Use of Triplets for Analytical Aerotriangulation

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ABSTRACT: *The concept of Analytical Relative Orientation as known in Photogrammetry always incorporates two photographs. This paper introduces a new idea of using Triplets instead of Pairs of photographs as an improvement in Analytical Aerotriangulation. It shows why the adoption of this new line of thought is expected to introduce great economies in cost as a result of the considerable reduction in computing time involved.*

INTRODUCTION

ANALYTIC photogrammetry as used in this paper, is defined as the purely digital solution of the perspective problem which derives ground coordinates of objects from measured coordinates of corresponding images on photographs. Therefore, this method yields X, Y, Z coordinates of ground objects adjusted statistically to fit all control throughout a strip or a block and to retain the highest possible digital accuracy consistent with the basic data.

A method for treating the problem of analytic photogrammetry has been developed and is now in productive use by the U. S. Coast and Geodetic Survey.[1] This method is a variation of the approach developed by Dr. Hellmut Schmid.[2] A brief outline of this approach will be mentioned later. The general adaptation by C&GS is described in Revista Cartografica.[3]

The purpose of this paper is to outline a modification in the steps of relative orientation involved in the above mentioned method, so that the most time-consuming step of the program, namely the "grand adjustment," could be eliminated.

THE MATHEMATICAL BASIS[4,5]

Figure 1 shows the basic geometric relationships between camera station, image and object. x^*, y^*, z^* are rectified coordinates of the image, X, Y, Z and X_0, Y_0, Z_0 are coordinates based on a reference earth system. The rectified values of the image co-

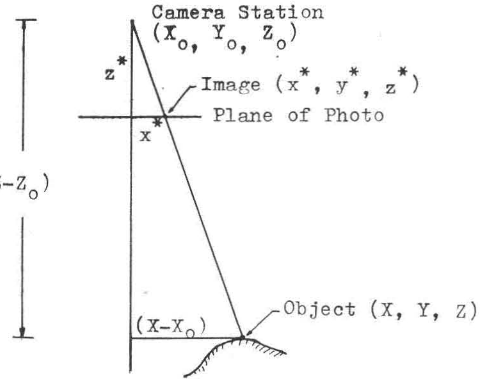

FIG. 1. Geometry of colineation condition.

EDITOR'S NOTE. This paper was submitted in the 1961 competition for the Wild Heerbrugg Instruments, Inc. Photogrammetric Award. The paper was the winner in the competition.

ordinates can be obtained from measured photo coordinates by the following matrix relation:

$$\begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

in which the a-matrix is the rotation matrix quite familiar to the reader.[4]

From the simple geometric relation of similar triangles and after substitutions and some manipulations, the following two basic equations as a condition for the collineation of the three points, station, image and object, are obtainable for every image on a photograph.

$$\frac{x}{z} = \frac{(X - X_0)a_{11} + (Y - Y_0)a_{12} + (Z - Z_0)a_{13}}{(X - X_0)a_{31} + (Y - Y_0)a_{32} + (Z - Z_0)a_{33}}$$

$$\frac{y}{z} = \frac{(X - X_0)a_{21} + (Y - Y_0)a_{22} + (Z - Z_0)a_{23}}{(X - X_0)a_{31} + (Y - Y_0)a_{32} + (Z - Z_0)a_{33}}$$

This pair of equations expresses the condition upon which Dr. Schmid's method is based.

The theoretical solution of the problem of analytic photogrammetry consists of computing all the observation equations for all the common images of selected points on all photographs of a strip or block in a single massive adjustment. But, since this solution is expensive and time consuming, because of the large number of unknowns encountered, ways are sought to reduce the computing time.

The attractive aspect of the C&GS scheme to reduce computing time is to refine the initial approximations to the point where the massive adjustment need be computed but once. This method consists of a scheme analogous to conventional aerotriangulation with a plotter: relative orientation, absolute orientation, cantilever extension, and adjustment of the results to fit control.

