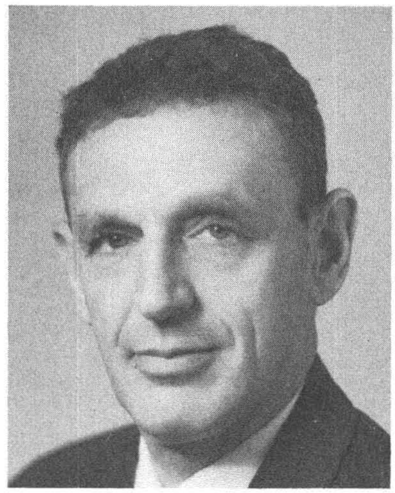
*Determining Exposure Point, Tilt, and Direction of Photograph From Three Known Ground Positions and Focal Length***

E. H. LEHMAN, JR.,
Senior Mathematical Statistician
Westwood Division, Houston Fearless Corporation†

INTRODUCTION

AMONG the many interesting photographic problems arising at Houston Fearless occurred the following: In a reconnaissance mission, one shot in particular created much excitement. A certain interpreter recognized with great surety several points in the picture whose exact location he knew, but with some changes of considerable importance. He wanted to know accurately the position, tilt, and direction of the lens at the moment that one frame was exposed. He knew the camera used and the approximate time and route over which the plane had flown. But was it possible to recover with precision the desired information? Yes, it was and the following pages tell how.



EUGENE H. LEHMAN, JR.

DISCUSSION

Available is a photograph (with its four fiducial points) upon which appear the images a, b, c [whose three space coordinates are i_x, i_y, i_z ($i=a$, b, or c)] of objects A, B, C [whose three space coordinates are I_x, I_y, I_z ($I=A$, B, C)]. The letter A is assigned to the lowest of the three objects. Our space coordinate system is set up with the origin at A, the $+x$ axis East from A, the $+y$ axis North from A, and the $+z$ axis Up from A. See Figure 1.

Ten quantities are known: the nine space coordinates $I_\alpha(\alpha=x, y, z)$ of the three objects, and the focal-length of the camera. On the photograph may be located the principal point p (whose three space coordinates are p_α) defined as the foot of the perpendicular from the exposure point L (whose three space coordinates are L_α) to the plane of the photo and located by the intersection of the two lines joining opposite fiducial points. All the distances may be measured on the photo between a, b, c, and p although their space coordinates are not known.* On the ground from the known coordinates the distances AB, AC, and BC are computed.

The focal-length f is defined to be the distance pL. pL is extended downward to an altitude of zero. This point is called $P(P_x, P_y, 0)$. We drop a vertical downward from L to the ground and call this the nadir point $N(N_x, N_y, 0)$. Of course $N_x=L_x$ and $N_y=L_y$.

The location of N and P is not yet known.

Call the angle NLP θ. It is the tilt of the photograph. The angle between the $+x$

† 11801 West Olympic Boulevard, Los Angeles 64, California.

* This measurement is performed to one micron by the Houston Fearless Dual Screen Measuring Projector.

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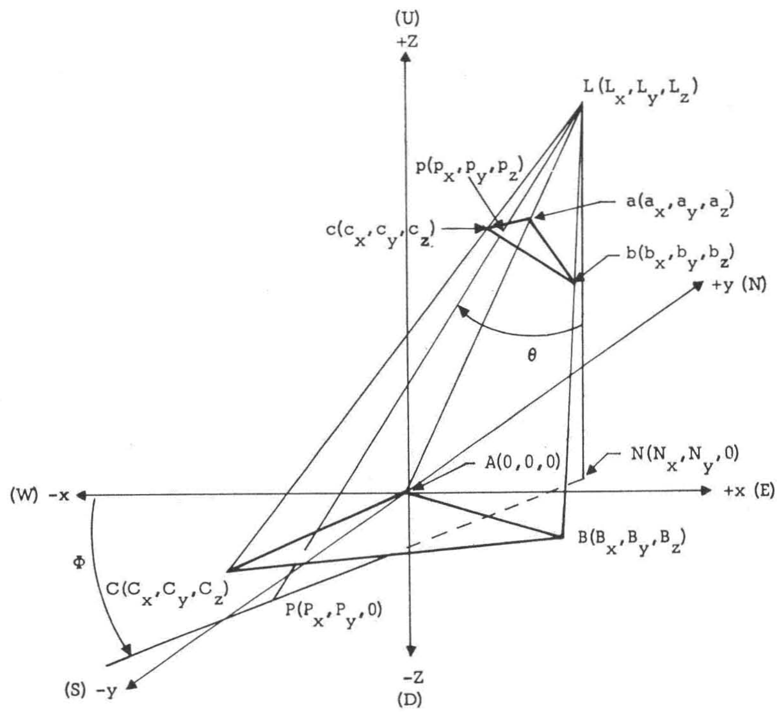

