The Effect of Tilt on the Measurement of Spot-Heights Using Parallax Methods

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ABSTRACT: *The effect of tilt on the determination of spot-heights is examined from a mathematical point of view. A mathematical model is developed in which the magnitude and direction of tilt, focal-length, and position of the object being measured are the independent variables. The influence of these variables is determined by an examination of the distribution pattern of signed deviations from known object heights, as the independent variables change value. This examination reveals that tilt does introduce an error into spot-height determinations and that, under certain circumstances, this error can be very large. Its magnitude becomes greater as the tilt and also the focal-length are increased. However, the magnitude and sign of the error is strongly influenced by the direction of tilt. The location of the point being measured has some effect on the pattern of errors but this effect is subordinate to the effect of focal-length.*

In addition to the preceding, the effect of using the average stereo-base method of height determination, within the framework of the aforementioned variables, was studied. This study revealed that this method introduced error into the height determinations but that as the focal-length is increased the error is lessened.

FROM time to time opinions are expressed concerning the effect of tilt on the precision of spot-height measurements made by use of parallax methods. Many workers are of the opinion that tilt has little effect on such spot-height measurements. In most cases the source of such opinions can be found in the following statement by Bagley (1941): "The full effect of tilt in causing error comes into play only when measurements are made entirely across the field of overlap. The error in parallax differences as measured between two objects whose images are close together on slightly tilted photographs is microscopic in magnitude, and hence the error in determining such local height differences is negligible provided an accurate value of the stereobase is available." Bagley then continues to derive a formula for the amount of error in absolute parallax caused by tilt *along the flight line.* He made no attempt to evaluate errors when the tilt is not along the flight line.

On the other side of the fence, Pope (1957) has stated that tilt appears to be one of the major causes for errors in tree-height measurement when using large-scale photographs taken with long focal-length cameras. In his discussion, he also assumed a condition where the tilt is along the flight line.

In contrast to Bagley and Pope, Fleming (1960) introduced the factor of direction of tilt (swing) into his study of the effect of tilt on spot-height measurements. However, because of the nature of his approach, which did not permit him to vary the magnitude of tilt, swing, or focal-length, he could not analyze the effect of these variables to any great depth. He did find that with absolute tilts of 3° and a relative tilt of 6°, transverse to the line of flight, errors on a stereo-triplet ranged from − 3.69 to +3.26%, depending on the distance and direction from the principal-point.

Because of these differences of opinion and the subsequent lack of a clear-cut answer to the problem, an investigation into the problem was initiated. The investigation took the form of a mathematical analysis in which the effects of magnitude of tilt, *direction of tilt*, and focal-length on the accuracy of spot height measurements were evaluated. The results, therefore, are based on mathematical theory and it is not certain that the apparent pattern would exist in practice. It is entirely conceivable that, in practice, the pattern would be considerably different, as has been the case with the effect of scale on the accuracy of parallax methods of measuring spot-heights (Johnson, 1958).

The basic three-dimensional coordinate system used in this study is shown in Figure 1. By definition the ground coordinate system has its origin at N_1, the ground nadir point for the first photograph. The positive X axis of the ground coordinate system is parallel to the flight-line, L_1L_2. There are two photographic coordinate systems, one for each photograph. These have their origins at their respective exposure stations L_1 and L_2, and their X axes coincide. The focal-length of the lens, f, and the altitude of the lens above the datum plane, H, are equal for both photographs. The base of an object of height h, lies in the datum plane and has the ground coordinates X_B and Y_B. The top of the object has the ground coordinates X_T, Y_T, and Z_T.

Tilt is introduced into this system by tilting photograph no. 2 and holding photograph no. 1 vertical. This admittedly is an arbitrary condition but it is the only one that permits relatively easy visualization of the effects of tilt. The introduced tilt is of the magnitude τ and has the direction Δ. As can be seen in Figure 2, Δ is defined differently from conventional swing. It is measured counterclockwise from the $+x$

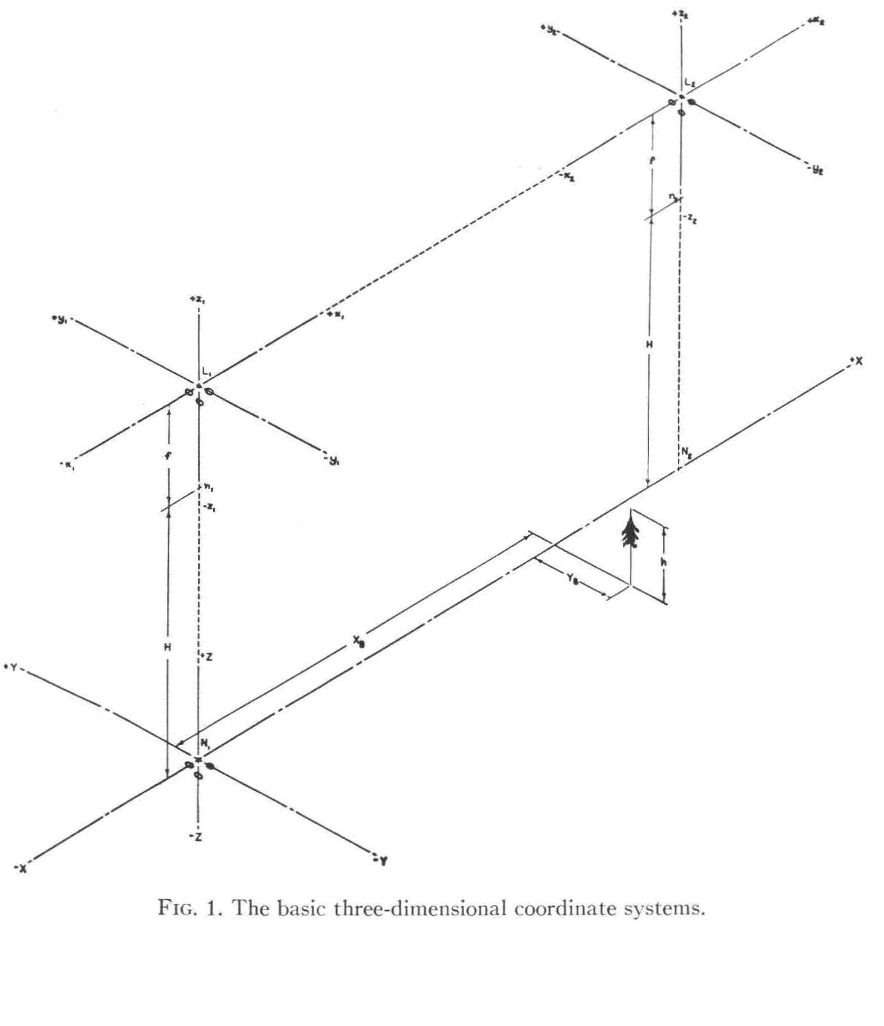

FIG. 1. The basic three-dimensional coordinate systems.

