*Practical Tests of the Theoretical Accuracy of Aerial Triangulation**

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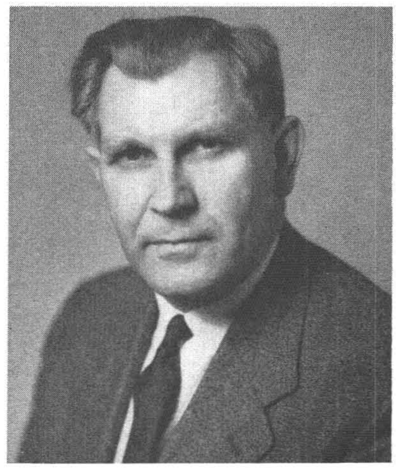
ABSTRACT: *This paper is a report on a practical aerial triangulation experiment on an Autograph A7, made to test the formulas for the accuracy to be expected in the results of a triangulation after application of the principles of numerical corrections.*

INTRODUCTION

THE general theory behind the experiment on an Autograph A7 was developed in a research project under a contract between ERDL, Fort Belvoir, Va. and Ohio State University. The author took part in this as a research associate in 1953–1954. In particular, the 7th interim technical report covers the principles to be applied. The well known error summation formulas of aerial triangulation according to *Bachmann* are used for the study of the correction of regular (systematic) errors in the basic operations, and of the propagation of the irregular errors. The theory was further developed in some papers in PHOTOGRAMMETRIC ENGINEERING. "The Principles of Numerical Corrections in Aerial Photogrammetry." April 1956; "Determination and Correction of Systematic Errors in the Fundamental Operations of Aerial Triangulation." September 1957; "A Theoretical Investigation of Aerial Triangulation as a Problem of Maxima and Minima." December 1958; and "Discussion about the Character of Errors in Spatial Aerotriangulation." Vol. XXVI, no. 2, April 1960. In particular, the principles treated in this publication have been used in the actual experiment.



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Further development of the theory has been made in contributions to the International Congress for Photogrammetry at Stockholm 1956 and at London 1960. Formulas for the final accuracy to be expected from photogrammetric triangulations, under different conditions and with different degrees of approximation, have been derived and presented. The formulas to be tested in this experiment were derived in the paper: "Investigations into the Accuracy of Various Methods for Photogrammetric Triangulation" which was presented at the London Congress in 1960. The formula systems and the principles of the derivations have also been shown in the textbook *Photogrammetry*, McGraw-Hill 1960.

Since the theoretical principles of the actual experiment have been published, and can be studied in detail in the papers mentioned, no detailed presentation will be made in this paper. Only a brief summary of the principles will be given as an introduction to the practical performance of the experiment and to the results obtained. Because grave misinterpretations of the basic principles of the procedures have occurred in certain research projects, and seem to exist in some photogram-

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† The author's employment at GIMRADA, Fort Belvoir, Va., has now been completed and he has returned to Stockholm.—EDITOR

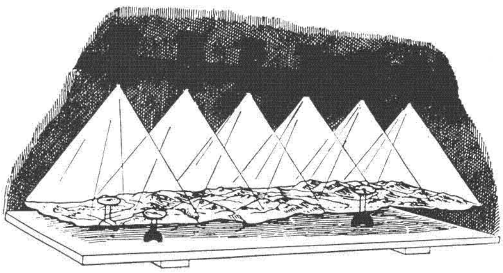

FIG. 1. Schematic diagram of the principles of aerial triangulation.

metric institutes, possibly unclear details will be especially discussed, even at the peril of being repetitious.

SUMMARY OF THE THEORETICAL PRINCIPLES OF THE EXPERIMENT

In principle, aerial triangulation can be conceived as a reconstruction of the photographed terrain obtained by reversing the geometrical procedure of the aerial photography. If all operations: the photography, the reconstruction of the bundles of rays, the relative and absolute orientation and the coordinate transformation could be performed without error the terrain could be correctly reconstructed, optically, mechanically or numerically—and no problems would in principle be present, see Fig. 1–3. But that ideal situation never occurs in practice. On the contrary there are errors involved in all operations and measurements and, since the triangulation in principle is a successive assembly of individual bundles of rays, the deformations of the bundles and the error, in the assembling operations will cause deviations of the reconstructed set of bundles from their correct positions. The reconstructed terrain model will suffer from deformations and will not become similar to the actual geometrical shape of the terrain. After a similarity transformation of the terrain model to the ground coordinate system, discrepancies in planimetry and elevation will be found in redundant control points. This is the normal situation after all aerial triangulation. The general problem is to make the discrepancies as small as possible through corrections and adjustment procedures and to find the laws of the creation and propagation of those errors which cause the discrepancies, in order to predict the accuracy to be expected from aerial triangulation procedures under well defined conditions.

