*Test Measurements in Comparators and Tolerances for Such Instruments**

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ABSTRACT: *Comparator tests are assumed made with the aid of measurements of grid coordinates of high and known geometrical quality. Regular errors are determined as parameters in a least squares adjustment of the discrepancies, and irregular errors are estimated as standard errors of unit weight. The error propagation is studied. Tolerances of the regular errors, residuals and standard errors of unit weight are determined according to confidence limits from statistics. Practical examples.*

Certain regular errors of the grid coordinates can be distinguished from the errors of the comparator. Tests of some different glass scales are shown.

INTRODUCTION

THE concept, comparator, refers in this paper to an instrument for the measurements of plane orthogonal coordinates, primarily in photographic pictures. Such instruments have become very important for various measurements in photogrammetry, not only for practical restitution purposes but also in connection with tests and calibrations of other instruments. Comparators can therefore be regarded as instruments of basic importance for the entire photogrammetric activity. The geometrical quality of the results of the measurements in such instruments must therefore be of great interest.

As with all instruments, comparators change their geometrical quality over a period of time; therefore, check measurements must be made at certain time intervals from the delivery and during practical work. It is evidently necessary to make these measurements and the computations according to well-defined and established principles in order to get reliable information concerning the actual geometrical quality of the instrument, particularly concerning the need for mechanical adjustment. This problem is identical with the general problem in all calibration procedures for distinguishing as well as possible between regular or systematic errors of the measured data on one hand, and irregular or random errors on the other; also to estimate and express a statistical value of the irregular errors in a well-defined and unique way. The geometrical quality of the determination of the regular or systematic errors shall also be determined, together with tolerances for the magnitude of these errors.

The most effective method for determining these basic geometrical qualities of the comparator measurements is to use the instrument for measurements of data which are known with sufficiently high geometrical quality. A glass grid of sufficient density, and with the coordinates of the grid intersections given with high and known geometrical quality, is probably the best tool. Other means are also possible, for instance a linear glass scale of high geometrical quality, as proposed and practiced by Dr. H. Schmid and Mr. G. Rosenfield.

Here some general principles for comparator tests with the aid of glass grids will be discussed under different conditions concerning the geometrical quality of the given grid coordinates. Some results of test measurements in different instruments will also

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be shown, and the principles of determining tolerances for the regular and irregular errors will be discussed according to available statistical methods.

1. Test Measurements of Coordinates in Grids of High and Known Geometrical Quality

The grid is placed in the image holder of the comparator and adjusted so that the x- and y-directions of the grid approximately coincide with the corresponding directions of the comparator. The coordinates of a suitable number of grid intersections are then measured in the comparator under operational conditions. In order to increase the *precision* of the measured coordinates, it is suitable to replicate the measurements in each point a number of times, and to read, record, and register automatically the coordinates each time. Through a great number of replicated settings in one point (preferably at least 30) the standard deviation of one setting can be determined according to statistical principles. The standard deviation of the average of a number of replicated measurements is then found by dividing the standard deviation of one setting with the square root of the number of replications. Averages from four replications therefore have a standard deviation which is the half of the standard deviation of one measurement. In the experiments to be described here, the coordinate measurements were usually replicated three times. With a heavy heart it must be confessed that in all four instruments of different types, where test measurements were made, the electronic automatic registration failed in comparison with the direct reading.

After completing the main series of measurements, some points were remeasured in order to check that no significant changes had occurred. If so, the entire measuring procedure was repeated.

The averages of the measured coordinates were then compared with the given values. In order to facilitate the comparison, both sets of coordinates were referred to the center point as origin. The discrepancies between the measured and the given coordinates are defined as follows:

$$\begin{aligned} dx &= x_{\text{meas.}} - x_{\text{given}} \\ dy &= y_{\text{meas.}} - y_{\text{given}} \end{aligned} \tag{1}$$

These discrepancies were then interpreted as caused by regular and irregular sources in the measuring operations.

The following regular errors were considered:

1. Due to the adjustment of the grid in the plate holder: two translations dx_0 and dy_0 and one rotation $d\alpha$.
2. Due to the comparator: two scale errors dm_x and dm_y in the x- and y-directions respectively, lacking orthogonality $d\beta$ between the coordinate axes of the instrument.