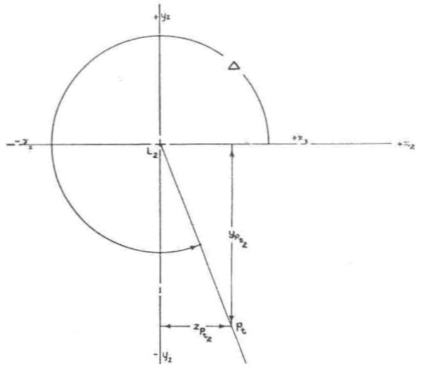

FIG. 2. The modified swing angle Δ.

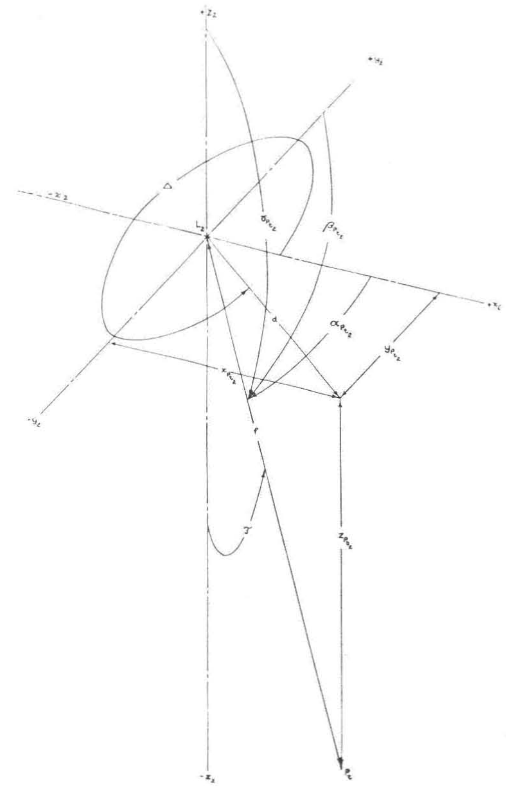

FIG. 3. The coordinates of the principal point of the tilted photograph.

axis of the untilted photograph no. 2. This modification is made to simplify the resulting computations.

Within the coordinate system of the untilted photograph no. 2, the principal-point of the tilted photograph no. 2, p_t, has the following coordinates (see Figure 3):

$$x_{p_{t_2}} = d \cos \Delta \tag{1}$$

$$y_{p_{t_2}} = d \sin \Delta \tag{2}$$

$$z_{p_{t_2}} = -f \cos \tau \tag{3}$$

where:

$$d = f \sin \tau \tag{4}$$

The direction cosines for the line connecting the exposure station, L_2, with the principal point, p_t, are:

$$\cos \alpha_{p_{t_2}} = x_{p_{t_2}}/f \tag{5}$$

$$\cos \beta_{p_{t_2}} = y_{p_{t_2}}/f \tag{6}$$

$$\cos \gamma_{p_{t_2}} = z_{p_{t_2}}/f \tag{7}$$

From these direction cosines and the length of the line $\overline{L_2p_t}$, which is equal to f, the equation of the plane of the tilted photograph is derived as follows:

$$x \cos \alpha_{p_{t_2}} + y \cos \beta_{p_{t_2}} + z \cos \gamma_{p_{t_2}} - f = 0 \tag{8}$$

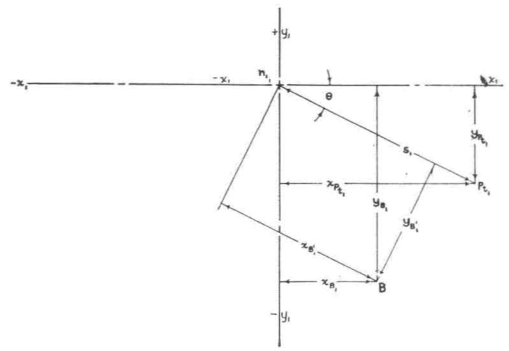

FIG. 4. The apparent flight line on photograph no. 1.

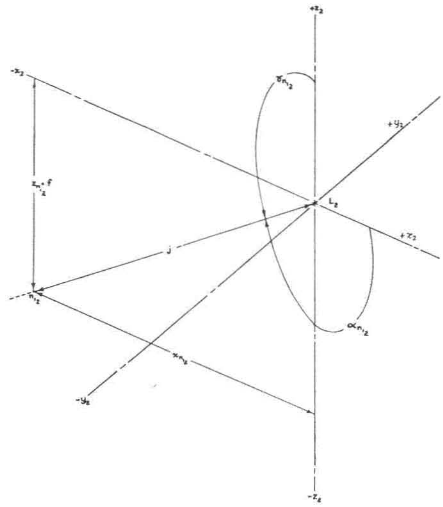

FIG. 5. Determination of the coordinates of n_{1_2}.

The ground coordinates of p_t are computed as follows:

$$X_{p_t} = \left(\frac{x_{p_{t_2}} H}{f}\right) + \overline{N_1 N_2} \tag{9}$$

$$Y_{p_t} = \frac{y_{p_{t_2}} H}{f} \tag{10}$$

From these ground coordinates the coordinates of p_t on photograph no. 1 are computed:

$$x_{p_{t_1}} = \frac{f X_{p_t}}{H} \tag{11}$$

$$y_{p_{t_1}} = \frac{f Y_{p_t}}{H} \tag{12}$$

The apparent flight line on photograph no. 1 now runs from n_{1_1} to p_{t_1} (see Figure 4). Its length is equal to s_1:

$$s_1 = \sqrt{x^2_{p_{t_1}} + y^2_{p_{t_1}}} \tag{13}$$

The coordinates of the base of the object on photograph no. 1, $x_{P_1}{}'$ and $y_{P_1}{}'$, using the coordinate system where the line $\overline{n_{1_1} p_{t_1}}$ forms the positive x axis, are computed by simultaneously solving the following two equations:

$$x_{B_1} = x_{B_1'} \cos\theta - y_{B_1'} \sin\theta \tag{14}$$

$$y_{B_1} = x_{B_1'} \sin\theta + y_{B_1'} \cos\theta \tag{15}$$

where:

$$x_{B_1} = \frac{f}{H} X_B \tag{16}$$

$$y_{B_1} = \frac{f}{H} Y_B \tag{17}$$

$$\theta = \arctan y_{p_{t_1}}/x_{p_{t_1}} \tag{18}$$

The coordinates of the top of the object, x_{T_1}', are obtained in the same manner:

$$x_{T_1} = x_{T_1'} \cos\theta - y_{T_1'} \sin\theta \tag{19}$$

$$y_{T_1} = x_{T_1'} \sin\theta + y_{T_1'} \cos\theta \tag{20}$$

where:

$$x_{T_1} = \frac{f}{H-h} X_T \tag{21}$$

$$y_{T_1} = \frac{f}{H-h} Y_T \tag{22}$$

The coordinates of the nadir of photograph no. 1, within the coordinate system of the untilted photograph no. 2, are computed as follows (see Figure 5):

