Finally from (8) obtain the direction

$$\tan \Phi = \frac{2.370 - 1.229}{-0.231 - 1.298} = -2.354$$

$$\Phi = 113°01' \text{ north of east,}$$

that is

$$\Phi = \text{N}66°59'\text{W}$$

Thus there have been determined exposure point $L(L_x, L_y, L_z)$, tilt θ, and direction of tilt Φ, as well as several other pieces of information such as P and N, given only three ground locations, the picture, and the focal-length.

II. Conclusion

The *PI* mentioned in the Introduction studied this paper in great detail and is now able to determine the exposure point, tilt, and direction of any photograph presented to him, provided he knows the focal length and recognizes three ground positions.

Differentiation of the Orientation Matrix by Matrix Multipliers

JAMES R. LUCAS
Senior Photogrammetrist,
The Autometric Corporation,
400 N. Washington St., Alexandria, Va.

ABSTRACT: *A method is presented for expressing the total derivative of any orientation matrix M as the sum of the products of M with three simple skew-symmetric matrices; the order of multiplication and form of the matrix multipliers being dependent only on the form of the orientation matrix. Equations are first developed in general form and the method is then illustrated by the differentiation of three types of orientation matrices in common use.*

Introduction

In several recent papers on analytical photogrammetry[1,2,3] reference has been made to the linearized form of the projective equations of von Gruber.[4] It has been stated that the linearization of these equations is accomplished by taking the partial derivatives of the measured photo coordinates with respect to each unknown variable. However, in most instances the derivation of these partials has been omitted, and rightly so, because of the complexity of the derivatives with respect to the elements of angular orientation. The reader, if he is so inclined, is then left the tedious task of term-wise differentiation of the transformation matrix. When this undertaking has been completed it is a careful worker indeed who has not committed at least one small error.

It is the purpose of this paper to show how the partial derivatives of any orthogonal transformation can be expressed as simple matrix products, thus reducing a time-consuming exercise in differential calculus to a systematic calculation in matrix algebra. The method will first be developed for the general orthogonal transformation and an example will then be given of its application to a specific photogrammetric system. Although this method is completely general, its application to some of the photogrammetric transformations in common use requires additional explanation which, if included in the text, might cause confusion. A complete discussion of these special cases is therefore presented in the appendix to this paper.

The General Orthogonal Transformation

In the following discussion an orthogonal transformation will be restricted to mean the transformation which expresses the Cartesian coordinates of a vector in another Cartesian coordinate system having the same origin, while leaving the length of the vector invariant. More precisely

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} m_{x'x} & m_{x'y} & m_{x'z} \\ m_{y'x} & m_{y'y} & m_{y'z} \\ m_{z'x} & m_{z'y} & m_{z'z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is an orthogonal transformation, if each element, m_{ij}, of the transformation matrix is the cosine of the angle between the i'-axis and the j-axis for each $i=x', y', z'$ and $j=x, y, z$.

Every orthogonal transformation is either a rotation of axes or the product of a rotation and a reflection of axes. Every rotation can be expressed as the product of three planar rotations. The most elementary form of an orthogonal transformation is a simple planar rotation.

PLANAR ROTATIONS

A planar rotation is a rotation of axes about a coordinate axis. Let $M_i(a)$ be a function of a defined to be the planar rotation, clockwise about the i-axis through an angle a. Three such functions exist:

$$M_x(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$$

$$M_y(a) = \begin{bmatrix} \cos a & 0 & -\sin a \\ 0 & 1 & 0 \\ \sin a & 0 & \cos a \end{bmatrix} \tag{1}$$

$$M_z(a) = \begin{bmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The derivative of each of these functions with respect to a is obtained by replacing each element in the matrix by its derivative with respect to a. The derivatives

$$\begin{aligned} \frac{d}{da}(\sin a) &= \cos a = \sin\left(a + \frac{\pi}{2}\right) \\ \frac{d}{da}(\cos a) &= -\sin a = \cos\left(a + \frac{\pi}{2}\right) \end{aligned} \tag{2}$$

suggest that each element of M_i that is a trigonometric function would be replaced by its derivative, if the rotation M_i were increased by $\pi/2$. Hence the following rule for forming the pre- (or post-) multiplier matrices:

