*Analytical Aerotriangulation at the University of Illinois**

A. A. El-Assal,
Univ. of Illinois
Urbana, Ill.

Abstract: *This article deals with analytical triangulation of single photographic strips. The different steps (analytical relative orientation, computation of the coordinates of the points in each model, assembly of the strip, and the absolute orientation of the assembled strip) are dealt with in detail. The general approach used by Mr. Schut of the National Research Council of Canada is followed, with a different mathematical formulation of the problem. Simple matrix operations are used, thus considerably facilitating the programming task for electronic computations. A program has been made for triangulation of single strips on IBM 650. The adjustment of photographic strips and blocks of strips is presently handled at the university of Illinois according to any of the existing methods of adjustment.*

Analytical Aerotriangulation could be defined as the process of control extension by means of two-dimensional measurements on photographs. Several methods and techniques are now in use for the solution of this problem. As could be expected, each of these methods has its advantages and disadvantages. After a thorough study of the different methods of analytical aerotriangulation, the general approach used in the method originated by Mr. G. H. Schut of the National Research Council of Canada was chosen. In Schut's approach the condition of resection of corresponding rays is used as a criterion. At the University of Illinois, a different mathematical formulation of the problem has been used. Simple matrix operations are used, thus facilitating considerably the programming task for electronic computations.

The present publication deals only with analytical triangulation of single strips. The problem has been programmed for the IBM 650. The residuals in the linearized condition equations were considered as random variables with equal weight. A rigorous mean square solution for the relative and absolute orientation will be the subject of another paper now being prepared.

As far as the adjustment of blocks of strips is concerned, this is being handled at the University of Illinois using any of the existing methods of adjustment (e.g. Zarzycki's method, Zeller's method of block adjustment, Karara's Cross Bases Method etc.)

Analytical aerotriangulation of single strips involve the following sequence of steps:

1. Obtaining the x and y coordinates of the different ground-control and pass-points in the successive photographs.
2. Applying corrections to these coordinates to eliminate as much as possible the errors due to lens distortion, film distortion (according to the method described in entry 2 of the bibliography) and atmospheric refraction (according to Leijonhufvud, See bibliography entry No. 8).
3. Analytical relative orientation of each two successive photographs.
4. Computation of the model coordinates of the points in each model.

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5. Assembly of the photographic strip.
6. Absolute orientation of the assembled strip (orthogonal transformation of model coordinates into a chosen ground-coordinates system).
7. Strip and block adjustment.

Step 1 is considered self explanatory and is left out of the present presentation. Step 7 has been explained above. In the following, steps 3 through 6 are dealt with in detail:

1. Analytical Relative Orientation

Let O' and O'' (see Figure 1) be two successive camera stations (projection-centers), and choose any right-handed orthogonal coordinate system with origin at O'. For any point I imaged in both pictures (i' and i''), let $\mathbf{p}_i' = (X_i', Y_i', Z_i')^T$ be the vector from O' to i', $\mathbf{p}_i'' + \mathbf{b} = (X_i'' + b_x, Y_i'' + b_y, Z_i'' + b_z)^T$ be the vector from O' to i'', and $\mathbf{b} = (b_x, b_y, b_z)^T$ be the vector from O' to O''.

As we all know, relative orientation is accomplished when corresponding rays (e.g. $O'\ i'$ and $O''\ i''$) intersect. A necessary and sufficient condition for such an intersection is:

$$\begin{vmatrix} b_x & b_y & b_z \\ X_i' & Y_i' & Z_i' \\ X_i'' + b_x & Y_i'' + b_y & Z_i'' + b_z \end{vmatrix} = 0 \tag{1}$$

Equation (1) can be written as follows:

$$\begin{vmatrix} b_x & b_y & b_z \\ X_i' & Y_i' & Z_i' \\ X_i'' & Y_i'' & Z_i'' \end{vmatrix} = 0 \tag{2}$$

Using the following notations:

$$D_i^1 = \begin{vmatrix} b_y & b_z \\ Y_i' & Z_i' \end{vmatrix}, \qquad D_i^2 = \begin{vmatrix} b_z & b_x \\ Z_i' & X_i' \end{vmatrix}, \qquad D_i^3 = \begin{vmatrix} b_x & b_y \\ X_i' & Y_i' \end{vmatrix}$$

in conjunction with expanding Equation (2), we get:

$$X_i'' \cdot D_i' + Y_i'' \cdot D_i^2 + Z_i'' \cdot D_i^3 = 0 = F_i \tag{3}$$

where F_i is a function of the elements of the lefthand side of the Equation (3).