The basic errors are to be found in the image coordinates of the photographs or diapositives, in the plotting instrument, in the observations made by the operator and in the calculations. These errors cause deforma-

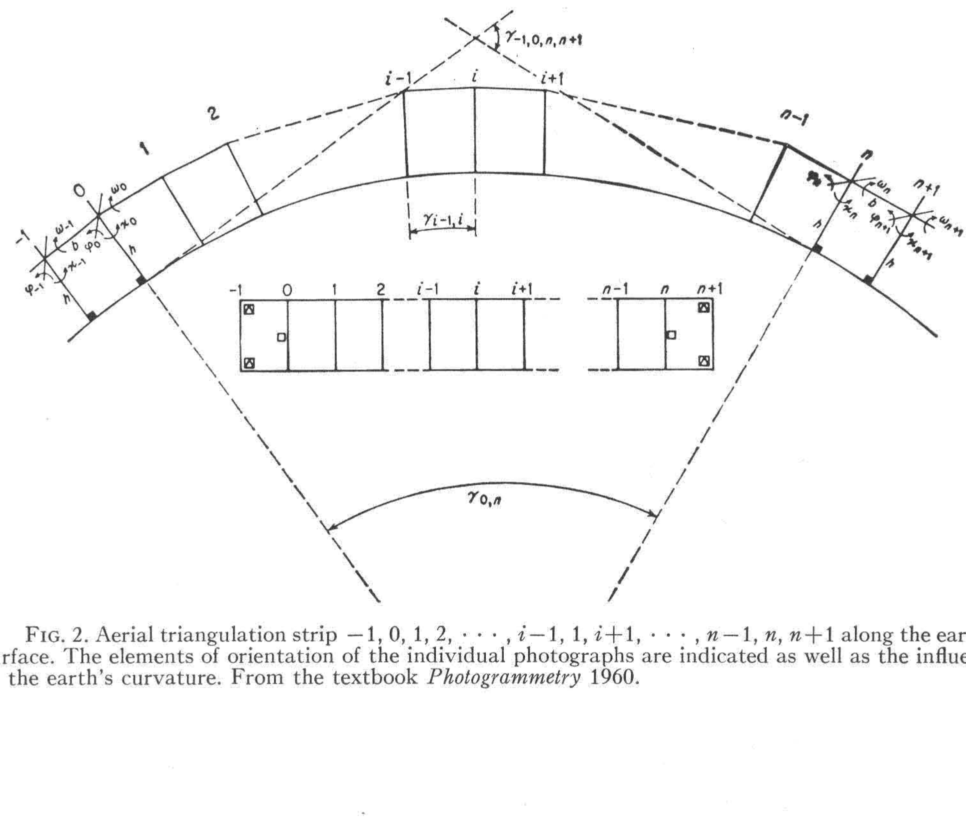

FIG. 2. Aerial triangulation strip $-1, 0, 1, 2, \cdots, i-1, 1, i+1, \cdots, n-1, n, n+1$ along the earth's surface. The elements of orientation of the individual photographs are indicated as well as the influence of the earth's curvature. From the textbook *Photogrammetry* 1960.

