The Conditions Necessary for Plotting Stereo Models in which the Set Principal Distance Is not the Same as that of the Taking Camera

J. A. EDEN,
*Directorate of Overseas Surveys**

1. Plotting Machines for Super Wide-Angle

There is considerable interest now in the new Swiss Super wide-angle cameras of 88 mm. focal-length, but before these can be used for the purpose of mapping relief, organizations must possess suitable plotting machines and that will involve further capital outlay.

Before going to the expense of purchasing new plotting equipment, most organizations who are in possession of plotting equipment suitable for plotting 6 inch wide-angle photographs might like first to know whether or not it would be possible to modify their existing equipment for the purpose of plotting Super wide-angle. This paper goes into the question of what modifications would be necessary to plotting machines if they are to be used in circumstances where the principal distances set are grossly different from that of the taking camera. The conclusions are that though these modifications might be made, they would be considerably complicated, and the expense in making the modifications would undoubtedly be greater than the cost of purchasing new plotters.

The paper examines the conditions necessary for plotting photographs in this manner, and shows that in the presence of tilts in the order normally experienced in near-vertical photography or of any larger size, the condition would necessitate the following additional devices on the plotting machine.

1. An arrangement whereby the principal points of the photographs are automatically decentered in a fixed relation to the set tilts.
2. An arrangement whereby each projector is moved in a horizontal plane by a fixed relation dependent on the tilts set and the depth of the model surface below the projection centres.

Although it would be possible to build a plotting machine on these lines, there would appear to be no simple way by which existing plotting machines could easily be modified to work in this way.

It would seem therefore that there is no means of avoiding the expenditure of purchasing new plotting machines if contouring is to be undertaken with unrectified super wide-angle photographs. In this circumstance it would be more suitable to purchase available plotting machines designed to conserve the geometry of the taking camera.

A possible alternative is to use rectified diapositives in plotting machines already in the possession of most departments, but this would involve carrying out the aerial triangulation in a comparator, so as first to obtain the tilts with which to make the rectifications.

2. Conditions in the Absence of Tilts

Figure 1 shows the taking conditions in the absence of tilts. It is clear from similar triangles that if the photographs *adb* and *acd*, exposed in a camera with principal distance m are removed and placed in projectors with principal distances M as shown in Figure 1A, then if all other conditions are similar, the vertical scale of the projected model will differ from its horizontal scale by the ratio M/m and this is the only effect.

3. Conditions in the Presence of Tilts

In order to determine the conditions in the presence of tilts, it is necessary first of all to consider the projection of a single photograph.

3/1. Relations between Planes and Centres of Projection on which any Given Projections are Obtained

In Figure 2 let p and p' be two planes on which the perspectives of figures in plane P are identical, and let L and L' be the corresponding perspective centres.

* Kingston Road, Tolworth, Surbiton, Surrey, England.

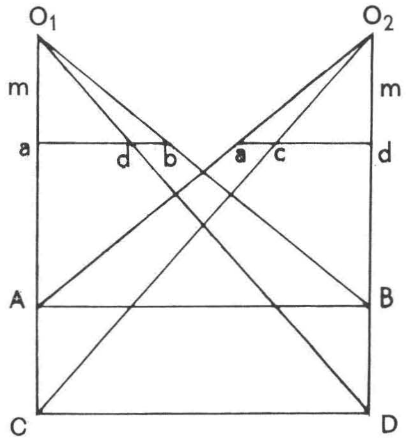

FIG. 1

Let $ABCD$ be a figure in plane P, $abcd$ its perspective in plane p, and $a'b'c'd'$ its perspective in plane p'.

Let XX' be the intersection of planes P and p.