The value of κ', ϕ' and ω' of the left camera (O'), as well as b_x could be assumed to be any values. The object of the relative orientation is to compute the value of κ'', ϕ'' and ω'' of the right camera (O'') together with the base-components b_y and b_z.

Using Taylor's theorem to expand F_i in equation (3) in terms of differential changes in κ'', ϕ'', ω'', b_y, and b_z we get:

$$F_i = F_{i0} + \frac{\partial F_i}{\partial \kappa''} \cdot \Delta\kappa'' + \frac{\partial F_i}{\partial \phi''} \cdot \Delta\phi'' + \frac{\partial F_i}{\partial \omega''} \cdot \Delta\omega'' + \frac{\partial F_i}{\partial b_y} \cdot \Delta b_y + \frac{\partial F_i}{\partial b_z} \cdot \Delta b_z \tag{4}$$

An equation similar to Equation (4) could be written for each point used in the relative orientation process (n point result then in n condition equations.) Such a set of condition equations can be written in matrix notation as follows:

$$\mathbf{B}_{n\times 5} \cdot \mathbf{\Delta}_{5\times 1} = \mathbf{L}_{n\times 1} \tag{5}$$

A least square estimate of the vector $\mathbf{\Delta}$ is then obtained from the solution of the following matrix equation:

$$\tilde{\mathbf{\Delta}} = (B^T \cdot B)^{-1} \cdot B^T \cdot L \tag{6}$$

The value obtained for the column vector $\tilde{\Delta}$ is used to obtain better values for κ'', ϕ'', ω'', b_y, b_z to start a new iteration process. The process continues until the absolute value of each of the corrections to κ'', ϕ'', ω'', is less than a preassigned value ϵ.

To obtain the coefficients of Equation (4), we make use of the following six matrices:

$$\boldsymbol{A}_\kappa = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_\phi = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}, \quad \boldsymbol{A}_\omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}$$

$$\frac{d\boldsymbol{A}_\kappa}{d\kappa} = \begin{bmatrix} -\sin\kappa & \cos\kappa & 0 \\ -\cos\kappa & -\sin\kappa & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{d\boldsymbol{A}_\phi}{d\phi} = \begin{bmatrix} -\sin\phi & 0 & -\cos\phi \\ 0 & 0 & 0 \\ \cos\phi & 0 & -\sin\phi \end{bmatrix}$$

$$\frac{d\boldsymbol{A}_\omega}{d\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\omega & \cos\omega \\ 0 & -\cos\omega & -\sin\omega \end{bmatrix} \tag{7}$$

to arrive to the following basic matrices:

$$\boldsymbol{A} = \boldsymbol{A}_\kappa \cdot \boldsymbol{A}_\phi \cdot \boldsymbol{A}_\omega \tag{8.1}$$

$$\frac{\partial \boldsymbol{A}}{\partial\kappa} = \frac{d\boldsymbol{A}_\kappa}{d\kappa} \cdot \boldsymbol{A}_\phi \cdot \boldsymbol{A}_\omega \tag{8.2}$$

$$\frac{\partial \boldsymbol{A}}{\partial\phi} = \boldsymbol{A}_\kappa \cdot \frac{d\boldsymbol{A}_\phi}{d\phi} \cdot \boldsymbol{A}_\omega \tag{8.3}$$

$$\frac{\partial \boldsymbol{A}}{\partial\omega} = \boldsymbol{A}_\kappa \cdot \boldsymbol{A}_\phi \cdot \frac{d\boldsymbol{A}_\omega}{d\omega} \tag{8.4}$$

where:

Matrix $\boldsymbol{A}$ is the well known matrix representing successive rotations ω, ϕ, and κ about the orthogonal axes X, Y, and Z respectively, taking the X-axis, Y-axis and Z-axis as primary, secondary and tertiary axes, respectively. Matrix $\boldsymbol{A}'$ is the matrix evaluated for the assumed rotations of camera O'. This is done once during the solution of the orientation of camera O'' relative to camera O'.