The complete solution is outlined as follows:[3]

(1) Ground-control operations
(2) Aerial photography
(3) Transformation of ground-control data to local geocentric coordinate system to account for earth curvature
(4) Image coordinate measurement
(5) Transformation of observed coordinates, incorporating corrections for film distortion, lens distortion and atmospheric refraction
(6) Relative orientation of each stereo pair independently, seeking and eliminating grossly erroneous observations
(7) Cantilever transformation based on step (6) (but without the use of any ground-control) assumed nominal scale and datum in the first pair
(8) Cantilever adjustment transforming coordinates of ground points in step (7) to fit control
(9) Resection of each photograph to fit data of step (8). Thus steps (8) and (9) yield refined values for a, b, c, X_0, Y_0, Z_0, X, Y, Z for each photograph and object
(10) Grand adjustment

From all the ten steps stated above, the "grand adjustment" is the one that requires the longest computing time and hence is the most expensive. The scheme as set forth above by the C&GS is meant to refine all the first nine steps to the point that the grand adjustment is only needed once, iteration being unnecessary. This paper proposes a further refinement in the procedure. This refinement leads to the omission of the grand adjustment, except for very special cases where very precise results are required. It promises to be an appreciable improvement which will produce economics in analytical aerotriangulation.

A study of the above mentioned step-by-step procedure yields the following comments.

(1) Grand adjustment is most related to the step of relative orientation. If the quality of relative orientation is poor the grand adjustment might be required more than once. On the other hand, improvements in the relative orientation, certainly contribute to the possibility of eliminating the grand adjustment.

(2) In checking the results of relative orientation as obtained above, only residuals in the y-direction (y-parallax) could be interpreted as errors and minimized. The residuals in x-direction (x-parallax) are interpreted as differences in elevation; in other words, any residual error in x-coordinate of a selected point will only be noticed in its elevation which will be read higher or lower according to the sign of the residual (see Figure 2). The reason for the difficulty in discovering the x-residual is due to the fact that, in relative orientation only two photographs are considered at a time, and consequently, only two rays determine the position of the point.

(3) For a small-size electronic computer the time required to solve a set of equations is approximately proportional to the cube of the number of unknowns. For the relative orientation of a model the number of unknowns is reduced to five for each model instead of 6× number of photos. But still, if a strip is long, although the relative orientation will not require a long time, yet the grand adjustment—as still needed—will require as much computing time as all the other steps together.

Therefore, a modification done in relative orientation that enables the detection of x-parallaxes and leads to the omission of grand adjustment will be quite a successful accomplishment and economical improvement in the present program.

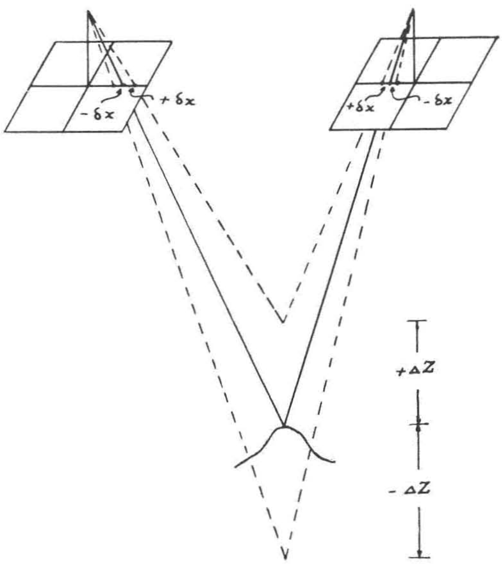

FIG. 2. In a stereopair, residuals in the x-direction cannot be inspected but only interpreted as differences in elevation.

The New Method

As mentioned before, two rays are not enough to detect any error in the x-direction and at least three will be required. This condition leads to the idea of using three photographs instead of just a pair in the step of relative orientation. Using three photographs, there will be an overlapped area of about 20%, which is common to all three photographs. Image points in this area will furnish a check on x-parallax and double check on y-parallax. It will help tie the first pair (photos 1 and 2) with the second pair (photos 2 and 3) which were oriented separately by the old method. The scheme will then be set as groups of three photographs (triplets), each oriented separately. The first triplet includes photographs 1, 2, and 3, the second includes photographs 2, 3 and 4; and so on as shown in Figure 3.

