Safety Topographic Aero Film Under Service Conditions," *Canadian Journal Research*, F, 509–517 (1946).
(4) Calhoun, J. M., "The Physical Properties and Dimensional Stability of Safety Aerographic Film," PHOTOGRAMMETRIC ENGINEERING, Vol. XIII: 163–221 (1947).
(5) Atwell, B. J., "Some Factors Affecting the Physical Quality of the Image in Air Photographs," *Photogrammetric Record*, Vol. 1, No. 6: 13–37 (1955).
(6) Filmer, R. W., "A Study of the Effect of Differential Film Shrinkage on the Space Resection and Orientation of an Aerial Photograph," *The Photogrammetric Record*, Vol. III, 13: 60–82 (1959).
(7) Eden, J. A., "Dimensional Stability Requirements for Photogrammetry," *Photogrammetric Record*, Vol. I, 1: 49–50 (1953).
(8) Sadler, L. E., "The Significance of Reseau Photography in Triangulation Operations," PHOTOGRAMMETRIC ENGINEERING, Vol. XXIV: 132–135 (1958).
(9) McNeil, G. T., "Film Distortion," PHOTOGRAMMETRIC ENGINEERING, Vol. XVII: 605–609 (1951).
(10) Gollnow, H. and Hagemann, G., "Displacements of Photographic Emulsions and a Method of Processing to Minimize this Effect," *Astronomical Journal*, Vol. 61: 399–404 (1956).
(11) Bruchlacher, W. A. and Luder, W., "Untersuchung über die Schrumpfung von Messfilmen und Photographischen Plattenmaterial (Investigation Concerning the Shrinkage of Topographic Film and Photographic Plates)," Deutsche Geodätische Komission bei der Bayerischen Akademie der Wissenschaften, München, Applied Geodesy, Series B, No. 31 (1956).
(12) Altman, J. H. and Ball, R. C., "On the Spatial Stability of Photographic Plates," *Photographic Science and Engineering*, Vol. V, No. 5, 278–282 (1961).
(13) Calhoun, J. M., Keller, L. E., and Newell, R. F., Jr., "A Method for Studying Possible Local Distortions in Aerial Films," PHOTOGRAMMETRIC ENGINEERING, Vol. XXVI: 661–672 (1960).
(14). Calhoun, J. M., Adelstein, P. Z. and Parker, J. T., "Physical Properties of Estar Polyester Base Aerial Films for Topographic Mapping," PHOTOGRAMMETRIC ENGINEERING, Vol. XXVII: 461–470 (1961).
(15) Harman, W. E., Jr., "Recent Development in Aerial Film," PHOTOGRAMMETRIC ENGINEERING, Vol. XXVII: 151–154 (1961).
(16) Tollenaar, D., "Moiré Interference Phenomena in Halftone Printing," Research Institute T.N.O. for Printing and Allied Industries, ter Gouwstraat 1, Amsterdam-O, The Netherlands, 1957.
(17) Eastman Kodak Company publication, "Manual of Physical Properties of Kodak Aerial and Special Sensitized Materials," (1961).

*Calibration of a Precision Coordinate Comparator**

GEORGE H. ROSENFIELD
Photogrammetric Analyst
Mathematical Services, RCA Data Processing

(*Abstract is on next page*)

SECTION I:—INTRODUCTION

THE comparator is a precision coordinate measuring instrument which has been manufactured and adjusted to meet a particular level of accuracy when used in accordance with the manufacturer's specifications. However, by proper calibration it is possible to achieve additional accuracy in the measurements. The calibration procedure consists of determining an appropriate error model to describe the systematic errors of the comparator. The correction is then performed by operating on the observations with the correction equations obtained from the computed error model. Since calibration is not entirely a stable condition, it is necessary that recalibration be performed at periodic intervals. In the meantime a comparator evaluation will determine whether significant changes in accuracy have taken place.

* This work was performed under the direction of the RCA Ballistic Camera Accuracy Review (BACAR) Project.

RCA Missile Test Project
AIR FORCE MISSILE TEST CENTER
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
Patrick Air Force Base, Florida
Prepared by the RCA Service Company, Missile Test Project, under sub-contract to Pan American World Airways, Incorporated.
Release date by author—15 February 1962.

A coordinate measuring comparator, especially one of the precision lead-screw type, is affected by several types of systematic errors. These include: periodic screw-error, scale-error and secular screw-error, curvature and weave of the ways, and nonperpendicularity of the axes.