FIG. 1. Three objects, their images, and the camera.

axis and NP measured counterclockwise from due East to the directed line NP is called Φ. It is the direction of the photograph.

The desire is to know five quantities: the space coordinates L_α of the exposure point, the tilt θ, and the direction Φ.

The action is as follows:

Determine distances iL from exposure point to images:

$$iL^2 = ip^2 + pL^2 \tag{1}$$

Determine angles iLj ($i \neq j$; $i, j = a, b, c$), the three vertical angles at L subtended by the triangle abc (or ABC) by applying the law of cosines:

$$\cos iLj = (iL^2 + jL^2 - ij^2)/2iLjL (= \cos ILJ) \tag{2}$$

Determine distances IL from exposure point to objects. This requires the simultaneous solution of these three quadratic equations formed by again using the law of cosines:

$$IJ^2 = IL^2 + JL^2 - 2ILJL \cos iLj \tag{3}$$

wherein the three unknowns are IL. ($I, J = A, B, C$ as usual, $I \neq J$).

This is done as follows:

In the quadratic involving A and B, solve for BL as a function of AL, using the conventional quadratic formula but rejecting the roots formed from the negative sign before the radical:

$$BL = AL \cos aLb + \sqrt{AB^2 - AL^2 \sin^2 aLb} \tag{3a}$$

In the quadratic involving A and C, similarly solve for CL as a function of AL;

$$CL = AL \cos aLc + \sqrt{AC^2 - AL^2 \sin^2 aLc} \tag{3b}$$

Substitute the values obtained from BL and CL in (3a) and (3b) into the third quadratic, that involving B and C, collect terms and simplify as much as possible:

$$\begin{aligned}
&(\cos^2 aLb + \cos^2 aLc - \cos aLb \cos aLc \cos bLc - 1)AL^2 \\
&+ \{(\cos aLb - \cos aLc \cos bLc)\sqrt{AB^2 - AL^2 \sin^2 aLb} \\
&+ (\cos aLc - \cos aLb \cos bLc)\sqrt{AC^2 - AL^2 \sin^2 aLc}\}AL \\
&- \cos bLc\sqrt{AB^2 - AL^2 \sin^2 aLb}\sqrt{AC^2 - AL^2 \sin^2 aLc} \\
&+ \tfrac{1}{2}(AB^2 + AC^2 - BC^2) = 0
\end{aligned} \tag{3c}$$

Solution of this pseudoquartic in AL may be performed on an electronic computer using Newton's method. If the original information is provided to four significant figures, about three or four iterations of Newton's method will give the four roots to that accuracy. Of these four roots, it is not difficult to determine the one desired.

Two of the roots will be complex; of the real roots, one will often be negative, or if positive, it will be patently too large or too small.

Using the result of (3c) as a value of AL, return to (3a) and (3b) to determine BL and CL. Check all three values in all three equations (3).

Now obtain the coordinates L_α of the exposure point by solving simultaneously these three spheres:

$$\sum (L_\alpha - I_\alpha)^2 = IL^2 \tag{4}$$

This set is considerably easier than set (3). Subtract the equation involving A from, where $\alpha = x; y, z$ those involving B and C. Thus the latter two become linear in L_x, L_y, and L_z. Solve these two to get L_x and L_y in terms of L_z and substitute those values in the equation involving A to obtain a quadratic in L_z. This can be solved and will give one positive root for L_z. Use this value of L_z to obtain L_x and L_y from the two linear equations.