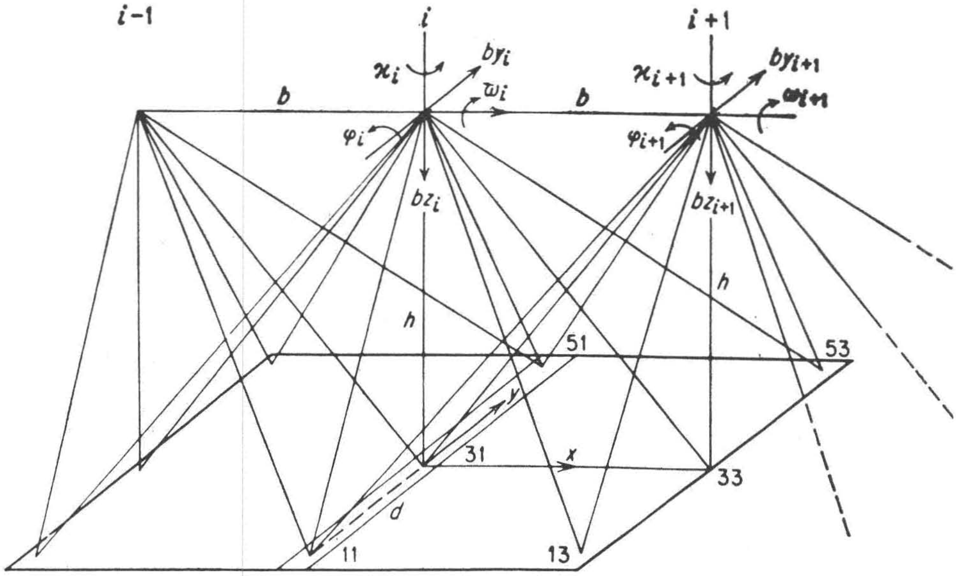

FIG. 3. Illustration for the derivation of the error summation formulas. If small changes are introduced to the orientation elements of the photograph i and the caused y-parallaxes of the model i, $i+1$ are corrected with the elements of $i+1$, the introduced changes will become transferred from i to $i+1$. This is the first, important step in the derivation. From the textbook *Photogrammetry* 1960.

tions of the reconstructed bundles of rays, false x- and y-parallaxes, which become partly compensated by the relative and absolute orientation operations. Thereby are introduced deviations in the elements of the external orientation, which in their proper turn cause deformations of the models in planimetry and elevation.

Since the triangulation strip is composed of individual models and bundles of rays, which are connected to each other by successive assembly, it is obvious that the terrain model deformations are caused by a rather complicated mechanism, in which the errors of the basic quantities—primarily image coordinates and parallaxes—are of fundamental importance. If these basic errors were known, however, it should be possible to correct the preliminary results of an aerial triangulation, provided that the numerical relation between the errors of the image coordinates and parallaxes, on one hand, and the deformations of the terrain model coordinates and elevations, on the other hand, are known. These relations are well known but are complicated in their inter-relation.

The differential formulas, however, which are linearizations of the general relations, are easier to handle and can be applied under normal, uniform conditions, although always with a certain approximation. The differential formulas which express these important relations in aerial triangulation are known as the *error summation formulas*.

For the application of the formulas, the basic errors must of course be known. As usual, distinction must be made between primarily two types of errors, namely

1. Regular (systematic) errors, the magnitude and direction of which obey well-defined laws and which consequently can be corrected individually and
2. Irregular (accidental or random) errors which do not show any regularities and which consequently cannot be corrected individually.

The relation between the magnitude and frequency of such errors is usually assumed to be identical with the *normal distribution*. This assumption is founded upon the *central limit theorem* from statistics—one of the most important theorems for all theory of errors. In fact, the normal distribution of the irregular errors is of importance for the laws of the propagation of such errors, for a correct adjustment of discrepancies, and for various statistical tests (confidence or significance tests).

The irregular errors cannot be determined individually in practice but an estimation of a statistical expression can frequently be made as root mean square value or as *standard error*

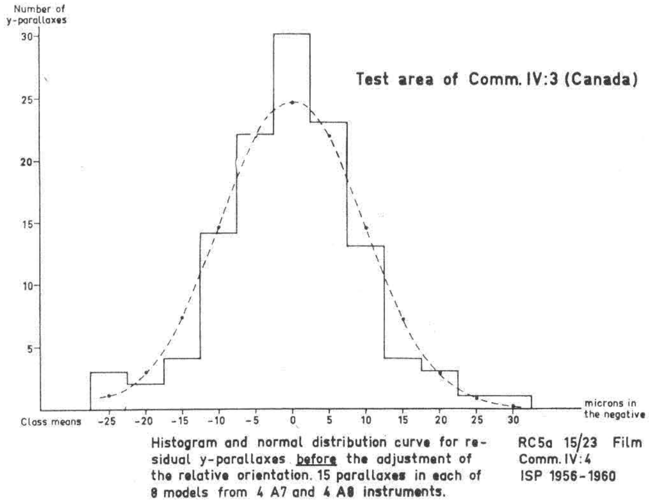

FIG. 4. An example of a normal distribution test of residual y-parallaxes in single models according to statistical methods. The numerical test is performed with the chi-square criterion. In this case the distribution is very close to a normal one.

of unit weight according to the method of least squares. Especially in connection with the calibration and test of cameras and measuring instruments and operators, it is most important that the irregular errors be estimated as standard errors of unit weight. In normal photogrammetric work an estimation of the magnitude of the basic irregular errors can usually be made from the residual y-parallaxes or from an adjustment of y-parallax measurements.