SECTION II:—PERIODIC SCREW-ERROR

Periodic screw-error is considered as the error associated with one complete revolution of the screw. These errors are also described by Bennett [1]. The method to be discussed was developed by Brown at Air Force Missile Test Center (AFMTC) [2], although it is fundamentally similar to that described by Bennett. However, Bennett concludes with a graphical evaluation of the periodic error. While this analysis has the purpose of both evaluating and calibrating the periodic error to achieve an increase in accuracy, it should be pointed out that an error model can be fitted to the data obtained in the method of Bennett, and thus be used for calibration purposes.

ABSTRACT: *The procedures for calibrating a screw-type, precision coordinate comparator are described. The errors to be calibrated are: periodic screw-error, scale-error and secular screw-error, curvature and weave of the ways, and non-perpendicularity of the axes. Periodic error is calibrated using a 2 mm. scale graduated into 0.1 mm. increments. The calibration model is formed by a least squares sine wave fit. Scale-error and secular screw-error are calibrated using a 240 mm. calibrated scale graduated into 1 mm. increments. The calibration model is formed by a least squares polynomial fit and a harmonic analysis. Curvature and weave of the ways are calibrated using a line divided into precision equal increments. The calibration model is formed using the principle of inversion, and a linear least square function fit and harmonic analysis. Non-perpendicularity of the axes is calibrated using non-calibrated points on a plate. The calibration model is formed using the principle of inversion.*

The calibration is performed for the individual corrections. The total correction is obtained by the cumulative summation of the individual corrections.

The periodic screw-error can be determined with the use of a graduated scale. It is not necessary that the scale be calibrated. The scale used at AFMTC is a 2 mm. scale graduated to 0.1 mm. Also available is a micrometer jig which will translate the scale a measured distance along its axis. The procedure is to make linear observations with the comparator on the scale graduations and to record the values. The scale is then moved 0.50 mm. with the jig and the set of observations repeated.

It is, of course, necessary that the observations be made with sufficient precision to make the calibration valid. A minimum of two sets of observations in each position is therefore necessary in order to establish this precision. Each set must be corrected for temperature fluctuations. The setting standard deviation is computed from the repeated readings and must not be greater than 0.5 microns. An average value for the observation is determined from the repeated readings.

An error model in the form of a sine-wave is then fitted to the averaged values of the observations. The sine-wave is described by the equation:

$$\delta x_1 = a \sin (u + b), \tag{2.1}$$

in which

$$u = 2\pi\Delta, \tag{2.2}$$

where Δ is the incremental comparator measurement deviating from the full mm. value.

The a coefficient represents the amplitude of the periodic error function and the b coefficient locates the origin of the sine-curve with reference to the full mm. graduation.

The description of the data handling and of the least squares adjustment as developed by Brown is given in Appendix A.

A satisfactory calibration has been achieved when the variance of the residuals from the adjustment is not significantly different from the standard error of the mean for the replicated readings. An F test may be used to establish the validity of the calibration.

The periodic-error may change slightly at different portions of the screw. Thus, it is advisable to evaluate and calibrate the periodic-error at several different places along the screw. An average value of the several calibrations can be used as the final-error model.

Similarly for the second comparator screw, the periodic-error function is computed by:

$$\delta y_1 = a' \sin (u' + b'). \tag{2.3}$$

Section III:—Scale Error and Secular Screw Error

Scale-error and secular screw-error are caused by variations in the pitch of the screw. These errors are also described by Bennett [1] as absolute cumulative error and as relative cumulative error. Scale or "absolute cumulative error is the difference between the true length of an object at standard temperature and its apparent length at the same temperature." Secular or relative cumulative error is "the difference between the apparent length of an object measured on the comparator screw and the apparent length of the same object measured on a perfectly uniform screw having a constant pitch." The method to be discussed is fundamentally similar to that described by Bennett. However, Bennett concludes with a graphical evaluation of the screw-errors, while this analysis has the purpose of both evaluating and calibrating the screw-errors to achieve an increase in accuracy. It should be pointed out that an error model can be developed for the data obtained in the method of Bennett, and thus used for calibration purposes.

Scale-error and secular screw-error can be determined with the use of a calibrated scale. For a 240 mm. screw, it is necessary to have a 240 mm. scale. The scale should be graduated at 1 mm. intervals and the graduation values recorded to 0.1 microns. The scale should be calibrated to have a standard error of not more than 0.3 microns. The scale is approximately aligned with the axis of the screw to be calibrated. Linear observations are made with the comparator on the scale graduations and the values recorded. The deviations of the comparator values from the calibrated scale for equal 1 mm. intervals are obtained by the expression:

$$e = S + \delta - M, \tag{3.1}$$

in which

S = the equal 1 mm. increment,
δ = the calibrated scale deviation,
M = the comparator value.