The matrices

$$\boldsymbol{A}'', \quad \frac{\partial \boldsymbol{A}''}{\partial\kappa''}, \quad \frac{\partial \boldsymbol{A}''}{\partial\phi''} \text{ and } \frac{\partial \boldsymbol{A}''}{\partial\omega''}$$

are evaluated for the camera O'' and have to be evaluated for every iteration cycle.

For every point i in the model, the planimetric coordinates (x_i' and y_i') in the photograph (O') and (x_i'' and y_i'') in the photograph (O'') are used to compute the following vectors:

$$(X_i', Y_i', Z_i')^T = \boldsymbol{A}' \cdot \boldsymbol{q}_i' \tag{9.1}$$

$$(X_i'', Y_i'', Z_i'')^T = \boldsymbol{A}'' \cdot \boldsymbol{q}_i'' \tag{9.2}$$

$$\left(\frac{\partial X_i''}{\partial\kappa''}, \frac{\partial Y_i''}{\partial\kappa''}, \frac{\partial Z_i''}{\partial\kappa''}\right)^T = \frac{\partial \boldsymbol{A}''}{\partial\kappa''} \cdot \boldsymbol{q}_i'' \tag{9.3}$$

$$\left(\frac{\partial X_i''}{\partial\phi''}, \frac{\partial Y_i''}{\partial\phi''}, \frac{\partial Z_i''}{\partial\phi''}\right) = \frac{\partial \boldsymbol{A}''}{\partial\phi''}\cdot\boldsymbol{q}_i'' \tag{9.4}$$

$$\left(\frac{\partial X_i''}{\partial\omega''}, \frac{\partial Y_i''}{\partial\omega''}, \frac{\partial Z_i''}{\partial\omega''}\right) = \frac{\partial \boldsymbol{A}''}{\partial\omega''}\cdot\boldsymbol{q}_i'' \tag{9.5}$$

where

$$\boldsymbol{q}_i' = (x_i', y_i', \pm f)^T$$

and

$$\boldsymbol{q}_i'' = (x_i'', y_i'', \pm f)^T$$

The sign of f is taken *Positive* if negative transparancies are used, and *Negative* if diapositives are used.

The elements of the ith row of the matrix $\boldsymbol{B}$ in equation (5)

$$= \left(\frac{\partial F_i}{\partial\kappa''}, \frac{\partial F_i}{\partial\phi''}, \frac{\partial F_i}{\partial\omega''}, \frac{\partial F_i}{\partial b_y}, \frac{\partial F_i}{\partial b_z}\right)$$

and the ith row of the matrix $L = -F_{oi}$ could now be evaluated as follows:

$$\frac{\partial F_i}{\partial\kappa''} = \frac{\partial X_i''}{\partial\kappa''}\cdot D_i^1 + \frac{\partial Y_i''}{\partial\kappa''}\cdot D_i^2 + \frac{\partial Z_i''}{\partial\kappa''}\cdot D_i^3 \tag{10.1}$$

$$\frac{\partial F_i}{\partial\phi''} = \frac{\partial X_i''}{\partial\phi''}\cdot D_i^1 + \frac{\partial Y_i''}{\partial\phi''}\cdot D_i^2 + \frac{\partial Z_i''}{\partial\phi''}\cdot D_i^3 \tag{10.2}$$

$$\frac{\partial F_i}{\partial\omega''} = \frac{\partial X_i''}{\partial\omega''}\cdot D_i^1 + \frac{\partial Y_i''}{\partial\omega''}\cdot D_i^2 + \frac{\partial Z_i''}{\partial\omega''}\cdot D_i^3 \tag{10.3}$$

$$\frac{\partial F_i}{\partial b_y} = X_i''\cdot Z_i' - Z_i''\cdot X_i' \tag{10.4}$$

$$\frac{\partial F_i}{\partial b_z} = Y_i''\cdot X_i' - X_i''\cdot Y_i' \tag{10.5}$$

$$-F_{oi} = -X_i''\cdot D_i' - Y_i''\cdot D_i^2 - Z_i''\cdot D_i^3 \tag{10.6}$$

Equations (10) thus represent the six coefficients of equation (4) for any point i.

2. Computation of Model Coordinates

Because of the errors in the observed x and y coordintes of the image points, the the vectors $\boldsymbol{p}_i'$ and $\boldsymbol{p}_i''$ will, in general, not lie in one plane, and hence will not intersect. The mid point of the line segment which represent the shortest distance between the two vectors will be considered as the point of intersection.