In this experiment the quality of the final results has been theoretically determined from such y-parallax measurements, but the y-parallaxes had to be corrected for regular errors of the image coordinates and instruments before the calculations. Normal distribution tests according to well-known statistical procedures have proved the central limit theorem in practice. In this connection reference is made to diagrams 1 to 4 in the report of the Sub-commission IV:4 at the London Congress in 1960: "Fundamental Questions in Relation to Controlled Experiments" see Figure 4 in this paper.

Returning to the actual aerial triangulation experiment, the determination of regular errors in the various operations, and the estimation of the statistical value of the irregular errors, will next be briefly treated.

The *image coordinates* of photographs from the aerial camera used in the experiment were tested according to the grid method after repeated photography of the *Öland* test area. The signalized and very accurately surveyed test points were located as the intersection points of a big grid, thus facilitating the computations considerably. Through these tests the most important regular errors of the image coordinates (primarily radial distortion effects and affine deformations) were repeatedly determined under operational conditions, and the irregular errors were estimated as standard errors of unit weight. These tests were made in photographs from the altitude of 5,000 meters above the ground, but the results were transformed to the altitude 1,500 meters to be used in the triangulation experiment, see Figure 5.

The *instrument* used in the triangulation experiment was carefully tested with the aid of a grid, the coordinates of which are known with a standard error of about 1 micron. The regular and irregular errors of the projectors of the instrument were determined according to the grid method and in the three positions —base zero, base-in and base-out. Some different enlargements were also tested.

From the regular errors of the projected coordinates the corresponding regular errors of the x- and y-parallaxes were determined for the different base settings. The standard errors of the regular errors were also determined from the standard errors of unit weight and the corresponding weight numbers.

The actual triangulation experiments were made with glass diapositives which were contact printed from the original film. The diapositives were dried in approximately horizontal positions since it had been found that drying in vertical position can introduce considerable errors into the image coordi-

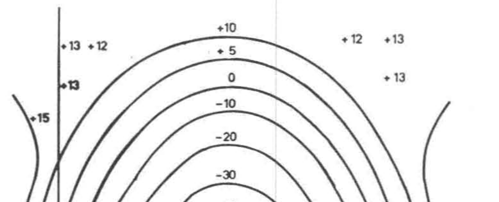

FIG. 5. Elevation correction diagram for individual models according to the radial distortion of the *photographs*, transformed to the actual flying altitude. The corrections are given in centimeters on the ground.

nates; this is probably due to emulsion creeping. The flatness of the diapositives was also especially tested even though possible flatness errors in this case are harmless.

The *operator* had to "warm up" his stereoscopic vision during about 10–15 minutes before starting stereoscopic elevation measurements, since it had been found that considerable regular changes of the stereoscopic settings can occur during the period mentioned, see Figure 6. The *precision* of observations of coordinates and parallaxes was furthermore determined by replicated settings and expressed as *standard deviation* of one observation. All observations were repeated three times and the averages were used. The standard deviation of the averages was thus considerably reduced (approximately one half of the standard deviation of one observation) and consequently the precision of the observed quantities was increased. Concerning the used terminology for accuracy and precision see the Appendix, where a summary of the terminology for the quality of these measurements is made.

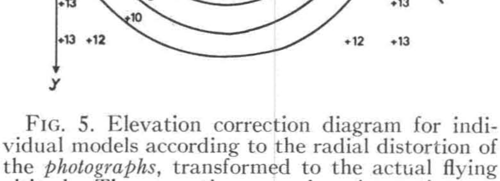

FIG. 6. Regular (systematic) variations of the stereoscopic elevation measurements in a photogrammetric model in the beginning of a series and in comparison with the elevation measurement in the same point the previous day. Averages of five operators from tests in a stereoplanigraph C5.