$$x_{n_{1_2}} = \frac{f}{H}(X_{N_1} - \overline{N_1N_2}), \quad \text{but,} \quad \text{since } X_{N_1} = 0,$$

$$= \frac{f}{H}(-\overline{N_1N_2}) \tag{23}$$

$$y_{n_{1_2}} = 0, \qquad \text{since } n_{1_2} \text{ is on the } x \text{ axis} \tag{24}$$

$$z_{n_{1_2}} = -f \tag{25}$$

The direction cosines of the line connecting the exposure station of photograph no. 2, L_2, to the ground nadir-point of photograph no. 1, N_1, (see Figure 5) are:

$$\cos\alpha_{n_{1_2}} = x_{n_{1_2}}/j \tag{26}$$

$$\cos\beta_{n_{1_2}} = y_{n_{1_2}}/j; \quad \text{however,} \quad \text{since } y_{n_{1_2}} = 0,$$

$$= 0 \tag{27}$$

$$\cos\gamma_{n_{1_2}} = z_{n_{1_2}}/j \tag{28}$$

where:

$$j = \sqrt{x^2_{n_{1_2}} + z^2_{n_{1_2}}} \tag{29}$$

These direction cosines are used in the derivation of the equation of the line $\overline{L_2N_1}$:

$$\frac{x}{\cos\alpha_{n_{1_2}}} = \frac{y}{\cos\beta_{n_{1_2}}} = \frac{z}{\cos\gamma_{n_{1_2}}} \tag{30}$$

The coordinates of the intersection of the line $\overline{L_2N_1}$ and the plane of the tilted photograph

$$(x_{n_{1_{t_2}}}, y_{n_{1_{t_2}}}, z_{n_{1_{t_2}}}),$$

in terms of the coordinate system of the untilted photograph no. 2, are computed as follows:

From the equation for $\overline{L_2N_1}$:

$$x = \frac{z \cos \alpha_{n1_2}}{\cos \gamma_{n1_2}} \tag{31}$$

$$y = \frac{z \cos \beta_{n1_2}}{\cos \gamma_{n1_2}}, \quad \text{but, since } \cos \beta_{n1_2} = 0,$$

$$= 0 \tag{32}$$

$$z = \frac{x \cos \gamma_{n1_2}}{\cos \alpha_{n1_2}} \tag{33}$$

Substituting these values in the equation for the plane of the tilted photograph (8):

$$x \cos \alpha_{p_{t_2}} + \left(\frac{x \cos \gamma_{n1_2} \cos \gamma_{p_{t_2}}}{\cos \alpha_{n1_2}}\right) - f = 0 \tag{34}$$

$$\left(\frac{z \cos \alpha_{n1_2} \cos \alpha_{p_{t_2}}}{\cos \gamma_{n1_2}}\right) + z \cos \gamma_{p_{t_2}} - f = 0 \tag{35}$$

These equations are solved as follows:

$$x_{n1_{t_2}} = f \Big/ \left[\cos \alpha_{p_{t_2}} + \left(\frac{\cos \gamma_{n1_2} \cos \gamma_{p_{t_2}}}{\cos \alpha_{n1_2}}\right)\right] \tag{36}$$

$$z_{n1_{t_2}} = f \Big/ \left[\left(\frac{\cos \alpha_{n1_2} \cos \alpha_{p_{t_2}}}{\cos \gamma_{n1_2}}\right) + \cos \gamma_{p_{t_2}}\right] \tag{37}$$

$y_{n1_{t_2}}$ remains equal to zero.

The apparent flight-line on photograph no. 2 now runs from p_t to n_{1_t} (see Figure 6). It forms the $-x$ axis of the tilted photograph. This is the coordinate system that would be used if the photographs were being applied for height determinations in the usual manner. The length of this line is equal to:

$$\overline{p_t n_{1_t}} = s_2 = \sqrt{(x_{p_{t_2}} - x_{n1_{t_2}})^2 + (y_{p_{t_2}} - y_{n1_{t_2}})^2 + (z_{p_{t_2}} - z_{n1_{t_2}})^2} \tag{38}$$

The stereobase that is often used in the parallax formula is the mean of the principal-point-conjugate principal-point distances on the two photographs:

$$s = \frac{s_1 + s_2}{2} \tag{39}$$

The direction cosines of the line connecting the exposure station of photograph no. 2, L_2, and the base of the object, B, (see Figure 7) are:

$$\cos \alpha_{B_2} = x_{B_2}/k \tag{40}$$

$$\cos \beta_{B_2} = y_{B_2}/k \tag{41}$$

$$\cos \gamma_{B_2} = z_{B_2}/k \tag{42}$$

where:

$$k = \sqrt{x_{B_2}^2 + y_{B_2}^2 + z_{B_2}^2} \tag{43}$$

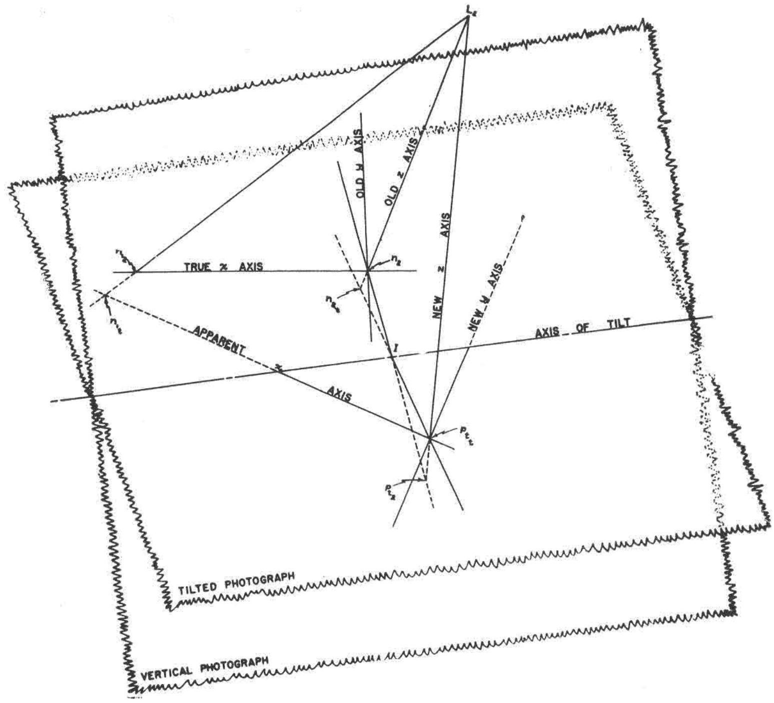

FIG. 6. The transformation of the coordinate systems on the tilted photograph.