The irregular errors were assumed to be caused primarily by the comparator and the operator. But the given coordinates of the grid cannot be regarded to be completely free from errors either, because they had been determined through measurements. Some of these irregular errors will be identical with the mentioned regular errors. But some of them will show up as residuals after the adjustment, and will be included in the concept, standard error of unit weight.

The basic error equations are, according to Figure 1:

$$\begin{aligned} dx &= -\, dx_0 + x dm_x + y(d\alpha + d\beta) \\ dy &= -\, dy_0 + y dm_y - x d\alpha \end{aligned} \tag{2}$$

The six unknowns can be determined if observations of the discrepancies dx and dy are determined in three suitably located points. More observations are, however,

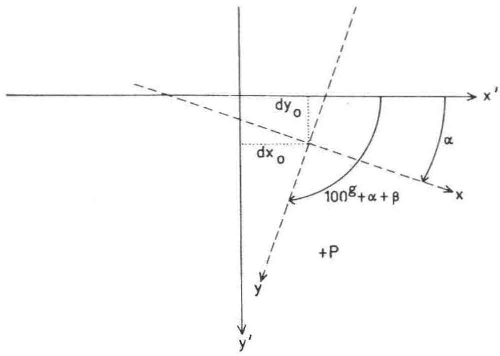

FIG. 1. Transformation of the coordinates of the point p between the two coordinate systems x, y and x', y' with the indicated parameters.

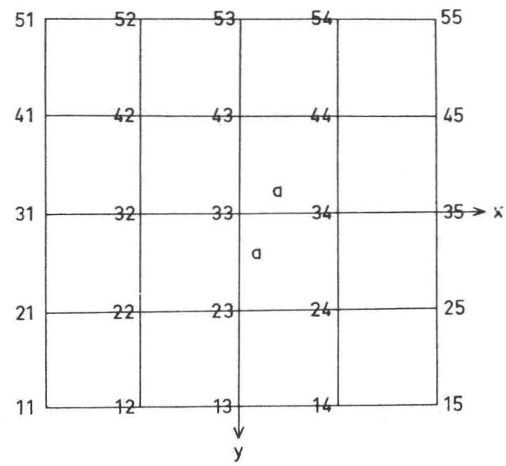

FIG. 2. Locations and notations of grid points used in the comparator tests.

always wanted and therefore the principles of least squares are applied in order to obtain a unique and convenient determination of the parameters. It is also suitable to obtain the parameters or regular errors as *corrections* directly. Therefore, the signs of the right side of the error Equations (2) are changed. Further, the residuals v_x and v_y are introduced, the sum of the squares of which is to be minimized.

Hence the following working correction equations are found:

$$\begin{aligned} v_x &= dx_0 - xdm_x - y(d\alpha + d\beta) - dx \\ v_y &= dy_0 - ydm_y + xd\alpha - dy \end{aligned} \tag{3}$$

The normal equations can then be formed according to well-known principles. It is immediately clear from the correction equations that the normal equations may be much simplified if the discrepancies are determined in points, which are regularly located, and if the coordinates of the points are given in a coordinate system, the origin of which is located in the point of gravity. One possible and suitable location scheme is shown in Figure 2. The 25 points are located in the corners of a grid. The coordinates of the points can be expressed in terms of the factor a, which is the side of the smallest square.

The correction equations are next applied to each of the points and the terms are tabulated in Table 1 for v_x and v_y.

The normal equations are next formed. They become very simple in this case and can be solved with elementary methods. The equations and the solutions are shown in the expressions (4) and (5). The symbols N_{251} etc., are simple combinations of the discrepancies and are shown in detail. The weight- and correlation-numbers have been derived according to their definitions and the expression $[vv]$ is determined according to well known formulas.

The weight- and correlation-numbers can be arranged in a matrix (the weight matrix or the variance-covariance matrix).