Since the method used for orienting any triplet is the same as for any other, the following discussion will consider the first triplet, or photographs 1, 2 and 3. In every photograph the coordinates of 9 images are measured (sometimes it is recommended to use twice that number to count for the eliminated points after the discovery of gross errors). It is known that every photograph absolutely encounters six unknown elements:

ω, ϕ, κ rotation elements and X_0, Y_0, Z_0 position elements. Also the reader is reminded that each image point on a photograph yields two observation equations as stated previously.

The Procedure

The general procedure previously followed in the C&GS method to orient a model (2 photos) was to hold the first photograph fixed (or $\omega_1=\phi_1=\kappa_1=X_{01}=Y_{01}=Z_{01}=0$), and choosing a value for the air base $X_{02}=1$ then work with the five left unknowns ω_2, ϕ_2, κ_2, Y_{02} and Z_{02}. The same principle could be extended here, except that when adding the third photograph, the first six elements $\omega_1, \cdots, Z_{01}$ should no longer be left untouched and need be changed. In such case a considerable change in the past procedure would be introduced which would make it quite difficult.

Therefore to make the work reasonably simple and to introduce no major changes in the old procedure, the following method is proposed. Photograph number 2 (the middle photo of the triplet) is fixed and both photographs 1 and 3 are tied to it simultaneously on both sides. Again, a value for either X_{01} or X_{03} is assigned and therefore seven of the eighteen elements are fixed. But in order to avoid negative values for X, X_{01} is fixed to zero and X_{02} fixed to unity. This will boil down to the following situation:

Photo #1	$X_{01}=0$	and	$\omega_1, \phi_1, \kappa_1, Y_{01}, Z_{01}$	initially assumed zeros
Photo #2	$X_{02}=1$	and	$\omega_2=\phi_2=\kappa_2=Y_{02}=Z_{02}=0$	always fixed
Photo #3	X_{03},		$\omega_3, \phi_3, \kappa_3, Y_{03}, Z_{03}$	initially assumed zeros

Consequently, and from Figure 4, we get the following count:

Photo No.	*No. of Unknowns*	*No. of Images*	*No. of Observation Equations*
1	5	6	12
2	0	9	18
3	6	6	12
	—	—	—
Total	11	21	42

So for every triplet, 42 observation equations are obtained then reduced through the method of least squares to eleven normal equations to be solved simultaneously for

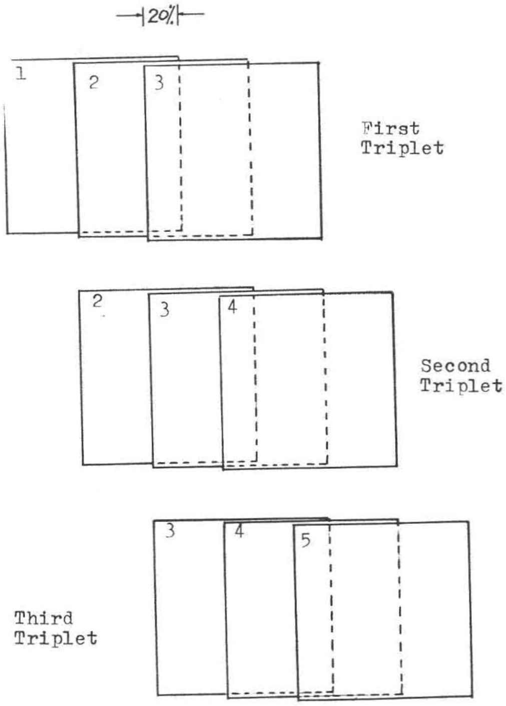

FIG. 3. Advance of triplets in relative orientation.