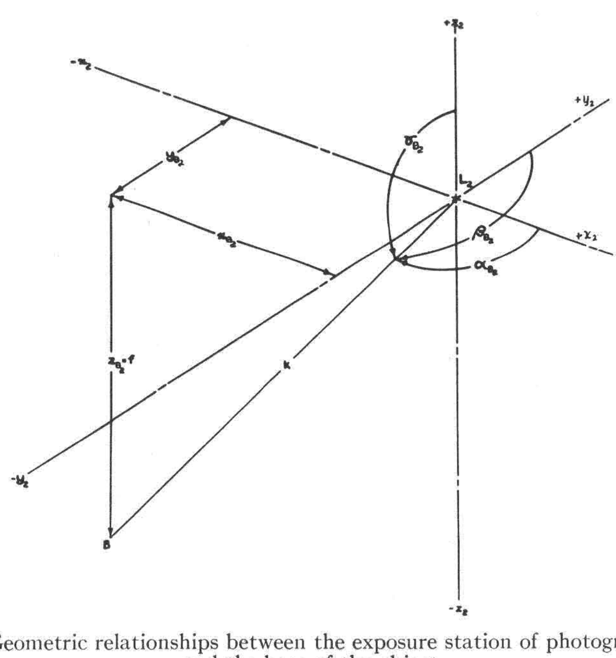

FIG. 7. Geometric relationships between the exposure station of photograph no. 2 and the base of the object.

$$x_{B_2} = \frac{f}{H}(X_B - \overline{N_1N_2}) \tag{44}$$

$$y_{B_2} = \frac{f}{H} Y_B \tag{45}$$

$$z_{B_2} = -f \tag{46}$$

The equation of line L_2B is:

$$\frac{x}{\cos \alpha_{B_2}} = \frac{y}{\cos \beta_{B_2}} = \frac{z}{\cos \gamma_{B_2}} \tag{47}$$

The space coordinates of the intersection of the line $\overline{L_2B}$ and the plane of the tilted photograph, in terms of the coordinate system of the untilted photograph no. 2, are computed as follows:

From the equation of $\overline{L_2B}$:

$$y = \frac{x \cos \beta_{B_2}}{\cos \alpha_{B_2}} \tag{48}$$

$$z = \frac{x \cos \gamma_{B_2}}{\cos \alpha_{B_2}} \tag{49}$$

Substituting these values in the equation of the plane of the tilted photograph:

$$x \cos \alpha_{p_{t_2}} + \left(\frac{x \cos \beta_{B_2} \cos \beta_{p_{t_2}}}{\cos \alpha_{B_2}}\right) + \left(\frac{x \cos \gamma_{B_2} \cos \gamma_{p_{t_2}}}{\cos \alpha_{B_2}}\right) - f = 0 \tag{50}$$

Solving this equation for x, yields $x_{B_{2_t}}$:

$$x_{B_{2_t}} = f \Big/ \left[\cos \alpha_{p_{t_2}} + \left(\frac{\cos \beta_{B_2} \cos \beta_{p_{t_2}}}{\cos \alpha_{B_2}}\right) + \left(\frac{\cos \gamma_{B_2} \cos \gamma_{p_{t_2}}}{\cos \alpha_{B_2}}\right)\right] \tag{51}$$

Following this pattern, but making the necessary substitutions, steps (48) through (51) are repeated in order to find

$$y_{B_{2t}} \quad \text{and} \quad z_{B_{2t}}.$$

The entire sequence from (40) through (51) is then repeated to yield the space coordinates of the top of the object,

$$x_{T_{2t}} \qquad y_{T_{2t}} \quad \text{and} \quad z_{T_{2t}}.$$

In the computations for the coordinates of the top of the object, (*H-h*) is substituted for *H*.

It is now necessary to convert the coordinates of p_t, n_{1_t}, B_1', and T_1', based on the coordinate system of the untilted photograph no. 2, to coordinates on the tilted photograph. In this case p_t has the coordinates $(0, 0, -f)$ and the $-x$ axis extends from p_t to n_{1_t} (see Figure 6). The coordinate axes of the untilted photograph must be rotated and translated to the new positions. To do this it is necessary to determine the direction cosines of the three new axes in terms of the old. Since we are primarily interested in x coordinates because of their importance in computing parallax, the following computation is limited to changes in the x coordinate values.

The direction cosines of $\overline{p_{t_2}n_{1_t}}$ are:

$$\cos \alpha_\mu = \frac{x_{p_{t_2}} - x_{n1_{t_2}}}{s_2} \quad (52)$$

$$\cos \beta_\mu = \frac{y_{p_{t_2}} - y_{n1_{t_2}}}{s_2} \quad (53)$$

$$\cos \gamma_\mu = \frac{z_{p_{t_2}} - z_{n1_{t_2}}}{s_2} \quad (54)$$

The x coordinate of B_t, in terms of the coordinate system of the tilted photograph is:

$$x_{B_t} = x_{B_{2t}} \cos \alpha_\mu + y_{B_{2t}} \cos \beta_\mu + z_{B_{2t}} \cos \gamma_\mu \quad (55)$$

This operation is repeated, using

$$x_{T_{2t}}, \quad y_{T_{2t}} \quad \text{and} \quad z_{T_{2t}},$$

in order to obtain the x coordinate of the top of the object, x_{T_t}.

The absolute parallaxes of the base and top of the object, using the coordinates on the tilted photograph, are:

$$AP_{B_t} = x_{B_1'} - x_{B_t} \quad (56)$$

$$AP_{T_t} = x_{T_1'} - x_{T_t} \quad (57)$$

The differential parallax is:

$$dP_t = AP_{T_t} - AP_{B_t} \quad (58)$$

The height of the tree is then computed using the parallax formula:

$$h_t = \frac{HdP_t}{AP_{T_t}} \quad (59)$$

The difference between the true height, h, and the height obtained from the tilted photograph, h_t, is termed the "error," ϵ_1:

$$\epsilon_1 = h_t - h \quad (60)$$

A standard procedure among foresters is to use the average stereobase, s (39), plus the differential parallax, in place of the absolute parallax of the top of the object, as the denominator in the parallax formula:

$$h_{t_s} = \frac{HdP}{s + dP} \quad (61)$$

The error using this formula is:

$$\epsilon_2 = h_{t_s} - h \quad (62)$$

When actual values are assigned to the several variables appearing in these equations a pattern of errors emerges. In the course of this study, the ϵ_1, and ϵ_2 values were computed for all the combinations of a series of arbitrarily selected values for the variables. These values were as follows:

1. The scale of the photography was set at 1:15,840.
2. The nadir-point of the second photograph, N_2, was assigned the ground-coordinates $X_{N_2} = 7{,}100.0$ feet and $Y_{N_2} = 0.0$ feet.
3. Three object positions were used, as is shown in Figure 8.

4. The height of the object, h, was set at 100.0 feet.
5. Three focal-lengths were selected. They were 4.00 inches, 8.25 inches, and 24.00 inches.
6. The tilts, τ, were set at 0°05′, 0°15′, 0°30′, 1°00′, 5°00′, and 10°00′.
7. Directions of tilt, Δ, were set at 30° intervals all the way around the circle.