(*See page 305*)

TABLE 1
WORKING CORRECTION EQUATIONS

Point	x	y	dx_0	dy_0	dm_x	dm_y	$d\alpha$	$d\beta$	dx	dy
v_x										
11	−2a	2a	1		2a		−2a	−2a	$-dx_{11}$	
12	−a	2a	1		a		−2a	−2a	$-dx_{12}$	
13	0	2a	1		0		−2a	−2a	$-dx_{13}$	
14	a	2a	1		−a		−2a	−2a	$-dx_{14}$	
15	2a	2a	1		−2a		−2a	−2a	$-dx_{15}$	
21	−2a	a	1		2a		−a	−a	$-dx_{21}$	
22	−a	a	1		a		−a	−a	$-dx_{22}$	
23	0	a	1		0		−a	−a	$-dx_{23}$	
24	a	a	1		−a		−a	−a	$-dx_{24}$	
25	2a	a	1		−2a		−a	−a	$-dx_{25}$	
31	−2a	0	1		2a		0	0	$-dx_{31}$	
32	−a	0	1		a		0	0	$-dx_{32}$	
33	0	0	1		0		0	0	$-dx_{33}$	
34	a	0	1		−a		0	0	$-dx_{34}$	
35	2a	0	1		−2a		0	0	$-dx_{35}$	
41	−2a	−a	1		2a		a	a	$-dx_{41}$	
42	−a	−a	1		a		a	a	$-dx_{42}$	
43	0	−a	1		0		a	a	$-dx_{43}$	
44	a	−a	1		−a		a	a	$-dx_{44}$	
45	2a	−a	1		−2a		a	a	$-dx_{45}$	
51	−2a	−2a	1		2a		2a	2a	$-dx_{51}$	
52	−a	−2a	1		a		2a	2a	$-dx_{52}$	
53	0	−2a	1		0		2a	2a	$-dx_{53}$	
54	a	−2a	1		−a		2a	2a	$-dx_{54}$	
55	2a	−2a	1		−2a		2a	2a	$-dx_{55}$	
v_y										
11	−2a	2a		1		−2a	−2a			$-dy_{11}$
12	−a	2a		1		−2a	−a			$-dy_{12}$
13	0	2a		1		−2a	0			$-dy_{13}$
14	a	2a		1		−2a	a			$-dy_{14}$
15	2a	2a		1		−2a	2a			$-dy_{15}$
21	−2a	a		1		−a	−2a			$-dy_{21}$
22	−a	a		1		−a	−a			$-dy_{22}$
23	0	a		1		−a	0			$-dy_{23}$
24	a	a		1		−a	a			$-dy_{24}$
25	2a	a		1		−a	2a			$-dy_{25}$
31	−2a	0		1		0	−2a			$-dy_{31}$
32	−a	0		1		0	−a			$-dy_{32}$
33	0	0		1		0	0			$-dy_{33}$
34	a	0		1		0	a			$-dy_{34}$
35	2a	0		1		0	2a			$-dy_{35}$
41	−2a	−a		1		a	−2a			$-dy_{41}$
42	−a	−a		1		a	−a			$-dy_{42}$
43	0	−a		1		a	0			$-dy_{43}$
44	a	−a		1		a	a			$-dy_{44}$
45	2a	−a		1		a	2a			$-dy_{45}$
51	−2a	−2a		1		2a	−2a			$-dy_{51}$
52	−a	−2a		1		2a	−a			$-dy_{52}$
53	0	−2a		1		2a	0			$-dy_{53}$
54	a	−2a		1		2a	a			$-dy_{54}$
55	2a	−2a		1		2a	2a			$-dy_{55}$

$$
\begin{array}{c|cccccc|}
 & Q_{x_0} & Q_{x_0} & Q_{m_x} & Q_{m_y} & Q_\alpha & Q_\beta \\
\hline
Q_{x_0} & \frac{1}{25} & 0 & 0 & 0 & 0 & 0 \\
Q_{y_0} & 0 & \frac{1}{25} & 0 & 0 & 0 & 0 \\
Q_{m_x} & 0 & 0 & \frac{1}{50a^2} & 0 & 0 & 0 \\
Q_{m_y} & 0 & 0 & 0 & \frac{1}{50a^2} & 0 & 0 \\
Q_\alpha & 0 & 0 & 0 & 0 & \frac{1}{50a^2} & -\frac{1}{50a^2} \\
Q_\beta & 0 & 0 & 0 & 0 & -\frac{1}{50a^2} & \frac{1}{25a^2}
\end{array}
$$