The vector equation of the line segment joining any two points on the vectors $\boldsymbol{p}_i'$ and $\boldsymbol{p}_i''$ is as follows:

$$\boldsymbol{b} + \mu\boldsymbol{p}_i'' - \lambda\boldsymbol{p}_i' \tag{11}$$

where μ and λ are variables to be determined.

The condition that this line segment be the shortest distance between the two vectors is that it is perpendicular to both $\boldsymbol{p}_i'$ and $\boldsymbol{p}_i''$. This condition is expressed by the following two equations:

$$(\boldsymbol{b} + \mu\boldsymbol{p}_i'' - \lambda\boldsymbol{p}_i')\cdot\boldsymbol{p}_i' = 0 \tag{12}$$

and

$$(\boldsymbol{b} + \mu \boldsymbol{p}_i'' - \lambda \boldsymbol{p}_i') \cdot \boldsymbol{p}_i'' = 0 \tag{13}$$

Or, in other words:

$$\lambda \boldsymbol{p}_i' \cdot \boldsymbol{p}_i' - \mu \boldsymbol{p}_i'' \cdot \boldsymbol{p}_i' = \boldsymbol{b} \cdot \boldsymbol{p}_i' \tag{14}$$

and

$$\lambda \boldsymbol{p}_i' \cdot \boldsymbol{p}_i'' - \mu \boldsymbol{p}_i'' \cdot \boldsymbol{p}_i'' = \boldsymbol{b} \cdot \boldsymbol{p}_i'' \tag{15}$$

Equations (14) and (15) are to be solved to get the values of μ and λ.

The vector from O' to the mid point of the shortest distance between the two vectors can be expressed as follows:

$$\tfrac{1}{2}(\lambda \boldsymbol{p}_i' + \mu \boldsymbol{p}_i'' + \boldsymbol{b})$$

The components of this vector are the spatial coordinates of the point i under consideration.

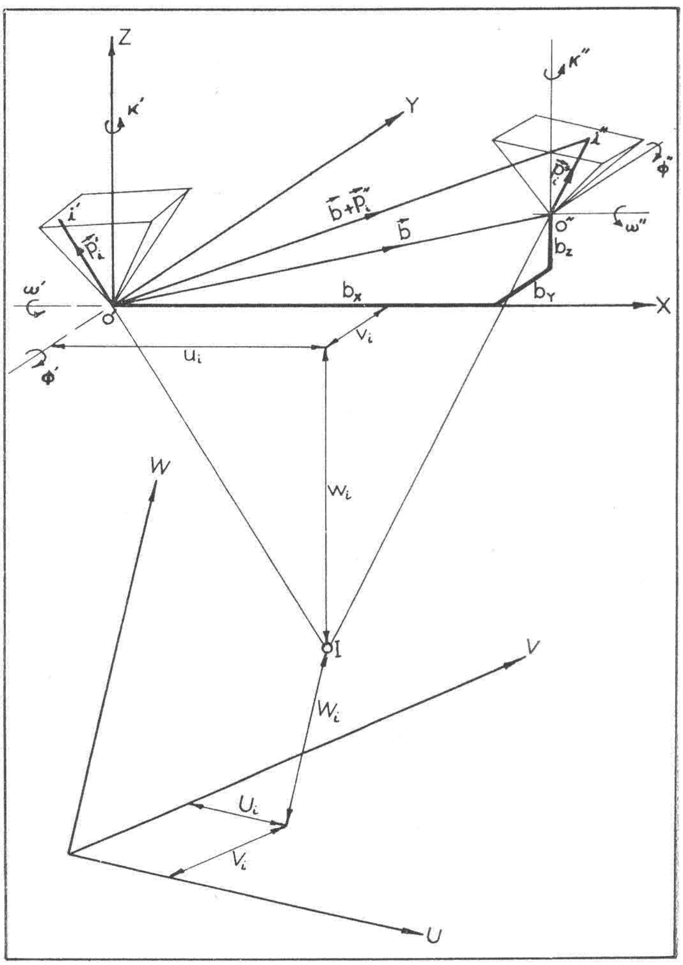

FIG. 1.