These computations resulted in a very large number of values, too large for inclusion in this paper. However, representative values are shown in Tables 1 through 3 and Figures 9 through 13. These tables and figures will be used to illustrate the items discussed in the remainder of the report.

Due to several factors actively influencing the magnitude of the errors, it is difficult to describe precisely the effect of individual variables. However, the following statements may be made:

1. As the tilt is made larger the error is increased. This is evident from Figures 9 through 13. In every case, when all factors except tilt were held constant, the error was increased with the larger tilts.
2. As the focal-length is increased the error is made larger. This, too, is substantiated from the evidence of Figures 9 through 13.
3. The direction of tilt influences the magnitude and sign of the error. This effect is caused by the influence of the direction of tilt on the absolute parallax values, AP_{B_t} and AP_{T_t}. For example, when the direction of tilt is 0°, the camera axis is swung forward, in the plane defined by the flight-line and the plumb-lines from the exposure stations, away from the companion photograph in the stereopair. This results in a lengthening of the X_{B_r} and the X_{T_t} coordinates. This can be seen in Figure 14. The lengthening of the X_{B_t} and X_{T_t} coordinates results in a corresponding increase in the absolute parallax values. There is some relative change between X_{B_t} and X_{T_t} but, with tilts up to 10°, this is so small that for the purpose of this discussion, it can be assumed that there is no change in the differential parallax, dP. Thus, when the data from

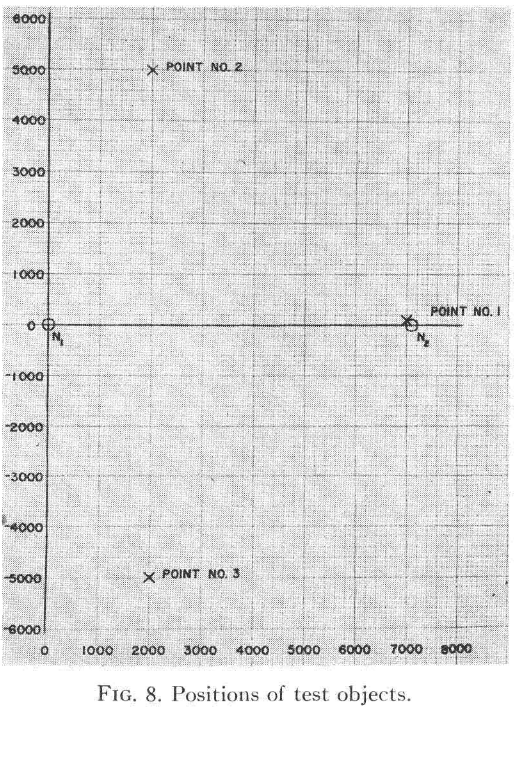

FIG. 8. Positions of test objects.

TABLE 1

VALUES OF ϵ_1 AND ϵ_2, FOR A 100 FOOT OBJECT, WHEN X_B=7,000 FEET AND Y_B=100 FEET

Δ=	ϵ_1 (*in feet*)				ϵ_2 (*in feet*)			
	0°	*90°*	*180°*	*270°*	*0°*	*90°*	*180°*	*270°*
τ=				f=4.00 inches				
0°05′	−0.11	0.00	0.11	0.00	−0.20	0.00	0.20	0.00
0°15′	−0.32	0.00	0.32	0.00	−0.61	0.00	0.61	0.00
0°30′	−0.63	0.00	0.64	0.00	−1.21	−0.01	1.22	−0.01
1°00′	−1.26	−0.02	1.29	−0.02	−2.41	−0.03	2.44	−0.02
5°00′	−6.00	−0.42	6.81	−0.41	−11.75	−0.60	12.61	−0.59
10°00′	−11.39	−1.64	14.75	−1.63	−23.04	−2.35	26.51	−2.31
				f=8.25 inches				
0°05′	−0.22	0.00	0.22	0.00	−0.27	0.00	0.27	0.00
0°15′	−0.66	0.00	0.67	0.00	−0.80	−0.01	0.81	0.00
0°30′	−1.31	−0.02	1.34	−0.02	−1.59	−0.02	1.63	−0.02
1°00′	−2.58	−0.07	2.72	−0.07	−3.13	−0.08	3.30	−0.08
5°00′	−11.73	−1.75	15.33	−1.74	−14.27	−1.93	18.47	−1.92
10°00′	−21.12	−6.67	36.57	−6.65	−25.76	−7.27	43.56	−7.24
				f=24.00 inches				
0°05′	−0.64	0.00	0.00	0.66	−0.66	0.00	0.67	0.00
0°15′	−1.90	−0.04	1.98	−0.04	−1.95	−0.04	2.03	−0.04
0°30′	−3.74	−0.15	4.04	−0.15	−3.83	−0.15	4.14	−0.15
1°00′	−7.20	−0.60	8.41	−0.60	−7.38	−0.61	8.63	−0.61
5°00′	−28.01	−13.15	63.67	−13.14	−28.67	−13.27	65.08	−13.25
10°00′	−43.94	−37.79	362.53	−37.78	−44.87	−37.92	359.37	−37.87

the tilted photograph are entered into the parallax formula, it can be seen that the numerator remains virtually unchanged from what it would be if the photograph had been truly vertical, while the denominator becomes larger. This results in a height value that is too small.

TABLE 2

VALUES OF ϵ_1 AND ϵ_2, FOR A 100 FOOT OBJECT, WHEN X_B=2,000 FEET AND Y_B=−5,000 FEET

Δ=	ϵ_1 (*in feet*)				ϵ_2 (*in feet*)			
	0°	*90°*	*180°*	*270°*	*0°*	*90°*	*180°*	*270°*
τ=				f=4.00 inches				
0°05′	0.01	−0.10	−0.01	0.10	0.00	−0.20	0.00	0.20
0°15′	0.01	−0.30	−0.02	0.30	0.00	−0.59	−0.01	0.58
0°30′	0.02	−0.60	−0.04	0.59	0.00	−1.19	−0.02	1.16
1°00′	0.02	−1.21	−0.11	1.16	−0.02	−2.43	−0.07	2.28
5°00′	−0.81	−6.64	−1.35	5.43	−0.94	−13.94	−1.15	10.15
10°00′	−4.23	−15.19	−4.69	10.12	−4.20	−33.96	−4.29	17.91
				f=8.25 inches				
0°05′	0.17	−0.05	−0.17	0.05	0.17	−0.09	−0.17	0.10
0°15′	0.51	−0.14	−0.52	0.15	0.51	−0.28	−0.52	0.29
0°30′	1.01	−0.27	−1.05	0.30	1.00	−0.57	−1.04	0.58
1°00′	1.99	−0.53	−2.14	0.63	1.97	−1.12	−2.13	1.17
5°00′	8.61	−1.69	−12.51	4.12	8.42	−5.21	−12.51	6.36
10°00′	14.38	−1.46	−30.94	10.47	13.84	−9.93	31.20	13.93
				f=24.00 inches				
0°05′	0.63	−0.01	−0.64	0.02	0.63	−0.03	−0.64	0.04
0°15′	1.85	−0.01	−1.93	0.09	1.85	−0.06	−1.93	0.13
0°30′	3.63	0.05	−3.94	0.25	3.63	−0.05	−3.93	0.34
1°00′	7.00	0.39	−8.21	0.78	6.99	0.17	−8.21	0.96
5°00′	26.95	11.73	−62.54	13.72	26.81	10.42	−62.69	14.25
10°00′	41.67	34.05	−350.86	38.31	41.25	31.52	−351.03	38.83