Let P_i be defined to be the result of replacing by zero the ii-element of $M_i(\pi/2)$. The resulting three skew-symmetric matrices

$$P_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad P_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3}$$

have the unique property

$$\frac{d}{da}[M_i] = [P_i][M_i] = [M_i][P_i] \tag{4}$$

for each i. The counterclockwise rotation corresponding to each M_i is the transpose of M_i. An equation formed by replacing each matrix in question (4) by its transpose would be correct, but since the P matrices are skew-symmetric, a simpler expression is

$$\frac{d}{da}[M_i]^T = -[P_i][M_i]^T = -[M_i]^T[P_i] \tag{5}$$

for each i. Hence the derivative of any planar rotation through any angle a can be found by multiplication by one of the skew-symmetric matrices (3), provided the direction of rotation is known.

MULTIPLE PLANAR ROTATIONS

As stated above every rotation can be expressed as the product of three planar rotations. In fact the general orthogonal transformation (the product of a rotation and a reflection of axes) can be expressed as the product of three planar rotations with the axis reflection embodied in one of them. This kind of planar rotation, called an improper rotation, in no way impairs the generality of the present discussion and is therefore covered in the appendix.

Consider now the rotation matrix M which is the product of three planar rotations, assumed for convenience to be clockwise. If subscripts are again used to denote the axis of rotation and a, b, c, the rotation angles, then

$$M_{ijk} = [A_i(a)][B_j(b)][C_k(c)] \tag{6}$$

Since each planar rotation is a function of a discrete variable, the partial derivatives of M are obtained by sequentially replacing a single term of the product by its derivative. Hence, by Equation (4)

$$\frac{\partial M}{\partial a} = \left[\frac{\partial A_i}{\partial a}\right][B_j][C_k] = [P_i][M] \tag{7}$$

$$\frac{\partial M}{\partial b} = [A_i]\left[\frac{\partial B_j}{\partial b}\right][C_k] = ([A_i][B_j])[P_j][C_k] \tag{8}$$

$$\frac{\partial M}{\partial c} = [A_i][B_j]\left[\frac{\partial C_k}{\partial c}\right] = [M][P_k] \tag{9}$$

If any of the above planar rotations were in the counterclockwise direction, it would only be necessary to attach a minus sign to the right-hand side of the corresponding derivative expression, as shown in the last section. The above equations, therefore, express the partial derivatives of any orthogonal transformation in terms of simple matrix products.

The utility of Equations (7) and (9) is readily apparent, but in some cases it may be inconvenient to precompute the product $[A_i][B_j]$. This deficiency leads to the development of the quasi-postmultiplier matrix Q. The Q matrix is so defined as to: (1) remove the tertiary rotation from the triple product, (2) multiply the secondary rotation by the appropriate P matrix, and (3) remultiply by the tertiary rotation.* The Q matrix obviously must be defined in terms of a specific set of rotations, and hence does not have the generality of the P matrix. Nonetheless, this matrix is easily formed and can be valuable in numerical work.

For the general transformation (6) the Q matrix is

$$Q_j = [C_k]^T[P_j][C_k]. \tag{10}$$

Equation (8) can now be rewritten as

$$\frac{\partial M}{\partial b} = [A_i]\left[\frac{\partial B_j}{\partial b}\right][C_k] = [M][Q_j] \tag{11}$$

and the total derivative of the transformation can now be written

$$dM = [P_i][M]da + [M][Q_j]db + [M][P_k]dc \tag{12}$$

Thus, the Q matrix provides a means for expressing the total derivative of any orthogonal transformation as the sum of products of the original matrix by simple matrix multipliers.