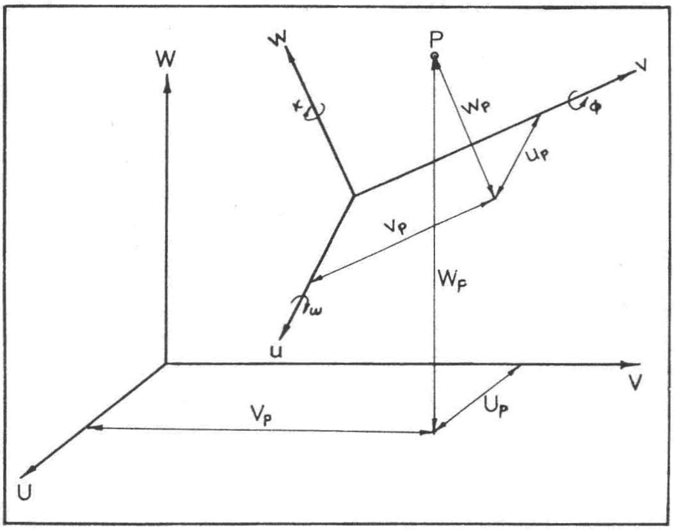

FIG. 2.

3. Assembly of the Strip

Cantilever assembly of the strip is done in the following fashion: κ, ϕ and ω of the first picture are chosen to be equal to zero, and b_x is assigned a convenient value depending on the chosen approximate scale of the strip. The coordinates of all the points in the first model are then determined. For the second model, the values of the rotations of the camera O' are taken equal to the rotations of the camera O'' in the previous model. The value of b_x is computed so that the elevation of a chosen set of points in the common overlap area between the two models agree in the two successive models.

4. Analytical Absolute Orientation

After the strip has been triangulated and assembled, a set of ground-control points is used to transform the strip coordinates of the points into the ground-coordinate system. Since the first photograph of the strip is not absolutely oriented, a three-dimensional orthogonal transformation is used in this step. As it is required to determine SEVEN parameters of transformation (3 rotations κ, ϕ, ω and four constants λ, U_0, V_0 and W_0), a minimum of SEVEN terrestrial coordinates are required for the solution (see Figure 2).

Three-dimensional Orthogonal Transformation

The transformation can be expressed in matrix notation as follows:

$$\boldsymbol{P}_i = \lambda \boldsymbol{A} \cdot \boldsymbol{p}_i + \boldsymbol{P}_0 \tag{16}$$

where

$\boldsymbol{P}_i(U_i, V_i, W_i)^T$ is a 3×1 vector whose components are the terrestrial coordinates of the point i.

$\boldsymbol{p}_i = (u_i, v_i, w_i)^T$ is a 3×1 vector whose components are the strip coordinates of point i.

$\boldsymbol{P}_0 = (U_0, V_0, W_0)^T$ is a 3×1 constant vector

λ = a scale factor

$\boldsymbol{A}$ = a 3×3 orthogonal matrix as defined in equation (8.1)

If n ground control points are available, n equations similar to Equation (16) can be written. Subtracting the first equation, corresponding to point 1, from every subsequent equation (corresponding to points 2 through n), we get the following matrix equation:

$$(\boldsymbol{P}_i - \boldsymbol{P}_1) = \lambda \boldsymbol{A}\cdot(\boldsymbol{p}_i - \boldsymbol{p}_1) \qquad i = 2, 3, \cdots, n \quad (17)$$

For the sake of clarity of presentation of the solution, the control-points will be considered to be given in full coordinates (U, V and W). The solution is still valid, however, for the other cases where some of the ground-control points may not be given in full coordinates (some points might be given in U and V only, others in W only.)

Expanding Equation (17), using Taylor's Theorem, we get:

$$(\boldsymbol{P}_i - \boldsymbol{P}_1 - \lambda \boldsymbol{A}(\boldsymbol{p}_i - \boldsymbol{p}_1) = \boldsymbol{A}(\boldsymbol{p}_i - \boldsymbol{p}_1)\cdot\Delta\lambda + \lambda \frac{\partial \boldsymbol{A}}{\partial \kappa}(\boldsymbol{p}_i - \boldsymbol{p}_1)\cdot\Delta\kappa$$

$$+ \lambda \frac{\partial \boldsymbol{A}}{\partial \phi}(\boldsymbol{p}_i - \boldsymbol{p}_1)\cdot\Delta\phi + \lambda \frac{\partial \boldsymbol{A}}{\partial \omega}(\boldsymbol{p}_i - \boldsymbol{p}_1)\cdot\Delta\omega \quad (18)$$

where $\boldsymbol{A}$, $\quad \frac{\partial \boldsymbol{A}}{\partial \kappa}, \frac{\partial \boldsymbol{A}}{\partial \phi}, \frac{\partial \boldsymbol{A}}{\partial \omega}$

are defined in Equation (7) and are evaluated using λ, κ, ϕ and ω as deduced from the previous iteration. The approximate values for λ and κ to start the iteration process can be evaluated from a two dimensional transformation of coordinates using any two control points given in U and V. The approximate values of ϕ and ω can be chosen as zero.