THE PRACTICAL PERFORMANCE OF THE EXPERIMENTS

The test strip was photographed by the Geographical Survey Office of Sweden using the camera Wild R.C. 5a Aviogon c = 152 mm. and over the Öland test area from an altitude of 1,500 meters above the ground. There were seven models covering a distance of about 6.3 km, and containing 29 signalized and very accurately surveyed control and check points. The triangulation was made as usual but the residual y-parallaxes were carefully measured in nine regularly located points of each model. In some models the parallaxes were measured in 15 points, and also along the fundamental circles for a check of radial distortion effects. Residual discrepancies in the scale transfer points were also carefully measured. The preliminary strip coordinates were then corrected according to the correction formulas and with respect to the measured y-parallaxes and elevation discrepancies; these first had been corrected for known regular errors due to the image coordinates and the instrument. The corrections were also computed with respect to each known type of regular error separately. Convenient forms were used for all computations. After the corrections, the strip coordinates and elevations were transformed to the geodetic system, with the aid of two control points at each end of the strip. The redundant control points were used for checks against gross errors and for affine corrections.

The discrepancies were then computed in the check points, see Figures 7–9. The discrepancies in the same check points *before* the corrections are also shown in Figures 10–12 for comparison. The correction procedure has evidently improved the accuracy considerably.