NORMAL EQUATIONS

$$
\begin{aligned}
&25dx_0 - [dx] = 0 \\
&25dy_0 - [dy] = 0 \\
&50a^2dm_x - aN_{251} = 0 \\
&50a^2dm_y - aN_{252} = 0 \\
&100a^2d\alpha + 50a^2d\beta - aN_{253} = 0 \\
&50a^2d\beta + 50a^2d\alpha - aN_{254} = 0
\end{aligned} \tag{4}
$$

The solution of this system is

$$
\begin{aligned}
dx_0 &= \frac{[dx]}{25} \\
dy_0 &= \frac{[dy]}{25} \\
dm_x &= \frac{N_{251}}{50a} \\
dm_y &= \frac{N_{252}}{50a} \\
d\alpha &= \frac{N_{253} - N_{254}}{50a} \\
d\beta &= \frac{2N_{254} - N_{253}}{50a}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
N_{251} = {} & 2dx_{11} + dx_{12} - dx_{14} - 2dx_{15} + 2dx_{21} + dx_{22} - dx_{24} - 2dx_{25} + 2dx_{31} + dx_{32} \\
& - dx_{34} - 2dx_{35} + 2dx_{41} + dx_{42} - dx_{44} - 2dx_{45} + 2dx_{51} + dx_{52} - dx_{54} - 2dx_{55}
\end{aligned} \tag{6}
$$

$$\begin{aligned} N_{252} = & -2(dy_{11}+dy_{12}+dy_{13}+dy_{14}+dy_{15})-(dy_{21}+dy_{22}+dy_{23}+dy_{24}+dy_{25}) \\ & +(dy_{41}+dy_{42}+dy_{43}+dy_{44}+dy_{45})+2(dy_{51}+dy_{52}+dy_{53}+dy_{54}+dy_{55}) \end{aligned} \tag{7}$$

$$\begin{aligned} N_{253} = & -2(dx_{11}+dx_{12}+dx_{13}+dx_{14}+dx_{15})-(dx_{21}+dx_{22}+dx_{23}+dx_{24}+dx_{25}) \\ & +(dx_{41}+dx_{42}+dx_{43}+dx_{44}+dx_{45})+2(dx_{51}+dx_{52}+dx_{53}+dx_{54}+dx_{55}) \\ & -2dy_{11}-dy_{12}+dy_{14}+2dy_{15}-2dy_{21}-dy_{22}+dy_{24}+2dy_{25}-2dy_{31}-dy_{32} \\ & +dy_{34}+2dy_{35}-2dy_{41}-dy_{42}+dy_{44}+2dy_{45}-2dy_{51}-dy_{52}+dy_{54}+2dy_{55} \end{aligned} \tag{8}$$

$$\begin{aligned} N_{254} = & -2(dx_{11}+dx_{12}+dx_{13}+dx_{14}+dx_{15})-(dx_{21}+dx_{22}+dx_{23}+dx_{24}+dx_{25}) \\ & +(dx_{41}+dx_{42}+dx_{43}+dx_{44}+dx_{45})+2(dx_{51}+dx_{52}+dx_{53}+dx_{54}+dx_{55}) \end{aligned} \tag{9}$$

$$\begin{aligned} & Q_{x_0x_0} = Q_{y_0y_0} = \frac{1}{25} && s_{x_0} = s_{y_0} = \frac{s_0}{5} \\ & Q_{m_xm_x} = Q_{m_ym_y} = \frac{1}{50a^2} && s_{m_x} = s_{m_y} = s_0\frac{\sqrt{2}}{10a} \\ & Q_{\alpha\alpha} = \frac{1}{50a^2} && s_\alpha = s_0\frac{\sqrt{2}}{10a} \\ & Q_{\beta\beta} = \frac{1}{25a^2} && s_\beta = \frac{s_0}{5a} \\ & Q_{\alpha\beta} = -\frac{1}{50a^2} \end{aligned} \tag{10}$$