It is, of course, necessary that the e_i values be obtained with sufficient precision to make the calibration valid. A minimum of two sets of observations is therefore necessary in order to establish this precision. Each set must be corrected for temperature fluctuations. The setting standard deviation is computed from the repeated readings and must not be greater than 0.5 microns. An average value of e_i is determined from these readings.

The e_i values are a function of 1) the scale-error of the comparator and 2) the secular screw-errors of the comparator. Analysis of these errors may be handled by any of several techniques. A suggested method is to first fit a polynomial in the form:

$$\delta x_2 = a_0 + a_1\bar{x} + a_2\bar{x}^2 + a_3\bar{x}^3, \tag{3.2}$$

in which

$$\bar{x} = x - x_0, \tag{3.3}$$

where x is the equal 1 mm. interval and x_0 is the origin of the calibration to the set of data. The constant term represents the displacement of the data at the origin; the linear coefficient represents the scale-error of the screw; and the higher order coefficients represent the long period secular errors. The residuals from this fit will be a function of the higher order harmonics of the secular screw-error. These n higher order harmonics may be evaluated from the residuals by a harmonic analysis [3] in the form:

$$\delta x_3 = A_0 + \sum_{k=1}^{n} A_k \sin k\bar{x}' + \sum_{k=1}^{n-1} B_k \cos k\bar{x}'. \tag{3.4}$$

The A_k, B_k coefficients which are significant represent the harmonics which are causing the remaining secular screw-errors, and

$$\bar{x}' = \frac{\pi}{n}\,\bar{x}. \tag{3.5}$$

A satisfactory calibration has been achieved when the variance of the residuals from the harmonic analysis is not significantly different from the combined variance of the scale calibration and the standard error of the mean for the repeated readings. The combined variance is obtained by:

$$\sigma^2 = \left[\sigma_s^2 + \sigma_r^2\right]. \tag{3.6}$$

An F test may be used to establish the validity of the calibration.

Similarly for the second comparator screw, the displacement, the scale-error, and long period secular-errors, are computed by:

$$\delta y_2 = a_0' + a_1'\bar{y} + a_2'\bar{y}^2 + a_3'\bar{y}^3 \tag{3.7}$$

in which

$$\bar{y} = y - y_0 \tag{3.8}$$

where y is the equal 1 mm. interval and y_0 is the origin of the calibration. The n higher order harmonics of the secular screw-errors may be evaluated from the residuals by the harmonic analysis:

$$\delta y_3 = A_0' + \sum_{k=1}^{n} A_k' \sin k\bar{y}' + \sum_{k=1}^{n-1} B_k' \cos k\bar{y}', \tag{3.9}$$

in which

$$\bar{y}' = \frac{\pi}{n}\,\bar{y}. \tag{3.10}$$

Section IV:—Curvature and Weave of the Ways

Curvature and weave of the ways is the deviation of the motion of the comparator stage from a straight line. Curvature and weave in a comparator way causes measuring error in the coordinate direction perpendicular to that way. Curvature and weave of the ways can be determined with a scribed straight line graduated at precision equal intervals. A specially scribed scale or the central line of a grid may be used. The line itself does not have to be calibrated since the principle of inversion is used in the error determination. Since the line is scribed on only one surface it is necessary to mount the line in contact with a similar unscribed glass (strip or plate) of the same thickness. All the glass used should be plane-parallel. This will allow the line to be returned to the same horizontal plane after inverting, and prevent any major refocusing of the miscroscope.

The line to be measured is aligned with the axis of the comparator way under investigation, and the perpendicular coordinate is measured at each equally-spaced graduation mark. The linear and perpendicular coordinate of the central point on the line are recorded. The plate is then inverted about the axis of the line and returned to its original position on the comparator. It is possible to return the central point on the line to within several microns of its original position. The line is again aligned with the axis of the comparator and the observations repeated.

It is, of course, necessary that the observations be made with sufficient precision to make the calibration valid. A minimum of two sets of observations in each position is therefore necessary in order to establish this precision. Each set must be corrected for temperature fluctuations. The setting standard deviation is computed from the replicated readings and must not be greater than 0.5 microns. An average value for the observations is determined from such readings.