For a comparison with the theoretical accuracy to be expected, the root mean

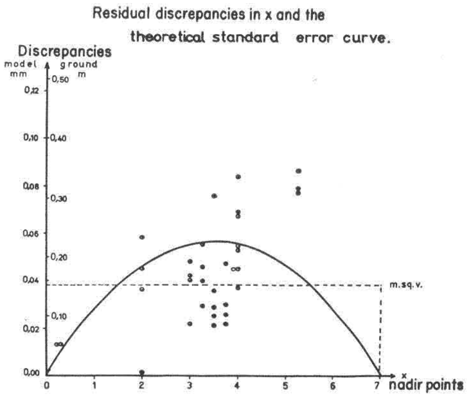

FIG. 7

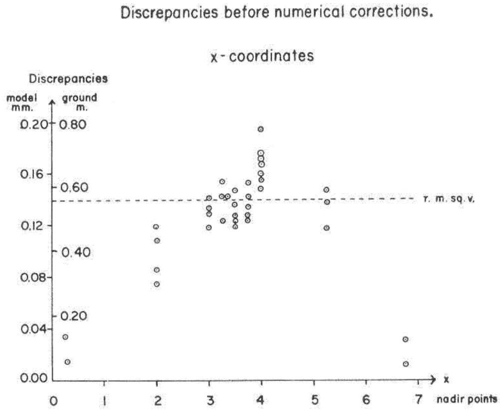

FIG. 10

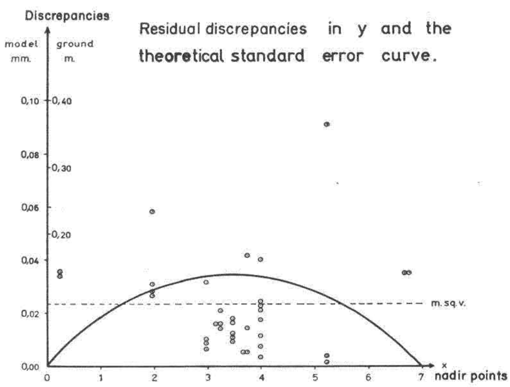

FIG. 8

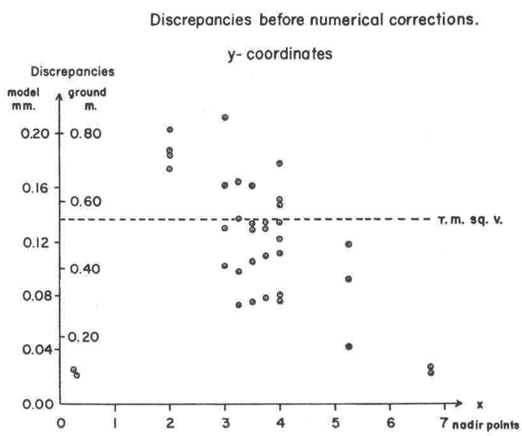

FIG. 11

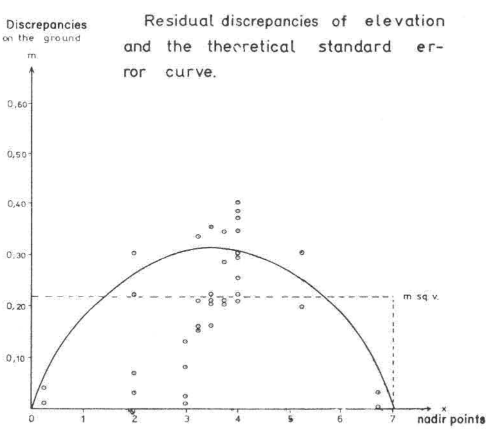

FIG. 9

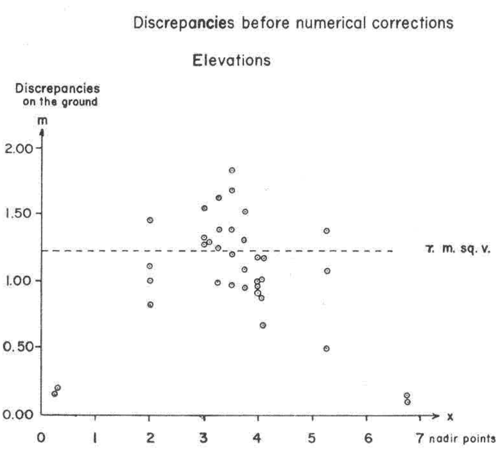

FIG. 12

TABLE 1

Coord.	*Theoretical root mean sq. values of st. errors, meter*	*Conf. limits, meter*	*Practical root mean square values, meter*	*Differences between theory and practice, meter*	*Per cent*
x	0.15	0.10–0.27	0.20	−0.05	25
y	0.10	0.07–0.18	0.11	−0.01	9
z	0.22	0.15–0.40	0.27	−0.05	19

square value of the discrepancies was computed for the x, y and z-coordinates.

COMPARISON BETWEEN THE THEORETICAL AND THE PRACTICAL ACCURACY

From the formulas for the theoretical accuracy to be expected in the results of the triangulation—see references [2] and [3]—the theoretical root mean square values of the standard errors along the strip were computed. From the results of y-parallax measurements, the value of the standard error of unit weight s_0 was found to be 0.006 mm. on the scale of the photographs. The number n was seven and the photographic scale 1:10,000. Further, the confidence limits of the standard error of unit weight were determined according to usual statistical procedures. For the level five per cent the confidence limits are $0.7s_0$ and $1.8s_0$ respectively. If the true root mean square values of the discrepancies between the photogrammetric and geodetic coordinates and elevations are located between the corresponding limits, the theory behind the formulas for the determination of the accuracy can be accepted on the five per cent level.

In Table 1 the results of the comparison and the corresponding differences between theory and practice are shown.

There is a rather good agreement between theory and practice, and this experiment indicates that the theoretical expressions for the accuracy of the results of the aerial triangulation, after application of the principles of numerical corrections, can be accepted.

DISCUSSION

The results obtained from this triangulation experiment are a confirmation of the laws of error propagation according to the method of least squares. The expressions for the accuracy to be expected after the bridging, under the assumed conditions, are founded upon the assumption that, for the most part, only irregular and normally distributed errors affect the fundamental operations, and that discrepancies in actual conditions, especially those of the relative orientation, are adjusted according to the method of least squares.

For obvious reasons certain approximations have been made in the derivation of the formulas for the accuracy to be expected. Among other assumptions it can be mentioned that the photography is performed under normal, uniform conditions, that the elevation differences on the ground are comparatively small in comparison with the flying altitude, and that the control and check points are uniquely identifiable and free from errors. The control points are assumed at the ends of the strip only. Due to the *arc sine law*, the length of the strip must be limited and should preferably not exceed 10 models. In long strips the irregular errors of the fundamental operations, according to the law mentioned, can cause deformations which have an apparent regular (systematic) character, and which consequently make all adjustment procedures more or less doubtful.

It is highly desirable that more experiments of the same character as reported in this paper be performed in order to check the procedure and the accuracy formulas. Separate tests of the basic operations are necessary prerequisites. If the formulas prove to be acceptable under the circumstances mentioned, the planning of aerial triangulation may become considerably facilitated, and the quality check of the routine work may be reduced primarily to measurements and simple calculations of y-parallaxes.

The formula systems for the numerical corrections and other calculations can be programmed for electronic computers; this is a task for applied mathematics and not primarily for photogrammetry. The accuracy formulas after minor changes can be applied to the results of analytical triangulation, provided that the necessary prerequisites concerning the errors of the fundamental operations are fulfilled.

A more detailed report on the performed triangulation experiment will be published as a GIMRADA paper.