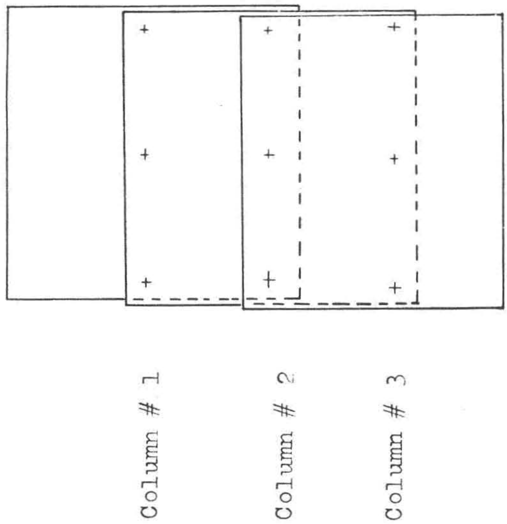

FIG. 4. Distribution of images in the triplets.

eleven unknowns. Also from Figure 4, columns 1 and 3 are common to two photographs therefore used to check X and Y only, while column 2 is common to all three photographs, hence used in checking X, Y and Z. When the second triplet is tied with the first, Z-values are taken from the first triplet in column 2 and from the second triplet in column 3, while X and Y values are checked together from both triplets.

OBSERVATION EQUATIONS

The basic condition equations that assure that each object-lens-image set of points is collinear are:

$$\frac{x_{i1}}{z_{i1}} = \frac{(X_{i1} - X_{01})a_{11} + (Y_{i1} - Y_{01})a_{12} + (Z_{i1} - Z_{01})a_{13}}{(X_{i1} - X_{01})a_{31} + (Y_{i1} - Y_{01})a_{32} + (Z_{i1} - Z_{01})a_{33}}$$ for photo-pair 1 and 2, or the first stereogram in the triplet

$$\frac{y_{i1}}{z_{i1}} = \frac{(X_{i1} - X_{01})a_{21} + (Y_{i1} - Y_{01})a_{22} + (Z_{i1} - Z_{01})a_{23}}{(X_{i1} - X_{01})a_{31} + (Y_{i1} - Y_{01})a_{32} + (Z_{i1} - Z_{01})a_{33}}$$

$$\frac{x_{i3}}{z_{i3}} = \frac{(X_{i3} - X_{03})d_{11} + (Y_{i3} - Y_{03})d_{12} + (Z_{i3} - Z_{03})d_{13}}{(X_{i3} - X_{03})d_{31} + (Y_{i3} - Y_{03})d_{32} + (Z_{i3} - Z_{03})d_{33}}$$ for photo-pair 2 and 3, or the second stereogram in the triplet

$$\frac{y_{i3}}{z_{i3}} = \frac{(X_{i3} - X_{03})d_{21} + (Y_{i3} - Y_{03})d_{22} + (Z_{i3} - Z_{03})d_{23}}{(X_{i3} - X_{03})d_{31} + (Y_{i3} - Y_{03})d_{32} + (Z_{i3} - Z_{03})d_{33}}$$

These equations are so complicated that a direct solution for all the unknowns ($\omega, \phi, \kappa, X_0, Y_0, Z_0, X, Y, Z$) is not attempted. Instead a logical approximation is made for each unknown, then corrections are computed which enforce the collineation condition, using equations formed from the above condition equations by means of partial differential calculus. These equations are then observed equations having the form:

$$\begin{aligned} Vx_1 &= P_{11} + P_{12}d\omega_1 + P_{13}d\phi_1 + P_{14}d\kappa_1 - P_{15}dX_{01} - P_{16}dY_{01} - P_{17}dZ_{01} \\ &\quad + P_{15}dX_{i1} + P_{16}dY_{i1} + P_{17}dZ_{i1} \\ Vy_1 &= P_{21} + P_{22}d\omega_1 + P_{23}d\phi_1 + P_{24}d\kappa_1 - P_{25}dX_{01} - P_{26}dY_{01} - P_{27}dZ_{01} \\ &\quad + P_{25}dX_{i1} + P_{26}dY_{i1} + P_{27}dZ_{i1} \end{aligned}$$

for the first stereogram

and

$$\begin{aligned} Vx_3 &= q_{11} + q_{12}d\omega_3 + q_{13}d\phi_3 + q_{14}d\kappa_3 + q_{15}dX_{03} - q_{16}dY_{03} - q_{17}dZ_{03} \\ &\quad + q_{15}dX_{i3} + q_{16}dY_{i3} + q_{17}dZ_{i3} \\ Vy_3 &= q_{21} + q_{22}d\omega_3 + q_{23}d\phi_3 + q_{24}d\kappa_3 - q_{25}dX_{03} - q_{26}dY_{03} - q_{27}dZ_{03} \\ &\quad + q_{25}dX_{i3} + q_{26}dY_{i3} + q_{27}dZ_{i3} \end{aligned}$$

