for these reductions have been widely distributed in references (3 through 8), and will not be presented herein. The reduction procedures utilize rigorous least squares adjustment techniques and a redundancy of data. For example: a 2-station reduction for position data already contains an overdetermination with one degree of freedom, and usually 3 stations or more are utilized in obtaining trajectory position data. The adjustment procedures for attitude reduction are analogous to that described above.

The random error of the measurements are propagated through the adjustment by covariance analysis techniques to obtain the standard deviation of the reduced trajectory data. Appropriate weighting factors for the various measurements are obtained from the inverse square ratio of the respective measuring errors as related by an arbitrary constant of proportionality referred to as the unit variance σ^2. (As a rule σ^2 is chosen to represent the measuring variance.)

Thus the weight of any observation j is computed from:

$$w_j = \frac{\sigma^2}{\sigma_j^2}. \qquad (1)$$

The unit standard deviation σ is normally considered to be:

1. for Ballistic Camera data: 3 microns,
2. for Fixed Camera data: 10 microns,
3. for Cine-theodolite data: 10 seconds of arc,
4. for V-angle attitude data: 30 minutes of arc.

It has been shown by Brown (3), that the relative co-variance matrix of the parameters determined by a least squares adjustment is given by the inverse of the coefficient matrix of the normal equations. It follows that the matrix N^{-1} multiplied by the unit variance σ^2 would represent the co-variance matrix of the reduced parameters. In particular, for triangulation, the N^{-1} matrix is of the order 3×3, since the X, Y, Z, coordinates of the triangulated position point are the only unknowns. If n_{kk}^{-1} denotes the diagonal element of N^{-1} corresponding to the kth unknown element of position, denoted here by u_k the standard deviation of u_k would be given by:

$$\sigma_{u_k} = \sigma\sqrt{n_{kk}^{-1}}. \qquad (1.2)$$

These standard deviations of the reduced parameters are commonly referred to as the *GDOP* (the Geometric Dilution of Precision) and are a function of the geometry of the photogrammetric model, and of the random error of the fundamental measurements. For position data, the co-variance matrix defines the confidence or error ellipsoid associated with the computed most probable position in space. A direct relationship exists between the *GDOP* and the random error of the observation. Thus, if the measuring error is changed by a given factor, the *GDOP* is also changed by the same factor, provided that other considerations of the photogrammetric model remain constant.

The nondiagonal elements of N^{-1} determine the correlations existing between the errors in the computed elements. The coefficient of correlation for the gth and kth elements is:

$$\rho_{gk} = \frac{n_{gk}^{-1}}{\sqrt{n_{gg}^{-1}\, n_{kk}^{-1}}} \qquad (1.3)$$

where n_{gk}^{-1} denotes the element in the gth row and kth column of N^{-1}. The coefficient of correlation provides an indication of the degree of mutual variation in a pair of statistical variables.

SYSTEMATIC ERRORS

The source and influence of systematic errors affecting a photogrammetric model are well known to the qualified optical engineer and photogrammetrist, and steps are usually taken to eliminate the influence of these errors in the final product of most photogrammetric operations. A comprehensive treatment of the systematic errors has been presented by Brown (3) in a manner amenable to analytical photogrammetric techniques.

Two groups of systematic errors are considered, the first group being those ordinarily corrected in the mathematical reduction. This first group may be further subdivided into those errors corrected directly and those corrected indirectly. Of the systematic errors corrected directly, radial-lens distortion and atmospheric refraction have the greatest effect upon the photogrammetric model, and elaborate reductions must be performed to effect their compensation. Another major source of systematic error is the comparator or film reader used for making the observational measurements. Not only must the comparator be a precision instrument in proper adjustment, but it must also be accurately calibrated to remove the effects of residual systematic errors. The major comparator errors to be considered are: nonperpendicularity of the comparator axes or ways, weave of the ways, periodic errors, and secular errors. Other systematic errors directly correctable in the data reduction process are those introduced by the peculiarities of the particular type of instrumentation. For example, certain cameras utilize a film moving across either a plane surface or a drum. The shutter may also be of the moving focal-plane type which must of necessity be located a finite distance from the actual focal-plane of the camera. Time series photography also introduces its own peculiarities into the data reduction process.

The other class of systematic errors are those indirectly corrected and whose compensation comes about as a by-product of the reduction. One such error is the constant measuring error of the operator. Selection of all measuring points to be of the same size and quality will eliminate this effect; but is obtainable only under rare and special circumstances, as with Ballistic Camera plates of light flashes against a star background. For normal conditions, the average of a set of direct and reversed measurements (reverse measurements are made with the plate rotated 180° from the original measuring position) will eliminate the operator bias. Scale errors are another class of indirectly correctable systematic errors. These include any errors which are linear in function and which can be corrected by a constant multiplication factor. Errors of this type include: dimensional changes in the emulsion and/or photographic base due to temperature and humidity fluctuations; the linear component of radial lens distortion; and linear comparator errors. The combined effect of all such errors can be compensated by use of a calibrated principal-distance. For differential changes in dimension, it might be necessary to utilize two calibrated principal-distances, one for the x coordinate and one for the y coordinate.

The second group of systematic errors are those not normally corrected in the data reduction process. Control of such systematic errors is mostly a prevention problem, and depends upon the establishment of suitable facilities for the camera, upon sufficiently refined instrumentation, upon proper photographic exposures, and upon reasonable care in the field work. A listing of systematic errors of this type would include: survey errors for the camera stations and control points, unflatness of the photographic plate, timing errors, identification bias, and camera instability between orientation and trajectory data recording. The causes of camera instability are many, but include: physical disturbance (accidental and deliberate) by the operating personnel; and environmental disturbances by the effects of wind, and temperature changes, and the effects of solar radiation.