Application to Photogrammetry

An orthogonal transformation that has received thorough treatment in photogrammetric literature[2,5] is the orientation matrix consisting of sequential planar rotations through roll, pitch, and yaw (ω, ϕ, and κ). In the references cited this transformation is employed to go from ground coordinates to photograph and is defined by

$$\begin{aligned} M &= \begin{bmatrix} m_{x'x} & m_{x'y} & m_{x'z} \\ m_{y'x} & m_{y'y} & m_{y'z} \\ m_{z'x} & m_{z'y} & m_{z'z} \end{bmatrix} \\ &= \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \end{aligned} \tag{13}$$

This equation has the form

$$M = [A_z][B_y][C_x].$$

By Equation (10),

$$\begin{aligned} Q_y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \\ &= \begin{bmatrix} 0 & \sin\omega & -\cos\omega \\ -\sin\omega & 0 & 0 \\ \cos\omega & 0 & 0 \end{bmatrix} \end{aligned} \tag{14}$$

* Q could just as easily have been defined for use as a premultiplier.

and since each planar rotation is in the clockwise direction, the partial derivatives with respect to the three rotation angles are given by

$$\frac{\partial M}{\partial \kappa} = [P_z][M] = \begin{bmatrix} m_{y'x} & m_{y'y} & m_{y'z} \\ -m_{x'x} & -m_{x'y} & -m_{x'z} \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

$$\frac{\partial M}{\partial \phi} = [M][Q_y] = \begin{bmatrix} -m_{x'y}\sin\omega + m_{x'z}\cos\omega & m_{x'x}\sin\omega & -m_{x'x}\cos\omega \\ -m_{y'y}\sin\omega + m_{y'z}\cos\omega & m_{y'x}\sin\omega & -m_{y'x}\cos\omega \\ -m_{z'y}\sin\omega + m_{z'z}\cos\omega & m_{z'x}\sin\omega & -m_{z'x}\cos\omega \end{bmatrix} \tag{16}$$

$$\frac{\partial M}{\partial \omega} = [M][P_x] = \begin{bmatrix} 0 & -m_{x'z} & m_{x'y} \\ 0 & -m_{y'z} & m_{y'y} \\ 0 & -m_{z'z} & m_{z'y} \end{bmatrix} \tag{17}$$

These partials are in agreement with the expressions obtained by Harris, Tewinkel, and Whitten[2] by term-wise differentiation.

Summary and Conclusions

Every orthogonal transformation, and therefore every photogrammetric orientation matrix, can be represented by a matrix triple product of the form

$$M = [A_i(a)][B_j(b)][C_k(c)]$$

where A_i, B_j, and C_k are planar rotations about the i, j, k-axes through angles a, b, and c respectively. The total derivative of this transformation can be written as the sum of matrix products by the equation

$$dM = \pm [P_i][M]da \pm [M][Q_j]db \pm [M][P_k]dc$$

where

$$Q_j = [C_k]^T[P_j][C_k]$$

and P_i, P_j, P_k are simple skew-symmetric matrices defined by Equation (3). The plus sign applies to the partial derivative of a clockwise planar rotation; the minus sign, to a counterclockwise planar rotation.

The differentiation of an orthogonal matrix is reduced by the method of matrix multipliers to an exercise in matrix algebra. This method is completely rigorous and general within the scope of orthogonal transformations. It provides the photogrammetrist with an efficient tool for analyzing coordinate transformations and permits a simplified mathematical approach to a variety of photogrammetric problems.