Determine the three vertical angles at L subtended by the three lines ip (or IP), where iL was found in (1) and pL is of course the focal-length.

$$\cos iLp = (iL^2 + pL^2 - ip^2)/2pLiL(= \cos ILP) \tag{5}$$

Find P, again using the law of cosines.

$$\sum (I_\alpha - P_\alpha)^2 = \sum (P_\alpha - L_\alpha)^2 + \sum (I_\alpha - L_\alpha)^2 - 2PLIL \cos iLp. \tag{6a}$$

Expand, cancel $P_\alpha{}^2$'s and $I_\alpha{}^2$'s, and divide out the "α".

$$\sum (L_\alpha - I_\alpha)P_\alpha = \sum L_\alpha{}^2 - \sum I_\alpha L_\alpha - PLIL \cos iLp \tag{6b}$$

This is a set of three linear equation in three unknowns P_α in terms of by-now-known constants and a fourth unknown PL for which a fourth equation is available, the ordinary formula

$$PL^2 = \sum (P_\alpha - L_\alpha)^2 \tag{6c}$$

However the set (6b) is mutually dependent since if distances BP_0, and CP_0 are given to any point P_0 on PL, then AP_0 is determined. Thus a fifth equation is needed. Since P is defined as being at the altitude of A,

$$P_z = 0 \tag{6d}$$

Using (6b) (6c) and (6d) and solving by determinants we have

$$P_x = L_x - \frac{PL}{D}\begin{vmatrix} AL\cos aLp & L_y - A_y & L_z - A_z \\ BL\cos bLp & L_y - B_y & L_z - B_z \\ CL\cos cLp & L_y - C_y & L_z - C_z \end{vmatrix} \tag{6e}$$

$$P_y = L_y - \frac{PL}{D}\begin{vmatrix} L_x - A_x & AL\cos aLp & L_z - A_z \\ L_x - B_x & BL\cos bLp & L_z - B_z \\ L_x - C_x & CL\cos cLp & L_z - C_z \end{vmatrix} \tag{6f}$$

$$D = \begin{vmatrix} L_x & L_y & L_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{6g}$$

The tilt θ is the angle between PL and NL, that is,

$$\cos\theta = \frac{L_z}{PL} \tag{7}$$

The direction Φ is the angle between the $+x$ axis and the projection NP upon the plane $z=0$ of LP,

$$\tan\Phi = (P_y - N_y) \div (P_x - N_x) = (P_y - L_y) \div (P_x - L_x) \tag{8}$$

An artificial example follows:

Given $A(0, 0, 0)$, $B(2, 1, \frac{1}{2})$, and $C(-.439, 2\frac{1}{2}, 1.028)$ and focal-length 5×10^{-4} where all distances are in kf (kilofeet $=10^3$ feet). On the photo measure $ab=2.5\times10^{-4}$, $ac=2.7\times10^{-4}$, and $bc=3.0\times10^{-4}$.

By observing p on the photo there can be measured

$$ap = .7062\times10^{-4}, \qquad bp = 2.3\times10^{-4}, \quad \text{and} \quad cp = 2.0\times10^{-4}.$$

(In composing the problem, bp and cp may be invented; then ap cannot be invented—p will be one of the at most two intersections of the circles about b and c of radius bp and cp respectively. In this problem p was placed inside Δabc; hence ap has the lesser of the two possible values.)

Then

$AB^2 = 5.250$	$AB = 2.291$
$AC^2 = 7.500$	$AC = 2.739$
$BC^2 = 8.477$	$BC = 2.912$
$ab^2 = 6.250\times10^{-8}$	$ab/AB = 1.0912\times10^{-4}$
$ac^2 = 7.290\times10^{-8}$	$ac/AC = .9858\times10^{-4}$
$bc^2 = 9.000\times10^{-8}$	$bc/BC = 1.0142\times10^{-4}$
$ap^2 = .4988\times10^{-8}$	$cp^2 = 4.000\times10^{-8}$
$bp^2 = 5.290\times10^{-8}$	$pL^2 = 25.00\times10^{-8}$

Now use (1) thrice:

$aL^2 = (.4988) + 25)\times10^{-8} = 25.50\times10^{-8}$	$aL = 5.050\times10^{-4}$
$bL^2 = (5.290 + 25)\times10^{-8} = 30.29\times10^{-8}$	$bL = 5.504\times10^{-4}$
$cL^2 = (4.000 + 25)\times10^{-8} = 29.00\times10^{-8}$	$cL = 5.385\times10^{-4}$

Next use (2) thrice:

$$\cos aLb = (25.50 + 30.29 - 6.25)/2(5.050)5.504 = .8912 \quad \angle aLb = 26°59'$$
$$\cos aLc = (25.50 + 29.00 - 7.29)/2(5.050)5.385 = .8483 \quad \angle aLc = 31°58'$$
$$\cos bLc = (30.29 + 29.00 - 9.00)/2(5.504)5.385 = .8679 \quad \angle bLc = 29°47'$$

Now (3) also thrice:

$$5.250 = AL^2 + BL^2 - 2ALBL(.8912)$$
$$7.500 = AL^2 + CL^2 - 2ALCL(.8483)$$
$$8.477 = BL^2 + CL^2 - 2BLCL(.8679)$$

Use (3a) and (3b):

$$BL = .8912AL + \sqrt{5.250 - .2058AL^2}$$
$$CL = .8679AL + \sqrt{7.500 - .2467AL^2}$$

where

$$-.2058 = .8912^2 - 1 = -\sin^2 aLb \text{ and } -.2467 = .8679^2 - 1 = -\sin^2 aLc$$

Now use (3c):

$$\begin{aligned}&[.8912^2 + .8679^2 - .8912(.8679).8483 - 1]AL^2\\ &+ \{[.8912 - .8679(.8483)]\sqrt{5.250 - .2058AL^2}\\ &+ [.8679 - .8912(.8483)]\sqrt{7.500 - .2467AL^2}\}AL\\ &- .8483\sqrt{5.250 - .2058AL^2}\sqrt{7.500 - .2467AL^2}\\ &+ \tfrac{1}{2}(5.25 + 7.5 - 8.477) = 0\end{aligned}$$

This simplifies to:

$$\begin{aligned}F(AL) \equiv &- AL^2 + [.6474\sqrt{25.51 - AL^2} + .5118\sqrt{30.40 - AL^2}]AL\\ &- 1.760\sqrt{25.51 - AL^2}\sqrt{30.40 - AL^2} + 19.67 = 0\end{aligned}$$

It is observed that AL must not exceed $\sqrt{25.51} = 5.051$. Starting with this value and using Newton's method it is discovered in only two iterations:

$$AL = 5.000 \qquad AL^2 = 25.00$$

Replacing AL by 5 in (3a) and (3b),

$$BL = 4.780 \qquad BL^2 = 22.84$$
$$CL = 5.494 \qquad CL^2 = 30.18$$

Next use (4)

$$(L_x - 0)^2 + (L_y - 0)^2 + (L_z - 0)^2 = 25.00 \tag{4a}$$
$$(L_x - 2)^2 + (L_y - 1)^2 + (L_z - \tfrac{1}{2})^2 = 22.84 \tag{4b}$$
$$(L_x + .439)^2 + (L_y - 2\tfrac{1}{2})^2 + (L_z - 1.028)^2 = 30.18 \tag{4c}$$

Subtract (4a) from (4b) and (4c) to obtain two linear equations in L_x, L_y, L_z; solve these two for L_y and L_z as functions of L_x and replace in (4a) to make a quadratic in

L_x. Of the two roots thus obtained, both will be real and possibly positive, but one will be unreasonable. In our case this process gives:

$$4L_x + 2L_y + L_z = 7.4016 \tag{4b'}$$

$$-.878L_x + 5L_y + 2.056L_z = 2.3155 \tag{4c'}$$

$$L_x^2 - 2.9511L_x + 2.1456 = 0 \tag{4a'}$$

and for roots of L_x: 1.2976 and 1.6535.

Substituting the latter into (4b′) and (4c′) gives a negative value for L_z. Hence the former is the correct one and there is obtained as the exposure point coordinates:

$$L = (1.298, -1.229, 4.670) \text{ and the nadir } N \text{ is}$$

$$N = (1.298, -1.229, 0)$$

Rechecking these three values in (4) gives us $AL^2 = 25.00$, $BL^2 = 22.85$, and $CL^2 = 30.18$ for the right hand of (4a), (4b), and (4c) respectively—close enough.