This line is its own perspective and can be considered in two aspects; one as the line XX'

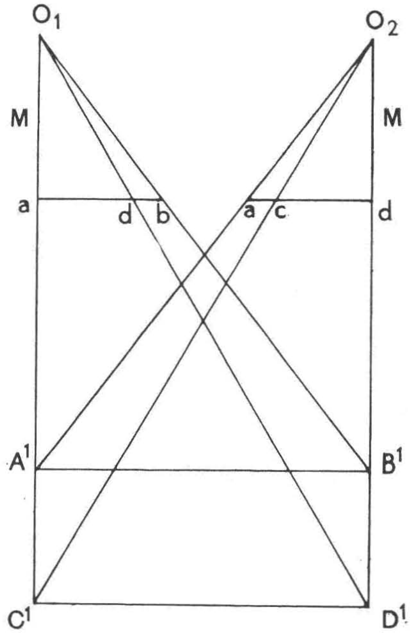

FIG. 1A

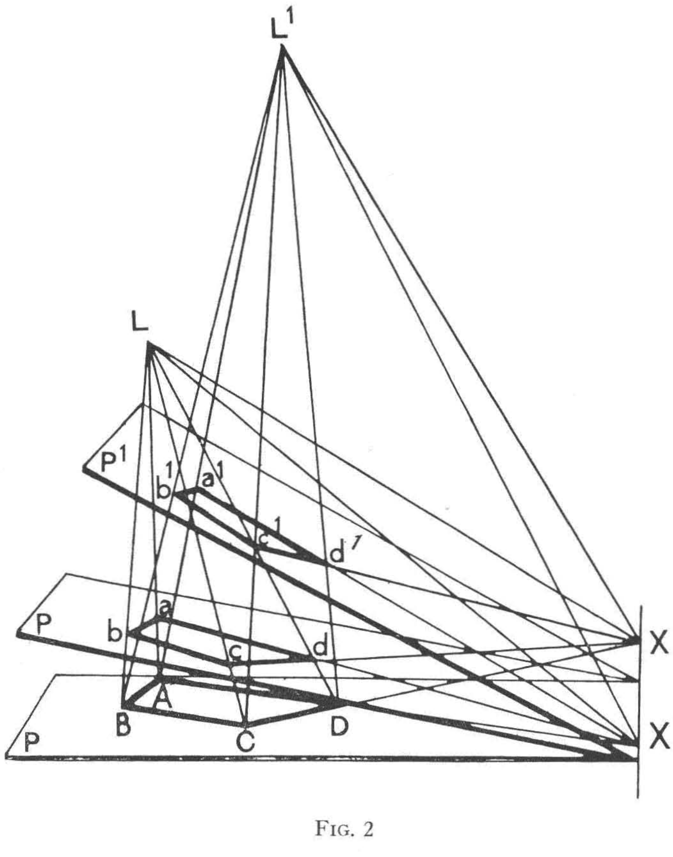

FIG. 2

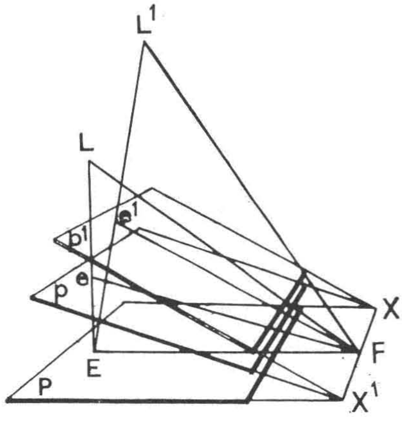

FIG. 3

in plane P, and the other as a line in plane p, which is the perspective in that plane of this line.

If there is considered a range of four points along this line considered as in plane P, the same range can be considered as belonging to plane p. The perspective of this range in plane p' is by hypothesis identical with that in plane p, and its component parts are therefore equal in all respects to the range in plane P. The perspective of the line XX' in plane p' must therefore coincide with that line in plane P or the perspective centre must be at infinity.

Since the perspective centre is not at infinity, these lines must coincide, that is to say, the plane p' must also intersect the plane P along this line.