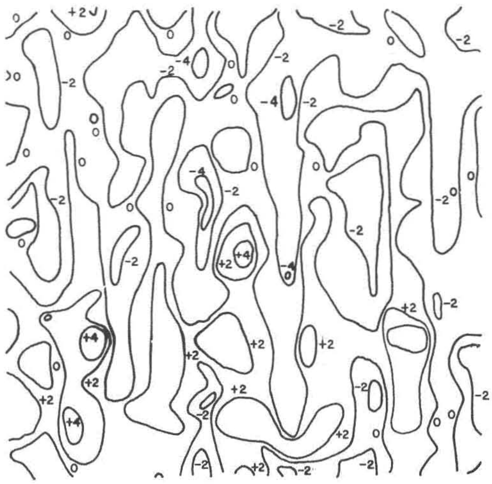

FIG. 13. Thickness variations of a film for aerial photography, shown as contour lines with the interval 2 microns. There are differences up to about 8 microns in this film sample. Greater differences have been found.

Further it can be mentioned that research concerning the sources of errors in the fundamental operations of photogrammetry is continuously being performed, primarily in order to distinguish, as far as possible, between systematic (regular) errors of various kinds, and to estimate the residual, irregular errors statistically. Distribution tests of the residuals show whether or not the important normal distribution is present. Tests of the flatness of surfaces of various kinds and of the thickness of films and emulsions have proved that considerable sources of errors are to be found in the deviations from flatness of the negative. See Figures 13 and 14. For triangulations such errors are of great importance. In fact, all of the more or less beautiful theories and procedures for the adjustment of aerial triangulation, and formulas for the error propagation, can be completely ruined if the film is not flat in the moment of exposure, or if there are large thickness variations of the film base or the emulsion. The real location of the image details (the contrasts) within the emulsion may also have a great influence upon the accuracy. Also this problem is now investigated as being of great importance for all kinds of photogrammetry. The final accuracy of photogrammetric procedures may be limited by these factors.

Even though it has been said repeatedly, it must be emphasized that much more research should be devoted to the fundamental operations of photogrammetry before the adjustment problem of discrepancies in redundant control points is treated. Most of the published experiments concerning the adjustment of aerial triangulation within ISP, OEEPE and elsewhere support this opinion.

It should also be repeated that no adjustment procedure can transform poor basic data and observations into results of high quality. If the real geometrical quality of these basic data is determined and is expressed in well defined terms, the geometrical quality of any function of the data and tolerance limits can be theoretically determined according to well-known laws. The main purpose of practical experiments should be a verification of the theoretical accuracy. When the laws are sufficiently checked, the ultimate goal—the establishment of tolerances—is not far away.

Concerning the theory of errors of single models, the results of the controlled experiments of Commission IV of the International Society for Photogrammetry, as shown at the London Congress in 1960, prove that the theoretical laws of error propagation agree very well with practice, even under unfavourable conditions.

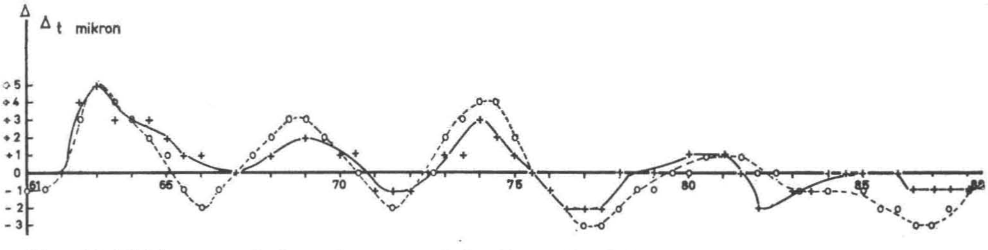

FIG. 14. Thickness variations along one of the diagonals of the film, fig. 13. One of the curves was taken from the curve diagram and the other one was directly measured. The agreement between the curves indicate the precision of the measurements. The standard deviation of one measurement is of the order of magnitude 1 micron.

This experiment with aerial triangulation is one of the first tests of the theoretical accuracy of such methods and should be followed by more.

ACKNOWLEDGMENTS

The aerial photography for this experiment was made by The Geographical Survey Office of Sweden over the Öland test area. Measurements and computations were primarily made by the following persons: C. O. Jonason, R. Helming and P. Kaasila.