Confidence limits of the corrections are: on the five per cent level $\pm 2.0s$ and on the one per cent level $\pm 2.7s$ where s is the corresponding standard error.

$$[vv] = [dx^2] + [dy^2] - \frac{[dx]^2 + [dy]^2}{25} - \frac{N^2{}_{251} + N^2{}_{252}}{50} - \frac{(N_{253} - N_{254})^2 + N^2{}_{254}}{50} \tag{11}$$

The standard error of unit weight is

$$s_0 = \sqrt{\frac{[vv]}{44}} \tag{12}$$

The standard error of the standard error of unit weight is

$$s_{s_0} = \frac{s_0}{\sqrt{88}} = 0.11s_0 \tag{13}$$

The confidence limits of s_0 are: on the five per cent level $0.9s_0$–$1.3s_0$ and on the one per cent level $0.8s_0$–$1.4s_0$.

If additional points had been measured besides the 25 points, in which the adjustment has been made, corrections to the measured coordinates could have been computed, with the aid of the expressions (2). The residuals in the points after the corrections could have been found with the aid of the expressions (3), where dx and dy denote the discrepancies before the corrections. In order to judge the significance of the residuals, their standard errors are of great interest. The tolerances of the residuals can then be determined with the aid of the confidence limits, which can be derived from the standard errors.

In order to determine the standard errors of v_x and v_y from (3), the general law of error propagation is applied, see Reference (1). We find directly from (3):[1]

$$
\begin{aligned}
Q_{v_xv_x} &= Q_{x_0x_0} + x^2{}_{m_xm_x} + y^2Q_{\alpha\alpha} + Qy^2Q_{\beta\beta} + 2y^2Q_{\alpha\beta} + 1 \\
Q_{v_yv_y} &= Q_{y_0y_0} + y^2Q_{m_ym_y} + x^2Q_{\alpha\alpha} + 1
\end{aligned} \tag{14}
$$

After substitution of the weight-and-correlation numbers from (10), the following expression is obtained:

$$
Q_{v_xv_x} = Q_{v_yv_y} = \frac{26}{25} + \frac{x^2 + y^2}{50a^2} \tag{15}
$$

The standard errors of the corrected coordinates are then

$$
s_x = s_y = s_0\sqrt{\frac{26}{25} + \frac{x^2 + y^2}{50a^2}} \tag{16}
$$

There is evidently a certain variation of the standard errors depending upon the location of the actual points, see Figure 3.

The root mean square value of the expression (16) can be determined analytically within a certain area. For the square $-2a < x < +2a$ and $-2a < y < +2a$ the value $1.06s_0$ is found, which corresponds well with Figure 3. With a minor approximation the standard error of the residuals in additional points within the grid can therefore be regarded to be

$$
s_{\text{res.}} = 1.1s_0 \tag{17}
$$

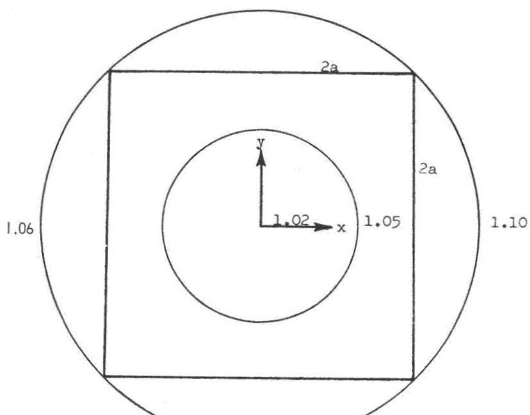

FIG. 3. Distribution of standard errors of corrected coordinates; corrections determined from 25 points.

From closer investigations of the residuals, further regular errors of the basic measurements may be detected, as for instance, periodic variations of the screws of the comparator, etc. A graphical summary of the residuals generally facilitates this investigation.

The adjustment procedure can be repeated including such a detected or assumed regular error as an additional parameter. If the standard error of unit weight after such a repeated adjustment procedure is significantly decreased, if the magnitude of the parameter is significant, and if the regular pattern of the residuals has disappeared, the introduced parameter can be regarded as a regular error. It is always advisable to repeat the entire measuring and adjustment procedure.