Produce any two sides, say CD and AD, of the figure $ABCD$ and the corresponding sides of the figure $abcd$ to meet the line XX'.

Since CD and cd are in the same plane, namely the plane LCD, they will meet in some point X, and this point being in the plane of $abcd$ and also of $ABCD$ must lie on the intersection of these planes. The point X must therefore lie on the line XX'. In a similar way it may be shown that the lines $c'd'$ and CD in the planes p' and P also meet in X on the line XX' (which is the intersection of these planes also) and that the lines AD, ad and $a'd'$ meet in one point X' on the same line.

The figure shows that dXX' and $d'XX'$ are the perspectives in planes p and p' respectively of the figure DXX' in plane P. These figures are by hypothesis equal. The points d and d' therefore lie on a circle whose centre is on the line XX' and whose radius is the perpendicular from these points to that line.

These relations between the planes on which identical perspectives are obtained may be summarized thus:—

(1) *All possible perspective planes intersect the plane of origin in the same straight line. This line is called the perspective axis.*

(2) *Homologous lines intersect on the perspective axis.*

(3) *The locus of homologous points is a circle whose centre is on the perspective axis, and whose radius is the perpendicular from the point to the axis.*

There remains to be considered the position of the perspective centre L and L' (see Figure 3). Consider a plane through L perpendicular to the line XX', cutting it at F, cutting the plane P along the line FE and plane p along the line Fe, E and e being homologous points.

XX' being perpendicular to the plane LFE is perpendicular to the lines eF and EF.

Then if e' is the perspective figure of E in plane p', it must lie on the circle of centre F, described by the revolution of the point e round the axis XX'; and since the line Ee' passes through L', the new perspective centre, L' must lie in this plane.

L' must therefore lie in the plane passing through L perpendicular to the line XX'.

In this plane draw LH parallel to plane p (Figure 4), cutting plane P in H.

Then since LH is parallel to plane p, the line LH will cut that plane at infinity. The perspective figure of H in plane p is therefore at infinity, and since the perspectives on planes p and p' are by hypothesis identical, the perspective figure of H in plane p' must also be at infinity.

Therefore $L'H$, the line joining H to the new perspective centre, must be parallel to plane p'.