Equations (18) can be briefly written in matrix notations as follows:

$$\boldsymbol{B}_{3(n-1)\times 4}\cdot \boldsymbol{\Delta}_{4\times 1} = \boldsymbol{L}_{3(n-1)\times 1} \quad (19)$$

where $\boldsymbol{\Delta}$ is the column vector of corrections, in other words:

$$\boldsymbol{\Delta} = (\Delta\lambda, \Delta\kappa, \Delta\phi, \Delta\omega)^T \quad (20)$$

A least square estimate $\tilde{\boldsymbol{\Delta}}$ of the correction vector $\boldsymbol{\Delta}$ is then obtained from the equations:

$$\tilde{\boldsymbol{\Delta}} = (\boldsymbol{B}^T\cdot\boldsymbol{B})^{-1}\cdot\boldsymbol{B}^T\cdot\boldsymbol{L} \quad (21)$$

Successive iteration cycles are performed until the absolute value of each of the corrections $\Delta\kappa$, $\Delta\phi$ and $\Delta\omega$ is less than a pre-assigned value ϵ.

After determining the final values of λ and $\boldsymbol{A}$, the value of $\boldsymbol{P}_0$ is determined from the equation:

$$\boldsymbol{P}_0 = \lambda\boldsymbol{A}\cdot\boldsymbol{p}_1 - \boldsymbol{P}_1 \quad (22)$$

ACCURACY OF THE TRANSFORMATION:

A comparison between the transformed and the field-determined values of the coordinates of the ground-control points gives an idea about the accuracy of the transformation.

The following formula is generally used to estimate the accuracy (Mean Square Error) of the transformed position of the set of ground-control points used in determining the transformation elements.

$$\text{m.s.e. } p = \sqrt{\frac{(\Delta U\cdot\Delta U) + (\Delta V\cdot\Delta V) + (\Delta W\cdot\Delta W)}{(3n-7)}} \quad (23)$$

where

$\Delta U_i = U_i$ (Given)—U_i (Computed)

$\Delta V_i = V_i$ (Given—V_i (Computed)

$\Delta W_i = W_i$ (Given)W_i (Computed)

n = number of points used in the transformation (i.e., the number of points common to the two coordinate systems).

Remarks on the IBM 650 Program:

The above described problem has been programmed for the IBM 650. Due to the limited storage capacity of this electronic computer, no attempt was made to deal with the case of unequal weights of observations. The case of unequal weights will be considered in a future program for analytical aerotriangulation. This, however, will be programmed for the IBM 7090 to be acquired in the immediate future by the University of Illinois.

The solution of the analytical relative orientation generally requires three iterations and can handle up to 21 points. The approximate time required for relative orientation using 9 points is about four minutes per model.

The absolute orientation also requires about three iterations and takes around four minutes for the determination of the transformation constants.

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Ortho-Contour Photography

KENZO TOISHI* and MITSUYOSI KUREYA

THE image produced by the usual method of photography is inevitably perspective. Its aspect is so similar to that which we observe visually that it can be said to be the simplest for grasping the concept of the object. In photogrammetry, however, great efforts are expended in order to get an ortho-projective diagram of contours of the object from the perspective images. This is the function of most items of "mapping equipment." It would be quite convenient if the ortho-projective image of the object itself, or even better, the ortho-projective image of its contours could be photographed directly. A method which enables us to accomplish the former process was invented by Cooke, and reported by Prickett and Morris in 1950.[1] This method necessitates the use of a on-spherical lens large enough to cover the object. Although they have ameliorated this defect by the attempt to widen the area to be

[1] R. Prickett and M. Morris: The Orthocamera: Orthogonal Photographic Scanning Camera. PHOTOGRAMMETRIC ENGINEERING, XVI (1950), pp. 823–830.

* Tokyo National Research-Institute of Cultural Properties, Ueno Park, Tokyo.