1.1 PRACTICAL EXAMPLE

In order to illustrate the theoretical development, the results of the computations of test measurements in a comparator will be shown. A glass grid was used, which had been checked carefully. The standard error of the grid coordinates had been determined to be about 1 micron. In order to determine the *precision* of the coordinate

[1] This simple procedure is identical with the variance-covariance technique from statistics and can also be expressed in matrix notations.

measurements in the instrument to be tested, 22 replicated settings were made and recorded. The standard deviation of one setting was found to be one micron. In the test measurements three settings were used to each point and the average was computed. The standard deviation of the average was, therefore, of the order of magnitude 0.5–0.6 microns.

Sixty-one points of the grid were measured, 25 of which were located in a regular grid pattern with the outer dimensions 200×200 mm. The factor a is 50 mm. The adjustment was made with the aid of the discrepancies in the 25 points and residuals were then computed in all points. All computations were made in suitable forms with the aid of a desk calculating machine. Computations in high-speed electronic computers can of course be used and are certainly suitable in the practical application of the procedure.

The following data were found:

$$[dx] = 63 \quad [dy] = 43; \quad N_{251} = 48; \quad N_{252} = 17; \quad N_{253} = -2756; \quad N_{254} = -1385$$

For $a = 50{,}000$ microns, the following corrections are found from the expressions (5):

$$dx_0 = +2.5 \text{ micron} \qquad s_{x_0} = 0.3 \text{ micron}$$

$$dy_0 = +1.7 \text{ micron} \qquad s_{y_0} = 0.3 \text{ micron}$$

$$dm_x = 19 \cdot 10^{-6} \qquad s_{mx} = 4 \cdot 10^{-6}$$

$$dm_y = 7 \cdot 10^{-6} \qquad s_{my} = 4 \cdot 10_{-}^{6}$$

$$dm_x - dm_y = 12 \cdot 10^{-6} \qquad s_{m(x-y)} = 5 \cdot 10^{-6}$$

$$d\alpha = -548 \cdot 10^{-6} \text{ radians} = -3^{c}\,49^{cc} \qquad s_\alpha = 2\overset{cc}{.}5$$

$$d\beta = -5.6 \cdot 10^{-6} \text{ radians} = -3\overset{cc}{.}6 \qquad s_\beta = 3\overset{cc}{.}2$$

$$[vv] = 76 \text{ micron}^2$$

$$s_0 = 1.3 \text{ micron}$$

The residuals are shown in Figure 4a.

The standard errors of the corrections were computed from the expressions (10). The standard error of the difference $dm_x - dm_y$ was derived from the basic expressions of the difference.

The standard error of the grid coordinates is about 1 micron. The standard error of the comparator therefore becomes $\sqrt{1.69-1}$ or about 0.8 microns. The results indicate certain scale errors and in particular a difference of the scale errors in x and y. This means a certain affine deformation. The orthogonality is very good.

After computing corrections to the results of the measurements in the additional points, the root mean square values of the residuals (figure 4b) were found to be:

in x 1.8 micron

and in y 1.9 micron

2. Tolerances

The corrections dm_x, dm_y and $d\beta$ are referred to the comparator and indicate lacking adjustment of the instrument. If the corrections are sufficiently large, mechanical adjustments of the instrument should be made. It is therefore of importance to determine the magnitude of the corrections (tolerances) which require that adjustment. Further, the residuals in the grid points after the numerical adjustment of the discrepancies and corrections should also be subjects to tests concerning their magnitude. If they are too large, gross errors or lack of adjustment of the instrument may be the