But since $L'H$ and LH are parallel to planes

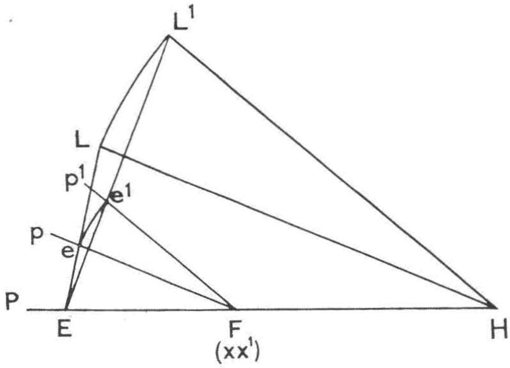

FIG. 4

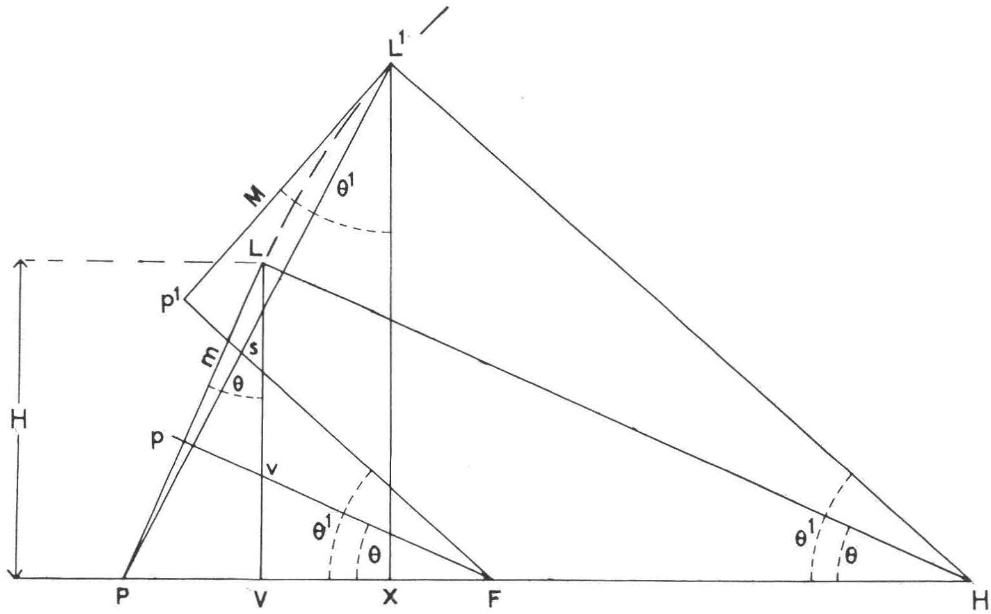

FIG. 5

p' and p respectively, we have:

$$\frac{LH}{eF}=\frac{EH}{EF} \quad \text{and} \quad \frac{L'H}{e'F}=\frac{EH}{EF}$$

(e and e' being homologous points of E in the two planes); but $eF=e'F$, therefore $LH=L'H$.

The relations, therefore, between the perspective centres from which identical projections are obtained and their corresponding planes are:

(1) *All possible perspective centres lie in one plane passing through the original perspective centre and perpendicular to the perspective axis.*

(2) *All perspective centres lie on the circumference of a circle whose centre is the point in the plane of origin whose perspective is at infinity.*

(3) *For any position of the perspective centre, the corresponding perspective plane is parallel to the line joining that centre to the point on the plane of origin whose perspective is at infinity.*

The above theorem is taken directly from "Simple Methods of Plotting from Air Photographs," N. F. I. King, H. M. Stationery Office 1925. It is now proposed to enlarge on this theorem.

3/2. THE RELATIONS BETWEEN THE TILTS OF THE TWO PERSPECTIVE PLANES AND THE DECENTERING OF THE PRINCIPAL POINT.

Consider Figure 5 which is a special case of Figure 4. Let m and M be the principal distances of the two perspectives and p and p' their principal points. Let H be the height of perspective centre L above PFH.

3/2/1. *Determination of* θ'

$$Lv = m/\cos\theta$$

$$v\text{N} = H - m/\cos\theta$$

$$\text{N}F = v\text{N}\cot\theta = \left(H - \frac{m}{\cos\theta}\right)\cot\theta$$

$$\text{N}H = H\cot\theta$$

$$FH = \text{N}H - \text{N}F = (H\cot\theta) - \left(H - \frac{m}{\cos\theta}\right)\cot\theta$$

$$M = FH\sin\theta'$$

$$= \sin\theta'\left((H\cot\theta) - \left(H - \frac{m}{\cos\theta}\right)\cot\theta\right)$$

$$M = \sin\theta'\left((H\cot\theta) - H\cot\theta + \frac{m\cot\theta}{\cos\theta}\right)$$

$$M = \frac{\sin\theta' m\cot\theta}{\cos\theta} = m\frac{\sin\theta'}{\sin\theta}$$

Whence

$$\sin\theta' = \frac{M}{m}\sin\theta \tag{1}$$

3/2/2. *Determination of the Vertical Scale*

In Figure 5 drop a perpendicular from L' to meet PH' in X.

$$\frac{XL'}{LV} = \frac{XL'}{H}$$

is then the vertical scale.