References Cited

1. Mikhail, Edward M., "Use of Triplets for Analytical Aerotriangulation," Photogrammetric Engineering, Vol. XXVIII, No. 4, September, 1962, pp. 625–632.
2. Harris, Wm. D., G. C. Tewinkel, and Charles A. Whitten, "Analytical Aerotriangulation in the Coast and Geodetic Survey," Photogrammetric Engineering, Vol. XXVIII, No. 1, March, 1962, pp. 44–69.
3. Brown, Duane C., *A Solution to the General Problem of Multiple Station Analytical Stereotriangulation,* RCA Data Reduction Technical Report No. 43, February, 1958.
4. von Gruber, Otto, *Photogrammetry, Collected Lectures and Essays,* Chapman and Hall, London, 1932.
5. Rosenfield, George H., "The Problem of Exterior Orientation in Photogrammetry," Photogrammetric Engineering, Vol. XXV, No. 4, September, 1959, pp, 536–553.

Appendix

In many photogrammetric techniques, angles are so defined that one or more of the planar rotations is through the complement or supplement of the angle of interest. A further complication arises from the inclusion of a reflection of axes in one or more of the planar rotations. In working with an unfamiliar transformation one may have difficulty in recognizing the direction of a particular planar rotation. This Appendix is included, therefore, to provide a method for establishing whether a given planar rotation is clockwise or counterclockwise, and to show how the general technique for differentiating matrices can be applied to representative photogrammetric systems.

Determination of the Direction of a Planar Rotation

Any clockwise planar rotation can be reduced to the form M_i of Equation (1) and any counterclockwise rotation can be reduced to the form M_i^T by one or more of the following operations. These operations, when performed in the order given (omitting any that do not apply), will transform any rotation, proper or improper, into one of the forms M_i or M_i^T. It must be emphasized that these operations are performed only to determine the direction of a given planar rotation and, hence, the algebraic sign of its derivative. The skew-symmetric matrix then multiplies or is multiplied by the original planar rotation and not the transformed rotation matrix.

1. If one element of the principal diagonal has a minus sign, change the algebraic signs of either the row or column containing that element.

In this case the rotation can have either direction, depending on whether a row or column is changed. However, if the signs of a row are changed, the indicated skew-symmetric matrix must be used as a post multiplier; if a column is changed, the indicated P matrix must premultiply the rotation. This dual polarity is characteristic of an improper rotation, and is illustrated by the following example:

$$A_z = \begin{bmatrix} -\cos a & \sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos a & -\sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\frac{\partial A}{\partial a} = -[A_z][P_z] = [P_z][A_z]$$

2. If two elements of the principal diagonal have minus signs, change the algebraic signs of all elements of the cofactor of the ii-element, where i is the axis of rotation.

Here the rotation is in reality through the angle $(a+\pi)$ or,

$$A_z = \begin{bmatrix} -\cos a & \sin a & 0 \\ -\sin a & -\cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos a & -\sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \pi & -\sin \pi & 0 \\ \sin \pi & \cos \pi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\frac{\partial A}{\partial a} = -[A_z][P_z] = -[P_z][A_z]$$

3. If the elements on the principal diagonal are sines of the angle, exchange each trigonometric function for its cofunction and change the algebraic signs of the off-diagonal elements.

In this case the rotation is through the angle $(a+3\pi/2)$

$$A_z = \begin{bmatrix} \sin a & \cos a & 0 \\ -\cos a & \sin a & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos a & -\sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} & 0 \\ +\sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\frac{\partial A}{\partial a} = -[A_z][P_z] = -[P_z][A_z]$$

If the rotation were through $(a+\pi/2)$, both operations (2) and (3) would be required.

These three rules are sufficient to determine the direction of any planar rotation. The foregoing principles of matrix differentiation can now be applied, without confusion, to transformations of the type described above. The differentiation of two common photogrammetric transformations which illustrate these principles will now be given without further explanation. A complete treatment of each transformation, but not its differentiation, can be found in the literature cited.