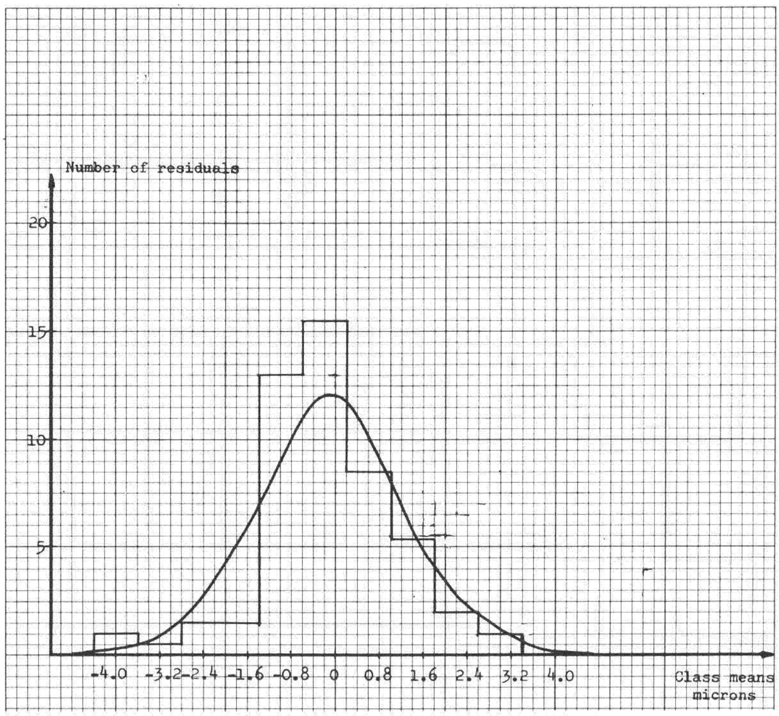

FIG. 4a. Histogram and normal distribution curve of residuals after adjustment of grid coordinate measurements in a stereocomparator.

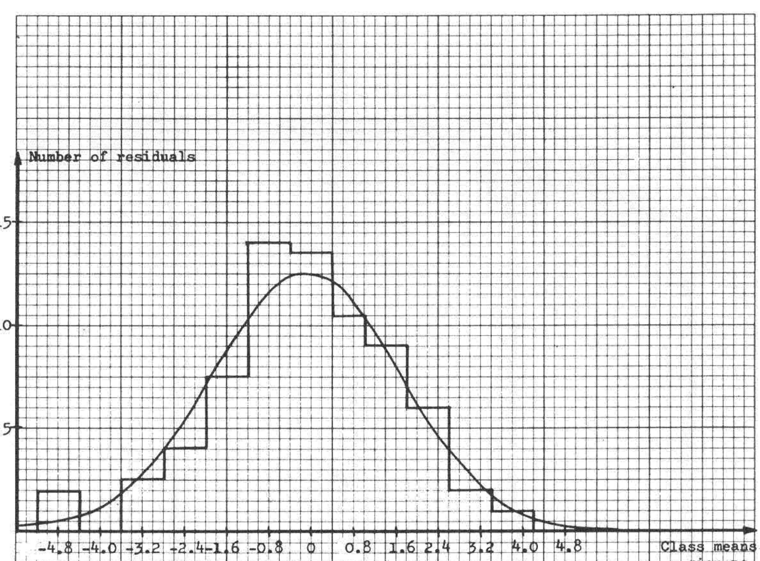

FIG. 4b. Histogram and normal distribution curve of residuals in additional points after adjustment of grid coordinate measurements in a stereocomparator.

cause. The root mean square value of the residuals in the additional points is an indication of the geometrical quality of the instrument and tolerances for this value are therefore of great interest. Also, the standard error of unit weight is an indication of the geometrical quality and should not exceed certain values, which may be furnished by the instrument manufacturer as a characteristic of the type of instrument in question.

In all these cases it is of great interest to find well-defined procedures to determine the tolerances. Here the principles of confidence limits from statistics will be applied. See Reference [2]. There is a certain difference between the determination of tolerances for linear functions of the basic measurements (for instance the corrections dm_x, dm_y and $d\beta$ or the residuals v_x and v_y) or for root mean square values or standard errors.

2.1 LINEAR FUNCTIONS OF THE BASIC MEASUREMENTS

The tolerances of the corrections or residuals are determined with respect to the standard errors of the same quantities, and are simply found from multiplication of the standard errors with certain factors. These factors are determined with respect to the reliability of the determination of the standard error (the degrees of freedom) and the degrees of reliability of the statement that the errors should not exceed the actual tolerance (the confidence level).