TABLE 3

VALUES OF ϵ_1 AND ϵ_2, FOR A 100 FOOT OBJECT, WHEN $X_B=2{,}000$ FEET AND $Y_B=5{,}000$ FEET

Δ=	ϵ_1 (*in feet*)				ϵ_2 (*in feet*)			
	0°	*90°*	*180°*	*270°*	*0°*	*90°*	*180°*	*270°*
τ=				$f=4.00$ inches				
0°05′	0.00	0.10	−0.01	−0.10	0.00	0.20	0.00	−0.20
0°15′	0.01	0.30	−0.02	−0.30	0.00	0.58	−0.01	−0.59
0°30′	0.02	0.59	−0.04	−0.60	0.00	1.16	−0.02	−1.19
1°00′	0.02	1.16	−0.10	−1.21	−0.02	2.28	−0.07	−2.43
5°00′	0.81	5.43	−1.34	−6.64	−0.94	10.15	−1.14	−13.94
10°00′	−4.23	10.12	−4.69	−15.19	−4.20	17.91	−4.27	−33.96
				$f=8.25$ inches				
0°05′	0.17	0.05	−0.17	−0.05	0.17	0.10	−0.17	−0.10
0°15′	0.51	0.15	−0.52	−0.14	0.51	0.29	−0.52	−0.28
0°30′	1.01	0.30	−1.05	−0.27	1.00	0.58	−1.04	−0.57
1°00′	1.99	0.63	−2.14	−0.53	1.97	1.17	−2.13	−1.12
5°00′	8.61	4.12	−12.51	−1.69	8.42	6.36	−12.52	−5.21
10°00′	14.38	10.47	−30.94	−1.47	13.84	13.93	−31.19	−9.94
				$f=24.00$ inches				
0°05′	0.63	0.02	−0.63	−0.01	0.63	0.04	−0.63	−0.03
0°15′	1.85	0.09	−1.93	−0.01	1.85	0.13	−1.93	−0.06
0°30′	3.63	0.26	−3.94	0.05	3.63	0.34	−3.93	−0.05
1°00′	7.00	0.79	−8.21	0.39	6.99	0.96	−8.21	0.17
5°00′	26.95	13.73	−62.53	11.72	26.81	14.25	−62.68	10.41
10°00′	41.67	38.32	−351.29	34.04	41.25	38.83	−350.97	31.52

When the direction of tilt is reversed, to 180°, the camera axis is swung backward toward the companion photograph. This results in a shortening of X_{B_t} and X_{T_t} and a corresponding decrease in the absolute parallax values. Again there is virtually no change in the dP value. When these data are en-

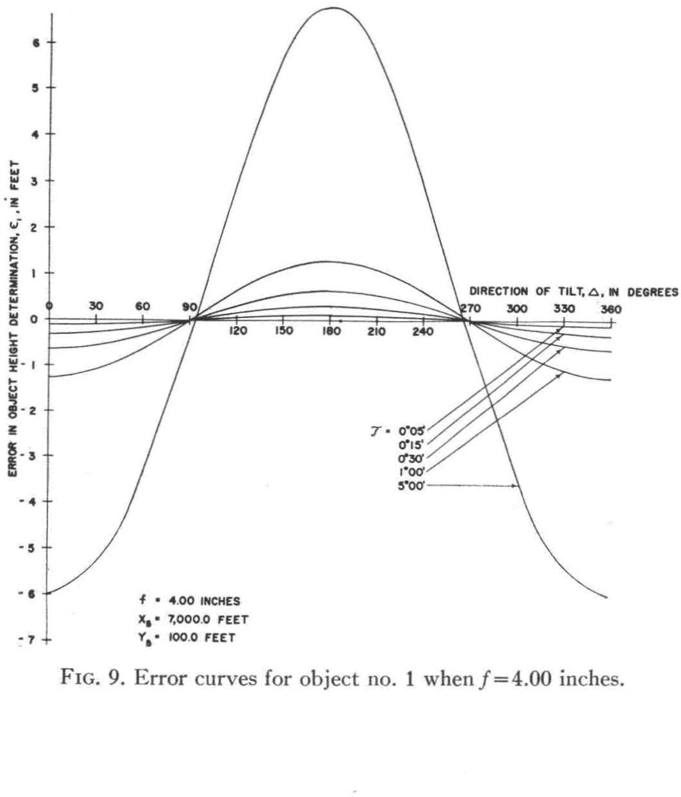

FIG. 9. Error curves for object no. 1 when $f=4.00$ inches.