Now in (5) the three vertical angles from L to a, b, c (or A, B, C) are obtained

$$\begin{array}{l|l}
\cos aLp = (29 + 25 - 4)/2(5.305)5 = .9285 & \angle aLp = 21°48' \\
\cos bLp = (30.29 + 25 - 5.29)/2(5.504)5 = .9084 & \angle bLp = 24°43' \\
\cos cLp = (25.50 + 25 - .4978)/2(5.050)5 = .9901 & \angle cLp = 8°4'
\end{array}$$

Next, of the equations under (6), use (6e):

$$D = \begin{vmatrix} 1.298 & -1.229 & 4.670 \\ 2 & 1 & \frac{1}{2} \\ -.439 & 2\frac{1}{2} & 1.028 \end{vmatrix} = 27.91; \qquad \frac{1}{D} = .03583 \tag{6g'}$$

$$P_x = 1.298 - .03583 \begin{vmatrix} 5\ (.9285) & -1.229 & 4.670 \\ 4.780(.9084) & -2.229 & 4.170 \\ 5.494(.9901) & -3.729 & 3.642 \end{vmatrix} PL \tag{6e'}$$

$$= 1.298 - .251PL$$

$$P_y = -1.229 - .03583 \begin{vmatrix} 1.298 & 5\ (.9285) & 4.670 \\ -.702 & 4.780(.9084) & 4.170 \\ 1.737 & 5.494(.9901) & 3.642 \end{vmatrix} PL \tag{6f'}$$

$$= -1.229 + .591PL$$

$$P_z = 0$$

Equation (6c) gives PL:

$$PL^2 = .251^2PL^2 + .591^2PL^2 + 4.670^L \quad \text{or} \quad .588PL^2 = 4.670^2$$

yields $PL^2 = 37.09$ and $PL = 6.090$.

Then the location of P from (6e′) and (6f′) is

$$P = (-.231, 2.370, 0)$$

From (7) obtain the tilt,

$$\cos\theta = \frac{4.670}{6.090} = .7668; \qquad \theta = 39°56'.$$

Finally from (8) obtain the direction

$$\tan \Phi = \frac{2.370 - 1.229}{-0.231 - 1.298} = -2.354$$

$$\Phi = 113°01' \text{ north of east,}$$

that is

$$\Phi = \text{N}66°59'\text{W}$$

Thus there have been determined exposure point $L(L_x, L_y, L_z)$, tilt θ, and direction of tilt Φ, as well as several other pieces of information such as P and N, given only three ground locations, the picture, and the focal-length.

II. Conclusion

The *PI* mentioned in the Introduction studied this paper in great detail and is now able to determine the exposure point, tilt, and direction of any photograph presented to him, provided he knows the focal length and recognizes three ground positions.

Differentiation of the Orientation Matrix by Matrix Multipliers

JAMES R. LUCAS
Senior Photogrammetrist,
The Autometric Corporation,
400 N. Washington St., Alexandria, Va.

ABSTRACT: *A method is presented for expressing the total derivative of any orientation matrix M as the sum of the products of M with three simple skew-symmetric matrices; the order of multiplication and form of the matrix multipliers being dependent only on the form of the orientation matrix. Equations are first developed in general form and the method is then illustrated by the differentiation of three types of orientation matrices in common use.*

Introduction

In several recent papers on analytical photogrammetry[1,2,3] reference has been made to the linearized form of the projective equations of von Gruber.[4] It has been stated that the linearization of these equations is accomplished by taking the partial derivatives of the measured photo coordinates with respect to each unknown variable. However, in most instances the derivation of these partials has been omitted, and rightly so, because of the complexity of the derivatives with respect to the elements of angular orientation. The reader, if he is so inclined, is then left the tedious task of term-wise differentiation of the transformation matrix. When this undertaking has been completed it is a careful worker indeed who has not committed at least one small error.

It is the purpose of this paper to show how the partial derivatives of any orthogonal transformation can be expressed as simple matrix products, thus reducing a