TILT, SWING, AND AZIMUTH[5]

$$M = \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} \begin{bmatrix} \sin s & \cos s & 0 \\ -\cos s & \sin s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= [A_z(\alpha)][B_y(t)][C_z(s)]$$

By rules (2) and (3) A_z is clockwise; by rule (3) C_z is counterclockwise. Therefore,

$$Q_y = [C_z]^T[P_y][C_z] = \begin{bmatrix} 0 & 0 & -\sin s \\ 0 & 0 & -\cos s \\ \sin s & \cos s & 0 \end{bmatrix}$$

and

$$\frac{\partial M}{\partial \alpha} = [P_z][M] = \begin{bmatrix} m_{y'x} & m_{y'y} & m_{y'z} \\ -m_{x'x} & -m_{x'y} & -m_{x'z} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial M}{\partial t} = -[M][Q_y] = \begin{bmatrix} -m_{x'z}\sin s & -m_{x'z}\cos s & m_{x'x}\sin s + m_{x'y}\cos s \\ -m_{y'z}\sin s & -m_{y'z}\cos s & m_{y'x}\sin s + m_{y'y}\cos s \\ -m_{z'z}\sin s & -m_{z'z}\cos s & m_{z'x}\sin s + m_{z'y}\cos s \end{bmatrix}$$

$$\frac{\partial M}{\partial s} = -[M][P_z] = \begin{bmatrix} m_{x'y} & -m_{x'x} & 0 \\ m_{y'y} & -m_{y'x} & 0 \\ m_{z'y} & -m_{z'x} & 0 \end{bmatrix}$$

AZIMUTH, ELEVATION ANGLE, AND AND ROLL[3,5]

$$M = \begin{bmatrix} -\cos\kappa & \sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\omega & \cos\omega \\ 0 & \cos\omega & \sin\omega \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [A_z(\kappa)][B_x(\omega)][C_z(\alpha)]$$

By rule (1) A_z must be clockwise if it is to be premultiplied, and by rules (1) and (3) B_x must be clockwise if it is to be postmultiplied. Hence,

$$Q_x = [C_z]^T[P_x][C_z] = \begin{bmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & \cos\alpha \\ -\sin\alpha & -\cos\alpha & 0 \end{bmatrix}$$

and

$$\frac{\partial M}{\partial \kappa} = [P_z][M] = \begin{bmatrix} m_{y'x} & m_{y'y} & m_{y'z} \\ -m_{x'x} & -m_{x'y} & -m_{x'z} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial M}{\partial \omega} = [M][Q_x] = \begin{bmatrix} -m_{x'z}\sin\alpha & -m_{x'z}\cos\alpha & m_{x'x}\sin\alpha + m_{x'y}\cos\alpha \\ -m_{y'z}\sin\alpha & -m_{y'z}\cos\alpha & m_{y'x}\sin\alpha + m_{y'y}\cos\alpha \\ -m_{z'z}\sin\alpha & -m_{z'z}\cos\alpha & m_{z'x}\sin\alpha + m_{z'y}\cos\alpha \end{bmatrix}$$

$$\frac{\partial M}{\partial \alpha} = -[M][P_z] = \begin{bmatrix} m_{x'y} & -m_{x'x} & 0 \\ m_{y'y} & -m_{y'x} & 0 \\ m_{z'y} & -m_{z'x} & 0 \end{bmatrix}$$

*Electronic Space Rods for Large Plotters**

CORWIN H. BRUMLEY and W. F. COOMBS,
Bausch & Lomb Incorporated,
Rochester 2, New York

ABSTRACT: *A servomechanism is described which automatically and continuously positions the illuminators for each of the projectors in a stereo plotter. The position of the plotting table is sensed by means of a lighting system located on the table platen which is monitored by photosensors associated with the illuminators. In this manner the optical axis of the illuminators is continuously directed onto the center portion of the plotting table platen for all movements of the table. Mechanical connections between the plotting table and the illuminators are thus replaced by the "space rods" of light emanating from the plotting table lights.*

SEVERAL mechanical systems have been used with direct projection plotters to couple illuminator movements to the motion of the tracing table. These mechanical couplings maintain the optic axes through the lamps, condensers, and projection lenses of the

* Presented at March 24–30, 1963 ASP-ACSM Convention, Hotel Shoreham, Washington, D. C.