An average value for the perpendicular coordinate at each graduation mark along the line is computed from the direct and inverted sets of observations. This set of average values represents an ideal straight theoretical line which is displaced from, and at an angle to, the comparator axis under investigation. A linear polynomial in the form:

$$y' = a_0 + a_1\bar{x}, \tag{4.1}$$

in which

$$\bar{x} = x - x_0 \tag{4.2}$$

where x is the equal 1 mm. interval and x_0 is the origin of the calibration, is fit to the set of data. The constant term represents the displacement of the line from the axis at the origin, and the linear coefficient represents the slope of the line with the axis. The residuals from this fit will be a function of the curvature and weave of the comparator way.

The n harmonics of curvature and weave of the comparator way may now be evaluated from the residuals by a harmonic analysis [3] in the form

$$\delta y_4 = A_0 + \sum_{k=1}^{n} A_k \sin k\bar{x}' + \sum_{k=1}^{n-1} B_k \cos k\bar{x}', \tag{4.3}$$

in which

$$\bar{x}' = \frac{\pi}{n}\,\bar{x}. \tag{4.4}$$

The A_k, B_k coefficients which are significant represent the harmonics which are causing the curvature and weave errors.

A satisfactory calibration has been achieved when the variance of the residuals from the harmonic analysis is not significantly different from the standard error of the mean for the replicated readings. An F test may be used to establish the validity of the calibration.

Similarly for the second comparator way, the displacement and slope of the ideal straight theoretical line are computed by:

$$x' = a_0' + a_1'\bar{y}, \tag{4.5}$$

in which

$$\bar{y} = y - y_0, \tag{4.6}$$

where y is the equal 1 mm. interval and y_0 is the origin of the calibration, and the n harmonics of curvature and weave evaluated from the residuals by the harmonic analysis:

$$\delta x_4 = A_0' + \sum_{k=1}^{n} A_k' \sin k\bar{y}' + \sum_{k=1}^{n-1} B_k' \cos k\bar{y}', \tag{4.7}$$

in which

$$\bar{y}' = \frac{\pi}{n}\bar{y}. \tag{4.8}$$

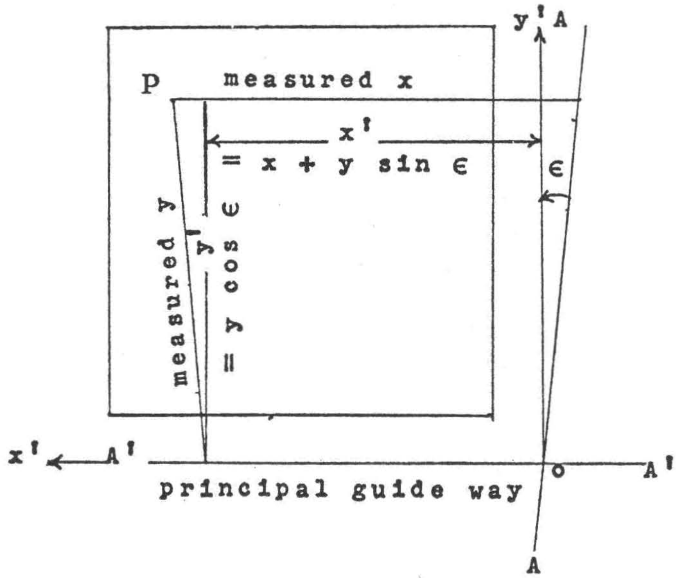

FIG. 1. Relation Between Comparator Coordinates x, y, and Rectangular Coordinates x', y' (after Zug).

SECTION V:—NONPERPENDICULARITY OF THE AXES

Nonperpendicularity of the comparator axes may be represented by the correction angle ϵ through which the secondary guide way must be rotated to be perpendicular to the principal guide way. Figure 1 illustrates the geometry of the coordinate comparator.

Determination of the nonperpendicularity error may be performed by any of several methods. One method presented by Zug [4] is based upon the principal of inversion of the plate about its axis parallel to the comparator primary axis. Another method is based upon using a precision or calibrated grid or graduated circle. In the former method the nonperpendicularity calibration is dependent upon the reading

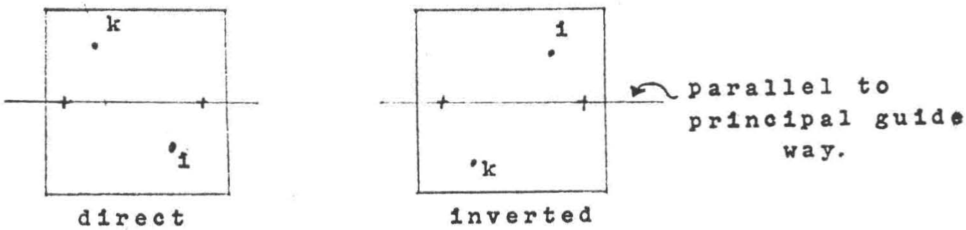

FIG. 2. Comparator Measures for Determination of ϵ (after Zug).

error. In the latter method, it is dependent upon both the reading error and the accuracy of the grid calibration.