$LH = H\sin\theta = L'H$ by construction.

$$XL' = L'H\sin\theta' = H\frac{\sin\theta'}{\sin\theta}$$

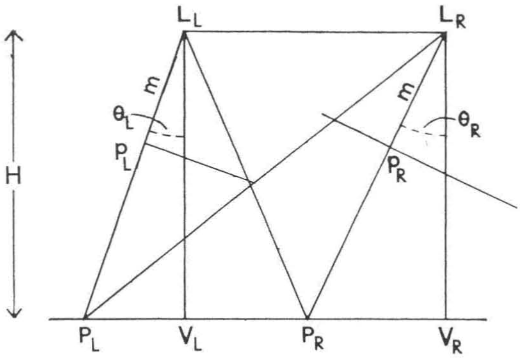

FIG. 6

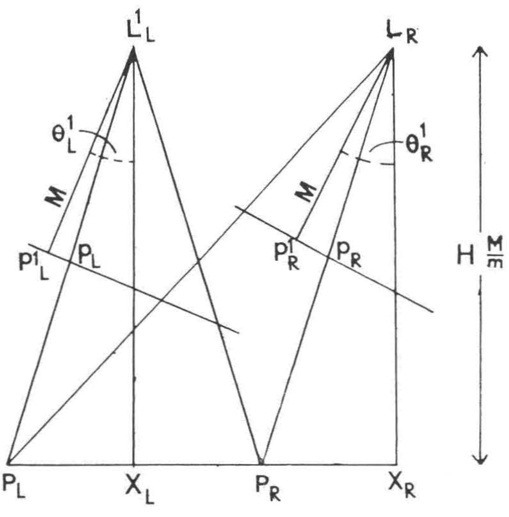

FIG. 6A

$$= H\frac{M}{m} \quad \text{from (1)}$$

Therefore the vertical scale

$$\frac{XL'}{H} = \frac{M}{m} \qquad (2)$$

and this is independent of θ and θ'

3/2/3. *Determination of the Decentering*

In Figure 5 p's is the decentering.

Determine first the value of tan $PL'X$

$$\tan PL'X = \frac{PX}{XL'} = \frac{PH - XH}{XL'} = \frac{Pv + VH - XH}{XL'}$$

$$= \frac{H \tan\theta + H \cot\theta - XL' \cot\theta'}{XL'}$$

and putting $XL' = H(M/m)$ from (2)

$$\tan PL'X = \frac{\tan\theta + \cot\theta - \frac{M}{m}\cot\theta'}{\frac{M}{m}}$$

and using

$$\frac{M}{m} = \frac{\sin\theta'}{\sin\theta} \qquad \text{from (1)}$$

$$\tan PL'X = \frac{1 - \cos\theta\cos\theta'}{\cos\theta\sin\theta'} \qquad (3)$$

The decentering

$$p\text{'s} = M \tan(\theta' - PL'X)$$

$$= M\frac{\tan\theta' - \tan PL'X}{1 + \tan\theta' \tan PL'X}$$

$$\left.\begin{aligned} &= M\frac{(\cos\theta - \cos\theta')}{\sin\theta'} \\ &\text{or} \quad M\frac{(\cos\theta - \cos\theta')}{\sin\theta} \end{aligned}\right\} \qquad (4)$$

3/3. THE FORMATION OF A MODEL FROM A PAIR OF PHOTOGRAPHS

It is now clear that if there are two cameras L_L and L_R both photographing a piece of horizontal ground $P_L P_R$ with the same principal distance m, and with fore and aft tilts θ_L and θ_R as shown in Figure 6, then the two photographs exposed in these cameras may be placed in two projectors of principal distance M (Figure 6a) which will then project an identical model P_L P_R in the following circumstances:—

(1) That the projectors are tilted θ'_L and θ'_R according to formula (1).
(2) That the points p_L and p_R are decentered from the projector principal points p'_L and p'_R by an amount Sp' according to formula (4).
(3) That the projection plane is placed a distance H M/m below L'_L L'_R according to formula (2).
(4) That the projectors L'_L and L'_R are placed a specific distance apart.