for the second stereogram

The values of the coefficients $P_{11} \cdots P_{27}$ and $q_{11} \cdots q_{27}$ in the above two sets of equations are derived by partial differentiation of the basic pairs of equations. The values of coefficients thus obtained are only preliminary values that are changed according to the Newton method of successive approximation through the iterative procedure of computing and adding corrections to the variables.

The unknowns ω, ϕ, κ, X_0, Y_0, Z_0, X, Y, Z involved in the equations above are of two different groups. The first six, though varying from one photograph to another, remain constant for all the images on a given photograph, whereas the last three differ for each image and hence often are called model coordinates as discussed in the next section.

MODEL COORDINATES

In conventional photogrammetry a three-dimensional model of a portion of the earth is reconstructed from the common area of two over-lapping photographs. The analytic model coordinates comprise computed three-dimensional coordinate values of model points corresponding to pairs of images, one from each photograph. By "relative orientation"—in the general sense—is meant the orientation of one photograph relative to an overlapping one. The case which is dealt with here, relative orientation is used in a broader sense, the middle photograph of the triplets is held fixed and both the first and third is oriented to it on each side. Consequently there will be two stereo models or stereograms.

The general formula for the elevation Z of an object in a model or stereogram is given by (5)

$$Z = \frac{(X_0'' - X_0')z^{*\prime}z^{*\prime\prime} + Z_0 x^{*\prime}z^{*\prime\prime} - Z_0'' x^{*\prime\prime}z^{*\prime}}{x^{*\prime}z^{*\prime\prime} - x^{*\prime\prime}z^{*\prime}}$$

in which the single and double primes refer to the first and second photographs in the stereogram respectively and the asterisks refer to rectified photo coordinates.

Applying this formula to both stereograms in the triplet and after substituting the values assigned to X_0'' Z_0'', X_0' as mentioned above ($X_0'' = X_{02} = 1$, $Z_0'' = Z_{02} = 0$, $X_0' = X_{01} = 0$) we get

$$Z_{i1} = \frac{z_{i1}^* z_{i2} + Z_{01} x_{i1}^* z_{i2}}{X_{i1}^* z_2 - X_{i2} z_{i1}^*} = \frac{z_{i1}^* + Z_{01} x_{i1}^*}{x_{i1}^* - (x_{i2}/z_{i2}) z_{i1}^*} \quad \text{in the first stereogram}$$

and

$$Z_{i3} = \frac{(X_{03} - 1)z_{i2}z_{i3}^* - Z_{03}X_{i3}^*z_{i2}}{x_{i2}z_{i3}^* - x_{i3}\cdot z_{i2}} = \frac{Z_{03}X_{i3}^* - (X_{03} - 1)z_{i3}^*}{x_{i3}^* - (x_{i2}/z_{i2})z_{i3}^*} \text{ in the second stereogram}$$

where the starred (*) terms are rectified image coordinates from the first and third photographs and x_{i2} and y_{i2} are unrectified coordinates of the corresponding image on the middle (second) photograph of the triplet. This allows

(1) An approximate Z to be computed for each model point in terms of the other approximate parameters Z_0, ω, ϕ, κ and observed image coordinates in both stereograms; and

(2) the other two horizontal coordinates X, Y of the object to be evaluated. Upon subsequent iterative modifications of the approximations Z, X and Y become valid model coordinates.