The development presented below is based upon the method of inversion. It is suggested that the observations be made on a grid plate.

Two widely separated points on the grid x-axis are selected as reference points. A sufficient number of points on both sides of the grid x-axis are selected as the data points. It is suggested that at least four points be used to give sufficient redundancy and strength to the solution. With the plate in the direct position, the grid x-axis is approximately aligned with the comparator principal-axis and the reference marks observed. The points i and k are identified and measured. The plate is then inverted about its own x-axis, the plate again aligned, and the reference marks observed. The points i and k are again measured. The measuring procedure is illustrated in Figure 2.

It is, of course, necessary that the observations be made with sufficient precision to make the calibration valid. A minimum of two sets of observations in each position is therefore necessary in order to establish this precision. Each set must be corrected for temperature fluctuations. The setting standard deviation is computed from the repeated readings and must not be greater than 0.5 microns. An average value for the observation is determined from these readings.

The average value of the observations in each position is corrected for the periodic and systematic errors of the screw and of the comparator ways. The corrected values are then rotated parallel to the comparator principal-axis, and centered at the comparator origin, by the equations:

$$x_i' = \frac{-(2y_i - y_r - y_l)(y_r - y_l) + (2x_i - x_r - x_l)(x_r - x_l)}{2M}, \tag{5.1}$$

$$y_i' = \frac{(2y_i - y_r - y_l)(x_r - x_l) + (2x_i - x_r - x_l)(y_r - y_l)}{2M},$$

in which

$$M = [(x_r - x_l)^2 + (y_r - y_l)^2]^{1/2}, \tag{5.2}$$

Determination of the nonperpendicularity error is performed using the observations as transformed into the comparator coordinate system. The development of the condition equations for the method of plate inversion is taken freely from Zug [4].

The relationship between the measured comparator coordinates x', y' and the rectangular coordinates x'', y'' of the point P is shown from Figure 1 to be in the form:

$$x'' = x' + y' \sin \epsilon, \qquad y'' = y' \cos \epsilon. \tag{5.3}$$

After the comparator coordinates x, y for images i and k are measured, corrected, and transformed, the rectangular coordinates x'' for images i and k are given by the first of Equation (5.3) as:

$$
\begin{aligned}
x_i'' &= x_i' + y_i' \sin \epsilon, \\
x_k'' &= x_k' + y_k' \sin \epsilon,
\end{aligned}
\tag{5.4}
$$

whence,

$$x_k'' - x_i'' = (x_k' - x_i') + (y_k' - y_i') \sin \epsilon. \tag{5.5}$$

It is obvious that the difference between the rectangular coordinates x'' for image i and k will be unaltered by inversion of the plate. If the comparator coordinates in the inverted position are denoted by $\underline{x}$, $\underline{y}$, we may then write for the inverted position:

$$x_k'' - x_i'' = (\underline{x}_k' - \underline{x}_i') + (\underline{y}_k' - \underline{y}_i') \sin \epsilon. \tag{5.6}$$

combining (5.5) and (5.6) there results as the expression for:

$$\sin \epsilon = - \left[\frac{(x_k' - x_i') - (\underline{x}_k' - \underline{x}_i')}{(y_k' - y_i') - (\underline{y}_k' - \underline{y}_i')} \right]. \tag{5.7}$$

An averaging solution to compute the value, $\sin \epsilon$, may be performed using redundant data. It is suggested that four points equally distributed on the plate be selected. Distribution of the points is shown in Figure 3. The selected points are to be used in pairs, each pair of points giving rise to a condition equation. Since the

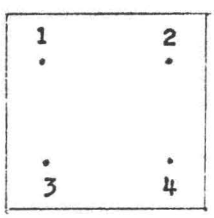

FIG. 3. Suggested Distribution of Points.

solution becomes indeterminate when the pair of points is parallel to the principal guide way, it is suggested that the following combinations be used:

1–3 2–3

1–4 2–4.