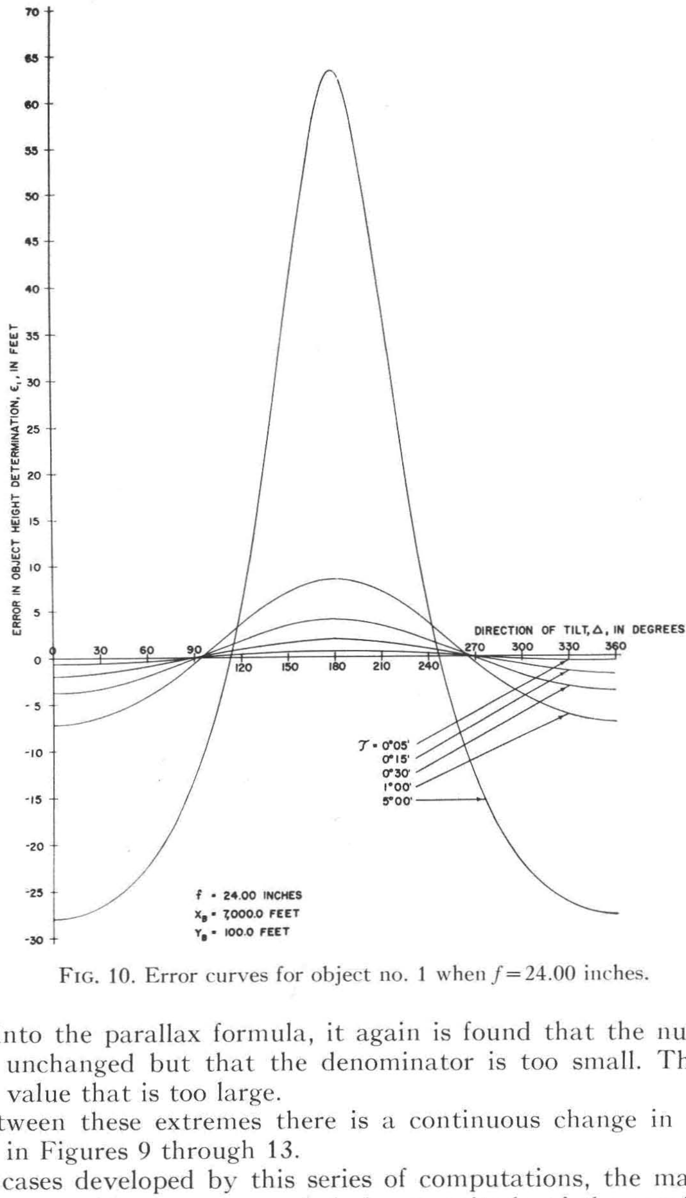

FIG. 10. Error curves for object no. 1 when $f=24.00$ inches.

tered into the parallax formula, it again is found that the numerator is virtually unchanged but that the denominator is too small. This results in a height value that is too large.

Between these extremes there is a continuous change in the error as is shown in Figures 9 through 13.

4. In all cases developed by this series of computations, the magnitude of the maximum positive error exceeded the magnitude of the maximum negative error. This is due to the parallax formula being a ratio. All ratios have the characteristic that their magnitudes are inversely proportional to the magnitude of their denominators. This proportionality, however, is not a linear function. Instead it is hyperbolic. In other words, a decrease in the magnitude of the denominator will have a greater proportional effect on the value of the ratio than an increase of the same magnitude. For example, if, in the parallax formula, $H-5{,}000$ feet, $dP=0.100$ inches, and AP_{T_2} (the absolute parallax of the top of the object on a truly vertical photograph) $=5.00$ inches, the height of the object is equal to 100.0 feet:

$$h = \frac{HdP}{AP_T} = \frac{5000(0.100)}{5.00} = 100.0 \text{ feet}$$

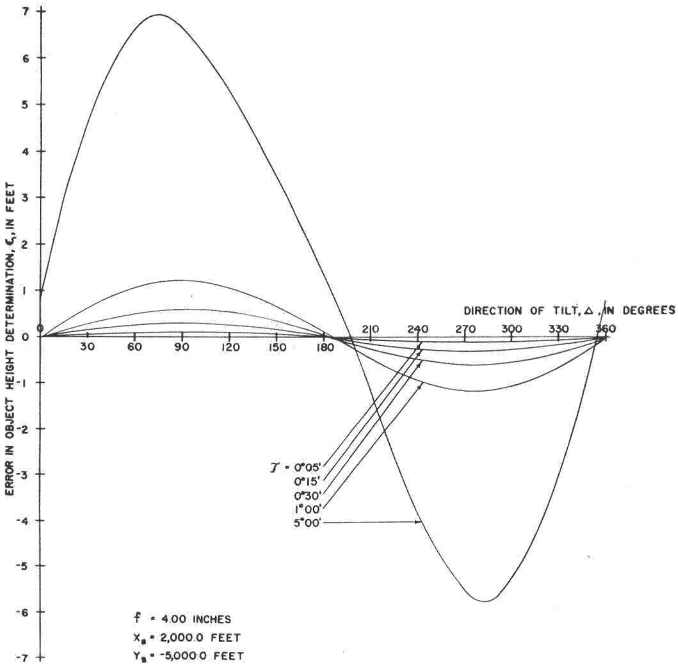

FIG. 11. Error curves for object no. 3 when $f=4.00$ inches.

If, due to tilt, the absolute parallax is reduced by 0.50 inches, the apparent height of the object would be:

$$h_t = \frac{HdP}{AP_{T_t}} = \frac{5000(0.100)}{4.50} = 111.1 \text{ feet}$$

and the error in height measurement is:

$$\epsilon = h_t - h = 111.1 - 100.0 = +\ 11.1 \text{ feet.}$$

If, on the other hand, the absolute parallax is increased by 0.50 inches, the apparent height of the object would be:

$$h_t = \frac{HdP}{AP_{T_t}} = \frac{5000(0.100)}{5.50} = 90.9 \text{ feet}$$

and the error is:

$$\epsilon = h_t - h = 90.9 - 100.0 = -\ 9.1 \text{ feet.}$$

Thus, when the tilt is in such a direction that the absolute parallax of the top of the object is increased, the resulting height determination will be too low. When the tilt is such that the absolute parallax is decreased the height determination will be too high. However, a decrease of a given magnitude has a greater absolute effect than an increase of the same magnitude. Consequently, the maximum positive errors run higher than the maximum negative errors.

5. The location of the object being measured has some effect on the pattern of errors. The argument presented under Item 3 above is based on the assumption that the object lies in the plane defined by the flight-line and the plumb-

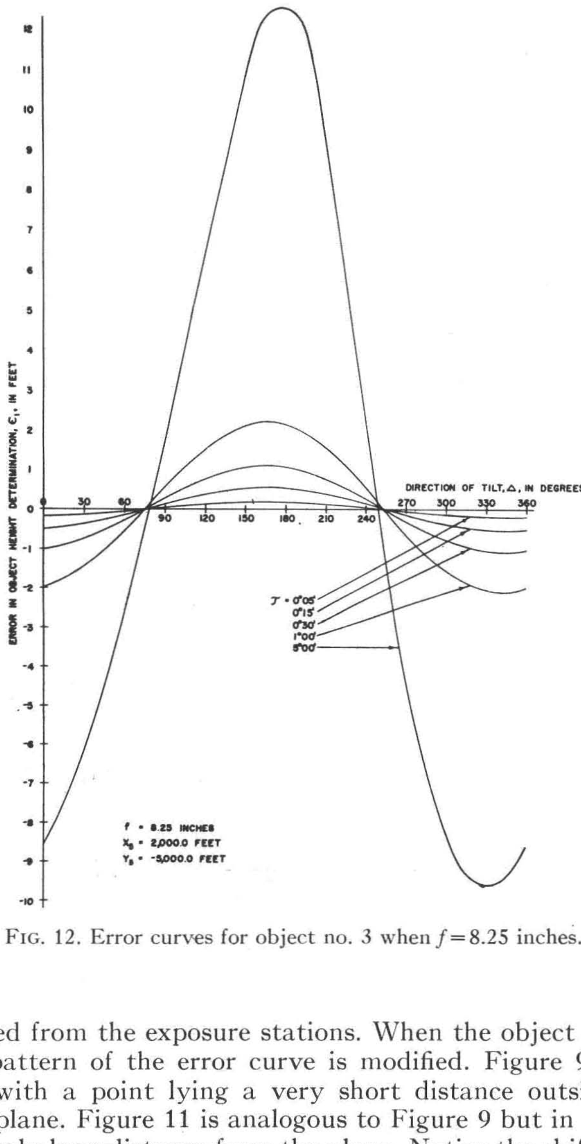

FIG. 12. Error curves for object no. 3 when $f=8.25$ inches.

lines dropped from the exposure stations. When the object lies outside of this plane the pattern of the error curve is modified. Figure 9 shows the errors associated with a point lying a very short distance outside the previously mentioned plane. Figure 11 is analogous to Figure 9 but in this case the point lies a relatively long distance from the plane. Notice the shift in the curve patterns. Figure 9 reveals a pattern that closely follows that developed theoretically in Item 3. The maximum positive error occurred when the tilt had a direction of approximately 180°, while the maximum negative error occurred when the tilt had an approximate direction of 0°. In contrast, the curves on Figure 11 do not follow this pattern. Here the maximum positive error occurred when the direction of tilt was transverse to the flight-line and in the direction away from the object.