It will now be shown that in the presence of relief where the distance H is varied, the positions of the projectors L'_L and L'_R must also be varied.

Consider Figure 7 which shows the left-hand camera photographing two points P_L and P_L' such that P_L' is at a distance dh vertically above P_L.

It is clear that ray L_L P_L misses point P_L' by $P_L'0$.

Where

$$P_{L'}0 = dh \tan\theta_L$$

Now consider the left-hand projector conditions in Figure 7a. In this case it is clear that the ray $L'_L\, P_L$ misses point $P'_L{}'$ by $P'_L{}'0'$

Where

$$P'_L{}'0' = dh \frac{M}{m} \tan \alpha_L$$

$$\alpha_L = PL'X \text{ in Figure 5.}$$

Hence from (3)

$$P'_L{}'0' = dh \frac{M}{m} \left(\frac{1 - \cos\theta_L \cos\theta'_L}{\cos\theta_L \sin\theta'_L} \right)$$

If the projector model is only to differ from the camera model in that its vertical scale is larger by M/m then the distance $P_L{}'0'$ must equal the distance $P_L 0$.

This may only be achieved by moving the projector L'_L bodily a distance

$$P'_{L'}0' - P_{L'}0 = dh \frac{M}{m} \frac{1 - \cos\theta_L \cos\theta'_L}{\cos\theta_L \sin\theta'_L} - dh \tan\theta_L$$

putting

$$\frac{M}{m} = \frac{\sin\theta'_L}{\sin\theta_L}$$

this reduces to

$$dh \frac{1 - \cos\theta_L \cos\theta'_L}{\sin\theta_L \cos\theta_L} - \frac{dh \sin\theta_L}{\cos\theta_L}$$

$$= dh \frac{(1 - \cos\theta_L \cos\theta'_L - \sin^2\theta_L)}{\cos\theta_L \sin\theta_L}$$

$$= dh \frac{\cos\theta_L - \cos\theta'_L}{\sin\theta_L} \qquad (5)$$

which is the tangent of the angular component of the decentering given in (4) multiplied by dh.

3/3/1. *Translation of projectors*

Therefore we have a general case:
Where the model has relief each projector must be moved along in a horizontal direction by

$$x \text{ projector translation} = dh \frac{\cos\theta_L - \cos\theta'_L}{\sin\theta_L} \qquad (5)$$

according to the value of dh.

3/2. THE CASE OF LATERAL TILTS

A glance at Figure 5 will show that the figure applies equally to fore and aft or lateral tilts. Therefore the conditions for lateral tilts are precisely the same as for fore and aft tilts and the principal-point decentrations and horizontal projector-translations must be made laterally with respect to the lateral tilts as they are made in the fore-and-aft direction with respect to fore-and-aft tilts.