An intermediate solution is computed for each jth pair of points. An average value is obtained from the intermediate solutions by:

$$\sin \epsilon = \frac{\sum_{j=1}^{4} \sin \epsilon_j}{4} \tag{5.8}$$

The value for $\cos \epsilon$ may then be computed by:

$$\cos \epsilon = [1 - \sin^2 \epsilon]^{1/2} \tag{5.9}$$

The standard deviations of the individual $\sin \epsilon$ about the average value is obtained by the usual method. A satisfactory calibration has been achieved when the variance is not significantly different from the standard error of the mean for the repeated readings. An F test may be used to establish the validity of the calibration.

With the values for $\sin \epsilon$ and $\cos \epsilon$ computed, the corrections to the data points for the nonperpendicularity error may be computed from equation (5.3) by:

$$\delta x_5 = y \sin \epsilon, \qquad \delta y_5 = y(\cos \epsilon - 1). \tag{5.10}$$

SECTION VI:—CORRECTION FOR COMPARATOR CALIBRATION

The total corrections for the entire comparator calibration are determined by summing the individual corrections from each phase of the calibration. Thus:

$$\begin{aligned} \delta x &= \delta x_1 + \delta x_2 + \delta x_3 + \delta x_4 + \delta x_5, \\ \delta y &= \delta y_1 + \delta y_2 + \delta y_3 + \delta y_4 + \delta y_5. \end{aligned} \tag{6.1}$$

The individual corrections are obtained using the calibration error model and the calibration coefficients determined independently for each error of the comparator, as follows:

Periodic Screw Error:

$$\delta x_1 = a \sin (u + b), \tag{6.2}$$

$$\delta y_1 = a' \sin (u' + b'). \tag{6.3}$$

in which

$$u = 2\pi\Delta, \tag{6.4}$$

where Δ is the incremental comparator measurement deviating from the full mm. value; b, b' are the origins of the calibrations with reference to the full millimeter graduations; a, a' are the amplitudes.

Scale Error and Secular Screw Error:

$$\begin{aligned} \delta x_2 &= a_0 + a_1\bar{x} + a_2\bar{x}^2 + a_3\bar{x}^3, \\ \delta x_3 &= A_0 + \sum_{k=1}^{n} A_k \sin k\bar{x}' + \sum_{k=1}^{n-1} B_k \cos k\bar{x}', \end{aligned} \tag{6.5}$$

in which

$$\bar{x} = x - x_0, \qquad \bar{x}' = \frac{\pi}{n}\,\bar{x}, \tag{6.6}$$

where x is the measurement and x_0 is the origin of the calibration

The a_i, A_i, and B_i, are the calibration coefficients.

$$\delta y_2 = a_0' + a_1'\bar{y} + a_2'\bar{y}^2 + a_3'\bar{y}^3, \tag{6.7}$$

$$\delta y_3 = A_0' + \sum_{k=1}^{n} A_k' \sin k\bar{y}' + \sum_{k=1}^{n-1} B_k' \cos k\bar{y}'.$$

in which

$$\bar{y} = y - y_0, \qquad \bar{y}' = \frac{\pi}{n}\,\bar{y}, \tag{6.8}$$

where y is the measurement and y_0 is the origin of the calibration.

The a_i', A_i', B_i' are the calibration coefficients.

Curvature and Weave of the Ways

$$\delta x_4 = A_0' + \sum_{k=1}^{n} A_k' \sin k\bar{y}' + \sum_{k=1}^{n-1} B_k' \cos k\bar{y}', \tag{6.9}$$

$$\delta y_4 = A_0 + \sum_{k=1}^{n} A_k \sin k\bar{x}' + \sum_{k=1}^{n-1} B_k \cos k\bar{x}'. \tag{6.10}$$

in which

$$\bar{x} = x - x_0, \qquad \bar{y} = y - y_0,$$

$$\bar{x}' = \frac{\pi}{n}\bar{x}, \qquad \bar{y}' = \frac{\pi}{n}\bar{y}, \tag{6.11}$$

where x, y, are the measurements and x_0, y_0, are the origin of the calibrations.
The A_i, A_i', B_i, B_i' are the calibration coefficients.

Nonperpendicularity of the Axes:

$$\delta x_5 = y \sin \epsilon, \tag{6.12}$$

$$\delta y_5 = y(\cos \epsilon - 1). \tag{6.13}$$

The values of the observations, corrected for comparator calibration are obtained by:

$$x' = x + \delta x,$$

$$y' = y + \delta y. \tag{6.14}$$

References

1. Bennett, J. M. "Method for Determining Comparator Screw-Errors with Precision," *Journal of the Optical Society of America*, Vol. 51, No. 10, October 1961, pp. 1133–1138.
2. Brown, D. C., unpublished document, RCA Missile Test Project, 1957.
3. Sokolnikoff, I. S. and Redheffer, R. M., "Mathematics of Physics and Modern Engineering," McGraw Hill Book Co., New York, 1958, pp. 711–715.
4. Zug, R. S., "High Altitude Range Bombing by the Aberdeen Bombing Mission, Using Ballistic Cameras," Laboratory Services Division, Report No. 1, Ordnance Research and Development Center, Aberdeen, Md., December 10, 1945, pp. 112–117.