Once Z is determined, the X and Y model coordinates can be determined by virtue of the similar triangles shown in Figure 1 and based on image coordinates of the second photograph (middle one of the triplet).

$$X_{i1} = (x_{i2}/y_{i2})Z_{i1}, \quad Y_{i1} = (y_{i2}/z_{i2})Z_{i1} \quad \text{for first stereogram}$$

$$X_{i3} = (x_{i2}/z_{i2})Z_{i3}, \quad Y_{i3} = (y_{i2}/z_{i2})Z_{i3} \quad \text{for second stereogram}$$

ELIMINATION OF THE MODEL COORDINATE TERMS

The general two observation equations for every image in each stereogram of the two forming the triplet, contain three terms dX, dY, dZ, corrections to the approximate model coordinates. These cannot be dealt with very well because of the increase in the number of unknowns with each added image (as stated previously X, Y, Z are variables that differ for each image). Therefore, an elimination of these variables is necessary. This is done by substituting their values as given above in terms of other variables that are fixed in value for every photograph, namely ω, ϕ, κ and Z_0.

The final form of the observation equations will then be in the form

$$Vx_1 = P_{11} + (P_{12} + K_1)dw_1 + (P_{13} + K_2)d\phi_1 + (P_{14} + K_3)d\kappa_1 - P_{16}dY_{01} - (P_{17} - K_4)dZ_{01}$$

$$Vy_1 = P_{21} + (P_{22} + H_1)dw_1 + (P_{23} + H_2)d\phi_1 + (P_{24} + H_3)d\kappa_1 - P_{26}dY_{01} - (P_{27} - H_4)dZ_{01}$$

$$Vx_3 = q_{11} + (q_{12} + K_1')dw_3 + (q_{13} + K_2')d\phi_3 + (q_{14} + K_3')d\kappa_3 - (q_{15} - K_4')dX_{03} - q_{16}dY_{03} - (q_{17} - K_5')dZ_{03}$$

$$Vy_3 = q_{21} + (q_{22} + H_1')dw_3 + (q_{23} + H_2')d\phi_3 + (q_{24} + H_3')d\kappa_3 - (q_{25} - H_4')dX_{03} - q_{26}dY_{03} - (9_{27} - H_5')dZ_{03}$$

in which $K_1 \cdots H_4$ are factors obtained through partial differentiation of model coordinate equations involving the variables ω_1, ϕ_1, κ_1 and Z_{01}.

And $K_1' \cdots H_5'$ are similar factors involving the variables ω_3, ϕ_3, κ_3 and Z_{03}.

The derivation of these coefficients is straight forward and can readily be seen, but is omitted here.

This completes, in brief, the presentation of the formulas and coefficients encountered in the observation equations. A routine for their evaluation is also set and ready to use. The reader is referred to reference (4), where upon comparisons he will find that the program set and already in productive use needs only minor changes to be adopted in the new technique.

CONCLUSION

The concept of relative orientation as known in photogrammetry always incorporates two photographs. The new idea of dealing with three photographs at a time opens a new line of thought in analytic photogrammetry. The most precise theoretical solution would result from the simultaneous solution of all the photographs in the strip or block. Mathematical logic demonstrates that dealing with three photographs yields far better results for relative orientation than handling only two photographs.

It ought to be mentioned that this improvement in relative orientation will need more computing time than that required for a two-photograph program. But, considering the time saved by the elimination of the grand adjustment, a very appreciable saving in computing time is expected. Also, an extra advantage is the simplicity in modifying an existing program for use according to the new concept which will require almost no programming effort. As an approximate comparison, a strip of 10 photographs will consume the following time:

relative orientation 2 photos at a time $\sim$9 models	$\times$(5 unknowns)$^3\sim$1125$\sim$1
relative orientation 3 photos at a time $\sim$8 triplets	$\times$(11 unknowns)$^3\sim$10648$\sim$9
grand adjustment $\sim$1 strip	$\times$(60 unknowns)$^3\sim$216000$\sim$192

So classical relative orientation combined with a grand adjustment will require approximately $(192+1)/9$ or 21 times the computing time for relative orientation of three photographs simultaneously with no grand adjustment.

The author believes that the same idea could be carried further. Should the strips become extremely long so that some doubt as to the results obtained by this technique exists, it would be possible to set up formulas for relative orientation of four photographs simultaneously to obtain more refined results. It is noteworthy, though, that the improvement in accuracy between the four-photos and three-photos cases is comparatively less than that between the triplet and two photos. This fact is mainly due to the invaluable advantage of checking and minimizing the x-parallaxes in case of triplets, which is impossible in the case of two photographs.

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