The magnitudes of the systematic errors for the different types of optical instrumentation data are given below. The comments and values are obtained from reference (1).

"With good data and a star calibration, Ballistic Camera systematic error is significantly less than the standard deviation of the random error and probably ranges from 1 to 2 microns. This assertion is based upon a thorough system analysis and is supported by the residuals resulting from orientation calibrations and from triangulations. With data of poor quality (over-sized, unsymmetric images) biases of the order of 5 to 10 microns (and occasionally even more) have been indicated by triangulation residuals. Significant accuracy dilution has also been found to arise from camera instability caused by wind and temperature change."

"The systematic error in reduced Cinetheodolite Postion data has been determined in practice to be of the order 30 to 40 seconds of arc. Proper calibration procedures which are in process of being instituted at the Atlantic Missile Range will reduce this bias to the values 5 to 10 seconds of arc."

"The systematic error in the Fixed Camera instrumentation system depends largely upon the method for determining the orientation of the individual cameras used in the triangulation reduction. With typical target board orientation (three to six targets), systematic error is equivalent to 20 to 40 microns. These figures may be doubled if target board distribution is especially poor. Bias in a dial orientation generally ranges from 180 to 240 seconds. Star orientation of the Fixed Camera will result in a lower bias than from the target board orientation, and proper calibration and operating procedures would reduce the bias of the dial orientations."

The concept of Intersection Photogrammetry, as stated earlier, considers that the orientations of the cameras are error-free. Obviously, this is not the case in actual practice, the error of the orientation varying from less than the 3 micron random error of Ballistic Camera, to the excessive amount for the uncalibrated Fixed Camera dial orientation. Thus the major source of bias in the reduced trajectory-measurements are caused by the influence of the incorrect orientations of the respective cameras. The utilization of Stereo-Photogrammetric procedures will eliminate this major disadvantage of Intersection Photogrammetry. In Stereo-Photogrammetry, the camera orientations and desired trajectory computations are carried simultaneously in the adjustment, and the computed orientations are forced to fit within the photogrammetric model. The computations for such adjustments and the associated error propagations are more complicated than for Intersection Photogrammetry, and have been derived by Brown (9). A compromise technique can be utilized in which only a minimum of selected trajectory determinations are carried in the Stereo solution. The mass of the trajectory data can then be determined by the intersection method using the orientations resulting from the initial Stereo solution. In this manner, the orientations must fit the photogrammetric model.

The least squares adjustment consists of determining the vector of measuring residuals, and the vector of parameter corrections, which satisfy the given condition equations while minimizing the weighted quadratic form of the residuals. The vector of residuals is computed from the elements of the least squares adjustment after the final corrections have been applied to the unknown parameters. The mean square error (the estimated unit variance) arising from the adjustment is obtained by dividing the weighted quadratic form of the residuals s by the statistical degrees of freedom f involved in the adjustment.

The degrees of freedom f for the system is equal to the total number of observations n minus the minimum n_0 required for solution:

$$= n - n_0. \qquad (1.4)$$

The minimum number of observations n_0 required to solve the existing condition can be determined from a consideration of the condition equations of the adjustment. The minimum number of observations are those required to solve the existing condition provided that no unknown parameters exist in the system.

The total number of independent condition equations m_0 arising from the system is equal to the degrees of freedom plus the number of parameters p:

$$m_0 = f + p. \qquad (1.5)$$

The least squares adjustment must consider the total number of independent condition equations only. Consideration of non-independent condition equations will lead to singularity of the determinant of the normal equation coefficient matrix, and thus prevent a solution to the problem.

The unit variance σ_0^2 estimated from the adjustment therefore becomes:

$$\sigma_0^2 = \frac{s}{f} \cdot \qquad (1.6)$$

Comparison of this with an accurate predetermined value of σ^2 based on the random error in the observations, provides an indication of the goodness of the adjustment, and in particular, of the presence of a significant degree of systematic error. This comparison may be made quantitative by utilization of the fact that the weighted quadratic form of the residuals divided by the unit variance has a *chi* square χ^2 distribution with f degrees of freedom.

Hence, if one computes:

$$\chi^2 = \frac{s}{\sigma^2}, \qquad (1.7)$$

the probability of obtaining a value of χ^2 as great as χ_0^2 with f degrees of freedom can be obtained from a table of the cumulative *chi* square distribution. If this probability is excessively small, the presence of a significant degree of systematic error would be indicated.

If one is interested in the mean error (the estimated standard deviation) arising from the adjustment of the kth unknown parameter, one should substitute σ_0 obtained from (1.6) for σ in equation (1.2).

The ratio of the mean error of the adjustment to the unit error results in the quantity known as the Bias Factor:

$$\beta_f = \frac{\sigma_0}{\sigma} \cdot \qquad (1.8)$$

The Bias Factor is a convenient tool for comparing systematic errors between instrumentation systems, and between different trajectory computations from the same type of instrumentation. The Bias Factor is also useful in propagating the systematic error through derivative data computations.

In order to obtain full benefit of the least squares adjustment as regards the associated error propagation, it is necessary that a redundancy of data be utilized for an overdetermined solution. A unique solution carrying only the minimum number of observations required for solution would lead to zero degrees of freedom, and make it impossible to estimate the degree of systematic error in the reduced data.

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