Acknowledgment

The author wishes to thank Mr. R. Gugel and Mr. F. Collen of RCA Optical Systems Data Reduction for their contributions and assistance in developing these calibration techniques. Mr. Gugel established the measuring procedures and supervised the measuring operation. Mr. Collen prepared the error model for calibrating periodic-error. Unsung praise must be given to the comparator operators for their skill and precision in making the measuring observations.

Mr. J. Duncan, RCA Mathematical Services, especially is given thanks for his contribution in the field of Numerical Analysis and Computer Programming. Mr. Duncan programmed the data handling and calibration computations. He also established the validity of the calibration error models.

Some very basic ideas concerning calibration of comparators were obtained from discussions with Dr. H. H. Schmid, Ballistic Research Laboratories, Aberdeen, Maryland.

Appendix A

Periodic Error Study

Prepared by F. Collen, *Data Reduction Analyst, Optical Systems Data Reduction, RCA Data Processing*

In the development of the mathematics and the computation of lead screw periodic error at the AFMTC, Brown considered only the simplest form of periodic-error. In the ideal case, the periodic-error can be expressed in terms of one turn of the lead-screw and in every complete rotation the characteristics of the error will be the same. It was only logical, then, that Brown used the sine-curve to describe the direction and magnitude of the error. To my knowledge Brown has not distributed this work in published form.

First, let $\bar{x}$ represent the true measurement and x, the measurement which includes the periodic error with amplitude a. Further, let u be a measure of the screw's rotation and let b locate the origin of the sine-curve with reference to the full mm graduation. Brown recommends the following expression as one practical for evaluation of the lead-screw, and convenient for correcting the readings which contain periodic error:

$$\bar{x} = x + a \sin (u + b) \tag{A.1}$$

in which x is the full comparator measurements and:

$$u = 2\pi\Delta, \tag{A.2}$$

where Δ is the incremental comparator measurement deviation from the full mm value.

In an effort to minimize scale-errors, the prescribed procedure is to first perform the readings of the scale in an initial position and then repeat the readings in an offset position in which the scale is moved 0.50 mm.

If we have several readings of the scale in the initial positions, these can be written as:

$$\begin{aligned} \bar{x}_{11} &= x_{11} + a \sin (u_{11} + b) \\ \bar{x}_{12} &= x_{12} + a \sin (u_{12} + b) \\ \bar{x}_{13} &= x_{13} + a \sin (u_{13} + b) \\ &\vdots \\ \bar{x}_{1n} &= x_{1n} + a \sin (u_{1n} + b). \end{aligned} \tag{A.3}$$

The expressions for the readings in the offset position are similar:

$$\begin{aligned} \bar{x}_{21} &= x_{21} + a \sin (u_{21} + b) \\ \bar{x}_{22} &= x_{22} + a \sin (u_{22} + b) \\ \bar{x}_{23} &= x_{23} + a \sin (u_{23} + b) \\ &\vdots \\ \bar{x}_{2n} &= x_{2n} + a \sin (u_{2n} + b). \end{aligned} \tag{A.4}$$

Subtract the readings in the initial position from the corresponding readings in the offset position. These differences can be expressed as:

$$\begin{aligned}
\bar{x}_{21} - \bar{x}_{11} &= x_{21} - x_{11} + a\sin(u_{21}+b) - a\sin(u_{11}+b) \\
\bar{x}_{22} - \bar{x}_{12} &= x_{22} - x_{12} + a\sin(u_{22}+b) - a\sin(u_{12}+b) \\
\bar{x}_{23} - \bar{x}_{13} &= x_{23} - x_{13} + a\sin(u_{23}+b) - a\sin(u_{13}+b) \\
&\vdots \\
\bar{x}_{2i} - \bar{x}_{1i} &= x_{2i} - x_{1i} + a\sin(u_{2i}+b) - a\sin(u_{1i}+b) \\
&\vdots \\
\bar{x}_{2n} - \bar{x}_{1n} &= x_{2n} - x_{1n} + a\sin(u_{2n}+b) - a\sin(u_{1n}+b).
\end{aligned} \tag{A.5}$$