This can be explained in much the same way as in Item 3 above. It must be remembered, however, that several factors are interacting to produce the computed results; consequently, the following statement may be oversimplified.

The evidence indicates that with very short focal-lengths the absolute parallax values are at a maximum when the tilt is directly away from the object. As was explained in Item 3, when this is the case, the height values are

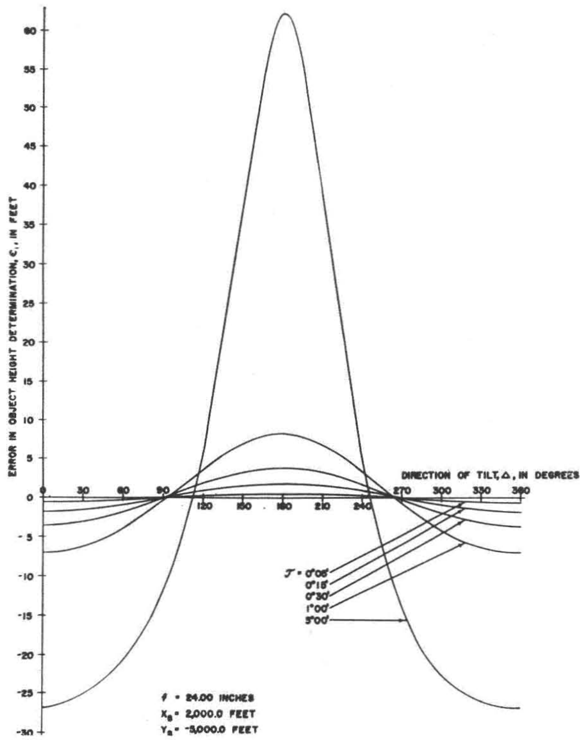

FIG. 13. Error curves for object no. 3 when $f = 24.00$ inches.

at a minimum. Conversely, when the tilt is directly toward the object, the absolute parallax values are at a minimum and the height values are at their maximum.

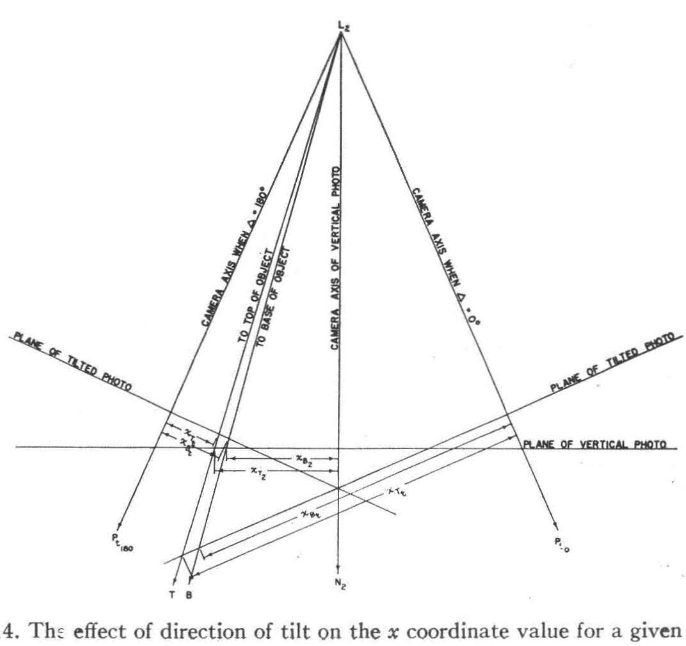

FIG. 14. The effect of direction of tilt on the x coordinate value for a given point.

6. As the focal-length is increased the effect of object location on the error curve pattern is decreased. This was implied in the preceding paragraph. The influence of focal-length can be seen if one compares Figures 11, 12, and 13. In Figure 11, where the focal-length equals 4.00 inches, the maximum positive error occurred when the direction of tilt was about 75°. In Figure 13, where the focal-length equals 24.00 inches, the maximum positive error occurred when the direction of tilt was approximately 180°. In Figure 12, where the focal-length equals 8.25 inches, an intermediate situation exists. Note that as the focal-length is increased, there was a relatively rapid shift in the curve pattern until at 24.00 inches there was little difference between that associated with Point 3 and that associated with Point 1, which is shown in Figure 10.

   This phenomenon appears to be due to the fact that as the focal-length is increased the ratio of the distance, $y_{B'}$ to the focal-length, f, becomes smaller. In other words, as the focal-length is increased the object comes relatively closer to the flight-line and the conditions described in Item 3 are more nearly met.
7. The use of the short cut method (average stereobase method) for computing object heights has a variable effect on the magnitude of the errors. In general, regardless of the position of the object being measured, as the focal-length is increased the magnitude of the error caused by the short-cut method is lessened. An examination of Tables 1, 2, and 3, reveals that with a focal-length of 4 inches, the errors resulting from the use of the short-cut method, ϵ_2, are about twice as large as those resulting from using the classical method. However, when the focal-length is increased to 8.25 inches the difference between the errors is reduced to about 20 percent of the error obtained by the classical approach. When the focal-length is increased to 24 inches the difference between the two groups of errors is negligible. It is not known if this is in the nature of a limit or if a further increase in the focal-length would result in a corresponding further change in the error resulting from the short-cut method. The author is inclined toward the belief that it is a limit, and that beyond a focal-length of 24 inches the method of computation would make no difference.

From these observations several conclusions may be drawn:

1. The earlier statements that tilt has little effect on the determination of spot heights, because there is little or no lateral distance between the upper and lower points, are essentially erroneous. The reason for this thinking appears to be that the earlier investigators considered only the effect of tilt on *differential parallax*. They indicated that the effect was slight. This is true. However, they neglected to consider the effect of tilt on the *absolute parallax* and here the effect may be very great.
2. The use of long focal-length lenses, in order to obtain large scales, is not desirable when spot heights are to be determined because long focal-lengths magnify errors caused by tilt.
3. The use of the average stereobase method of computing object heights is not desirable when the photographs have been taken with short or medium focal-length cameras.

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