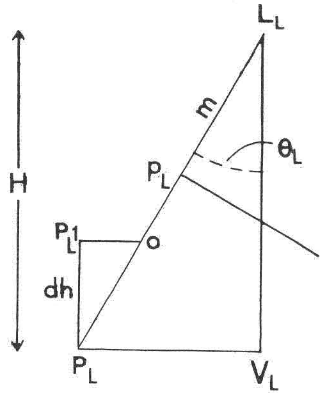

FIG. 7

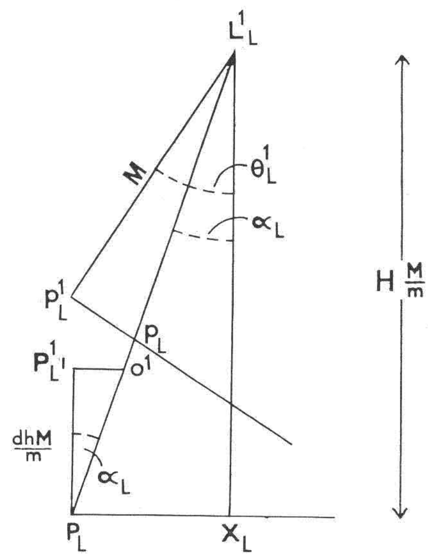

FIG. 7A

4. WORKING CONDITIONS FOR A PLOTTING MACHINE

A plotting machine whose principal-distances M differ from the principal distance m of the camera would need to be provided with an automatic couple between the ψ and ω tilts to effect a decentration of the principal-points of the diapositives away from the on axis position according to the relation . . . (4) para 3/2/3. The decentration in the downhill direction of the applied tilt, might be achieved by the means of cams. In addition to this, automatic couples would need to be provided between the measuring mark Z movement and each of the goniometers to shift the goniometers bodily in a horizontal plane according to relation . . . (5) para 3/3/1 in the direction in which the ψ and ω tilts are applied.

TABLE 1

	1	2	3	4	5
Model	*Tilts on True Model*		*Maximum Relief Range on Model*	*Maximum Errors on Model Reset at 1.72 P.D.*	
	Left Photo	*Right Photo*		*Correspondence*	*Height*
1	1°40′ ψ 2°20′ ω	0°44′ ψ 1°17′ ω	21 mms.	0.20 mms.	1.88 mms.
2	0°03′ ψ 0°50′ ω	0°36′ ψ 1°18′ ω	10 mms.	0.14 mms.	0.64 mms.

The amount of projection translation varies with the tilt applied as well as with the Z position of the measuring mark, but the angular component of the projection translation is equal to the angular component of the diapositive decentration. Therefore it should be possible to secure correct automatic goniometric translation by a simple couple between the measuring mark Z movement and the diapositives, instead of direct to the goniometers.

These automatic movements could be built in to the plotting machine only with respect to a fixed relation M/m unless provision was made for altering the cams and leverages for other conditions of M/m.

It is extremely doubtful whether the construction of an instrument to work on these lines would be worthwhile. In all circumstances it would seem simpler to use a plotting machine which conserved the camera geometry, such as are already available on the market.

5. The Probable Order of Model Errors when using Orthodox Plotting Machines without Modification and the Principal Distance Set is 1.72 Times Greater than the Taking Camera (as would be necessary for Super Wide-Angle).

To determine the nature of these errors by mathematical analysis would be extremely difficult owing to the possibility of distribution of residual errors in a large variety of different ways, by application of different machine settings.

It was decided therefore to make some empirical experiments.

Two RC5 camera models were erected in the Wild A5 Autograph set to correct camera principal distances. These went up well without significant error, the tilts and relief range being as indicated in columns 1, 2 and 3 of Table 1. The Autograph principal distances were then increased by 1.72 and an attempt was made to re-erect the models. This necessitated the application of larger ψs and ωs and was only possible when permitting residual wants of correspondence to the maximum extent indicated in column 4. Model heights were then examined on the assumption that the vertical scale would be increased by 1.72, and the maximum height errors were then as indicated in column 5.

It is clear from these experiments that the use of plotting machines in this way would give rise to residual errors too large generally to be permissible in the presence of tilts and relief ranges likely to be encountered in production mapping.

6. Conclusions

The use of unrectified super wide-angle photography in plotting machines incapable of accommodating the super wide-angle principal distance would give rise to unacceptably large residual errors in the presence of the kind of tilts and relief ranges likely to be met with in production usage.

The modification of such instruments to give a workable solution would be prohibitively difficult. Therefore where there is a requirement to utilize super wide-angle photography for contouring, the choice lies between the purchase of plotting equipment capable of conserving the super wide-angle geometry such as the Wild A9, B9, or B8 machines, or the Kern PG2, Zeiss Jena Super Wide Multiplex or the Zeiss Munich Aeromat—or alternatively to undertake the aerotriangulation of the super wide-angle photographs in a stereocomparator, and by this means determine the tilts which could be used to make rectified diapositives which might then be set up in plotting machines now held by most organisations to produce an acceptable solution.