The general Equation (A.5) may be rewritten as:

$$x_i' = K_i + a[\sin(u_{2i}+b) - \sin(u_{1i}+b)] \tag{A.6}$$

in which

$$K_i = x_{2i} - x_{1i}, \tag{A.7}$$

$$x_i' = \bar{x}_{2i} - \bar{x}_{1i}. \tag{A.8}$$

Trigonometric substitution leads to:

$$x' = K + a[\cos b(\sin u_2 - \sin u_1) + \sin b(\cos u_1 - \cos u_2)] \tag{A.9}$$

Equation (A.9) can be further reduced to:

$$x' = K + A(\sin u_2 - \sin u_1) + B(\cos u_1 - \cos u_2), \tag{A.10}$$

in which

$$A = a\cos b, \qquad B = a\sin b. \tag{A.11}$$

Trigonometric substitution again leads to:

$$\begin{aligned}
x' = K &+ A[2\cos\tfrac{1}{2}(u_2+u_1)\sin\tfrac{1}{2}(u_2-u_1)] \\
&- B[2\sin\tfrac{1}{2}(u_2+u_1)\sin\tfrac{1}{2}(u_2-u_1)].
\end{aligned} \tag{A.12}$$

Recalling that the displacement between the two sets of readings was specified as 0.50 mm. it follows that:

$$u_2 - u_1 = 180 \text{ degrees}. \tag{A.13}$$

Therefore:

$$\sin\tfrac{1}{2}(u_2 - u_1) = 1, \tag{A.14}$$

and Equation (A.12) simplifies to:

$$x' = K + 2A\cos\tfrac{1}{2}(u_2+u_1) - 2B\sin\tfrac{1}{2}(u_2+u_1). \tag{A.15}$$

One more step involving the above principle that:

$$u_2 = u_1 + 180°; \tag{A.16}$$

will simplify Equation (A.15) to the convenient form:

$$x' = K - 2A\sin u - 2B\cos u. \tag{A.17}$$

Least Squares Solution

From the basic condition equation described above:

$$x' = K - 2A \sin u + 2B \cos u, \tag{A.18}$$

may be derived the following set of normal equations:

$$* \begin{bmatrix} N & -\sum \sin u_i & -\sum \cos u_i \\ & \sum \sin^2 u_i & \sum \sin u_i \cos u_i \\ & & \sum \cos^2 u_i \end{bmatrix} \begin{bmatrix} K \\ 2A \\ 2B \end{bmatrix} = \begin{bmatrix} \sum x_i' \\ -\sum x_i' \sin u_i \\ -\sum x_i' \cos u_i \end{bmatrix}. \tag{A.19}$$

Solution of the normal equations takes the form:

$$* \begin{bmatrix} K \\ 2A \\ 2B \end{bmatrix} = \begin{bmatrix} N & -\sum \sin u_i & -\sum \cos u_i \\ & \sum \sin^2 u_i & \sum \sin u_i \cos u_i \\ & & \sum \cos^2 u_i \end{bmatrix}^{-1} \begin{bmatrix} \sum x_i' \\ -\sum x_i' \sin u_i \\ -\sum x_i' \cos u_i \end{bmatrix} \tag{A.20}$$

Finally,

$$a = \sqrt{A^2 + B^2}, \quad \text{and} \quad b = \tan^{-1} B/A \tag{A.21}$$

in the original form of the periodic error equation:

$$\bar{x} = x + a \sin (u + b), \tag{A.22}$$

The correction equation for the periodic error then takes the form:

$$\delta x = a \sin (u + b), \tag{A.23}$$

in which

$$\delta x = (\bar{x} - x). \tag{A.24}$$

Error Propagation

Brown utilizes the inverse of the normal equation coefficient matrix from the least squares solution in the study of errors associated with the computation of the coefficients K, A and B. The error bounds for the N points along the sine curve may be computed by the equation:

$$* \sigma_i^2 = \tfrac{1}{2}\sigma_0^2 (1 - \sin u_i - \cos u_i) \begin{bmatrix} N & -\sum \sin u_i & -\sum \cos u_i \\ & \sum \sin^2 u_i & \sum \sin u_i \cos u_i \\ & & \sum \cos^2 u_i \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -\sin u_i \\ -\cos u_i \end{bmatrix}, \tag{A.25}$$

in which

σ_0 is the mean error of the residuals obtained in fitting the sine curve through the empirical data.

Note: * represents a symmetric matrix.