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APPENDIX

SUMMARY OF METHODS USED IN THE TRIANGULATION EXPERIMENT FOR THE DETERMINATION OF QUALITY OF MEASUREMENTS AND THE CORRESPONDING EXPRESSION

1. *Direct, repeated measurements of* KNOWN *quantities* for the purpose of *calibration* and determination of the *accuracy* of instruments and methods. An *error* e (or discrepancy in the condition given by the known quantity) is defined as the measured value M minus the given (known) value G or in a formula

$$e = M - G$$

A *correction* v is the reverse of the error e and consequently

$$v = -e = G - M$$

The *accuracy* of one measurement in a series of n repeated, direct measurements of given (known) quantities (regarded to be errorless) is expressed as the *root mean square value* of the errors or discrepancies, i.e. in a formula

$$m = \sqrt{\frac{[ee]}{n}} \qquad ([ee] = e_1^2 + e_2^2 + \cdots + e_n^2)$$

If the discrepancies are regarded to be functions of regular (systematic) sources of errors, these errors may be determined as parameters in a least squares adjustment according to point 3 below.

2. *Direct, replicated or repeated measurements*[1] *of* UNKNOWN *quantities* for the purpose of determination of the *precision* of the measurements.

The average of the measurements is first computed and then the corrections v_1, $v_2, \cdots v_n$ to the individual measurements in order to make them coincide with the average.

The *precision* of *one* measurement is then defined as the *standard deviation* s according to the formula

$$s = \sqrt{\frac{[vv]}{n-1}}$$

The *precision of the average* of n measurements is obtained as the standard deviation of the average s_M according to the formula

$$s_M = \sqrt{\frac{[vv]}{n(n-1)}} = \frac{s}{\sqrt{n}}$$

3. *Indirect measurements of unknown quantities.*

The unknown quantities are assumed to be connected with the measurements via linear or linearized functions. For two unknown quantities x and y, and n measurements $l_1, \cdots, l_n$ of the linear function are obtained

$$l_1 = a_1x + b_1y$$
$$\cdots\cdots$$
$$l_n = a_nx + b_ny$$

where $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are known (errorless) coefficients. x and y may be regarded as regular errors and $l_1, \cdots, l_n$ as (measured) discrepancies in conditions according to point 1 above.

For unique determination of x and y, and, in order to make the computations as simple as possible, the method of least squares is applied. The individual observations $l_1, \cdots, l_n$ are corrected with the quantities $v_1, \cdots, v_n$ according to the expressions:

$$v_1 = a_1x + b_1y - l_1$$
$$\cdots\cdots\cdots$$
$$v_n = a_nx + b_ny - l_n$$

x and y are now to be determined under the condition that the sum of the squares $[vv]$ = minimum. This leads to the *normal equations*

$$[aa]x + [ab]y - [al] = 0$$
$$[ab]x + [bb]y - [bl] = 0$$

and further

$$-[al]x - [bl]y + [ll] = [vv]$$

[1] *Replicated* measurements are made at one place and at one period of time. *Repeated* measurements refer to different places and periods of time.

The *accuracy* of one measurement is then defined as the *standard error of unit weight* s_0 according to the formula

$$s_0 = \sqrt{\frac{[vv]}{n-2}}.$$

The standard error of the parameters x and y and of all functions and of all functions of functions of the basic measurements $l_1 \cdots l_n$ can then be determined from the standard error of unit weight and the *laws of error propagation*.* See for instance reference [2].

Tolerances for standard errors, standard error of unit weight, and all linear functions of the measurements are founded upon the statistical theory of *confidence limits*, admissible *confidence levels*, usually five per cent and the *degrees of freedom*. The *normal distribution* of the residuals after correction of regular errors is of basic importance for the laws of error propagation and the confidence theory. Tests of the normal distribution, founded upon the chi-square criterion, are therefore of great interest.

* These simple laws are sometimes denoted variance-covariance technique and are written in matrix notations.

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*Orthophotoscope Design Analysis**

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SECTION I—INTRODUCTION

THIS paper analyzes, quantitatively, the design parameters of an *automatic* orthophotoscope. It is assumed that:

1) the aerial photographs have previously been nominally rectified,
2) point elevations have already been found either manually or in a stereomat type apparatus,
3) ground-point positions had been computed and corrected for tilts, curvature, refraction, distortion, etc.
4) the photocoordinates, ground positions and elevations had been transcribed in digital form onto magnetic tape.

Thus, the input to the orthophotoscope consists of one picture and a tape. The function of the apparatus is to reposition image-points from their photo position to their correct map location.

Inasmuch as this is a somewhat unconventional approach to orthophotoscope design, a discussion of problems inherent to this scheme seems in order. In the traditional orthophotoscope, the projection of points from aerial photo to the orthophoto is a continuous process (at least along one direction). In the present system this is a (discontinuous) sampling scheme. The first question that poses itself is what sampling rate is required to achieve a given positional accuracy and resolution. This problem is analyzed in the next section.

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