For a limited number of redundant observations in the determination of the standard error of unit weight, the t-distribution is used. If there are many redundant observations (more than at least 30), the normal distribution can also be used. Moreover, the residuals from which the standard error of unit weight was determined and other residuals are assumed to be normally distributed on a reasonable level. Normal distribution tests should also be applied. The standard errors of the corrections and of the residuals are computed as the product between the standard error of unit weight and the square root of a weight number. See the expressions (10) and (16). The factors t_p with which the standard errors are to be multiplied in order to find the tolerance limits, are determined from a table of the t-distribution. See for instance Reference [2]. For the number 44 of redundant observations and on the five per cent confidence level, the factor t_p is 2.0. On the level one per cent, the value of t_p is 2.7. This means that the tolerances of the corrections and of the residuals under the mentioned conditions are ± 2.0 and ± 2.7 times their standard errors respectively. The risk that the errors of the distributions exceed these values are five and one per cent respectively. The tolerance statements can therefore be erroneous with the same percentages. In the example under point 1.1, the correction dm_x is larger than the tolerance limits on both levels but dm_y is smaller as well as $d\beta$. The difference $dm_x - dm_y$ exceeds the tolerance limits on the five per cent level but not on the one per cent level. The standard errors of the residuals are about 1.1 s_0 and the tolerances are consequently 2.2 s_0 and 3.0 s_0 on the five and one per cent levels respectively. This means in the actual case, 2.9 and 3.9 micron respectively.

2.2 STANDARD ERRORS

Standard errors of functions of the basic measurements have a chi^2-distribution and the tolerances must therefore be determined with respect to this distribution. See Reference [2]. The root mean square value of the residuals in additional points can, with some approximation, be regarded as the true value of the standard error of the residuals. If this standard error is determined as a mean value of the expression (16) and a comparison is made with the root mean square value of the true residuals, a certain difference is likely to be found. For the determination of tolerances for such a difference, the procedure derived in Reference [2] can be used. The confidence limits of the standard error of unit weight are:

$$s_0 \frac{\sqrt{n}}{\text{chi}_{p''}} \quad \text{and} \quad s_0 \frac{\sqrt{n}}{\text{chi}_{p'}}$$

where

s_0 is the standard error of unit weight,

n is the number of observations for the determination of s_0,

$\text{chi}_{p''}^2$ and $\text{chi}_{p'}^2$ are the p'' and p' per cent values of the chi^2-distribution for the actual degrees of freedom at the determination of s_0. A table of the chi^2-distribution can be used. See Reference [2].

In terms of the confidence level p per cent, the p'- and p''-values are defined as follows:

$$p' = 100 - \tfrac{1}{2}p \qquad p'' = \tfrac{1}{2}p$$

For $n=50$ and 44 degrees of freedom, the confidence (tolerance) limits of the standard error of unit weight are on the five per cent level 0.9 s_0 and 1.3 s_0 respectively. On the one per cent level, the limits are 0.8 s_0 and 1.4 s_0. Under point 1.1 the root mean square values of the residuals in additional points were found to be 1.8 and 1.9 micron in x and y respectively. For the standard error of the residuals of 1.43 micron according to (17), the tolerance limits become 1.3 and 1.9 micron on the five per cent level and 1.1 and 2.0 micron on the one per cent level. The tolerance limits are in this case nearly reached.

Similar considerations can be applied to the standard error of unit weight itself. If the manufacturer of an instrument has defined the quality of the instrument in terms of a certain value of the standard error of unit weight, the tolerance limits of this value may be computed according to the procedure shown above. There are certain approximations in this procedure but they may be acceptable.

3. Test Measurements with Grids of Low Geometrical Quality

Sometimes the regular errors of the grid have to be considered in connection with the tests of a comparator. The regular errors of the grid and of the comparator in such a case must be distinguished from each other as far as possible. This can be done through some simple tricks in connection with the measurements. The grid is measured in four positions and is rotated through a right angle between the series of measurements. See Figure 5. It can also be turned upside down and measured in four positions again. The regular errors of the comparator remain constant during all series of measurements but the directions of the regular errors of the grid become changed between the different positions.

The computations can be made in two different ways. The results of the measurements in each position can be computed separately and the regular errors can be separated from suitable combinations of the results of the individual computations. If the regular errors, as determined from the different positions I, II, III, and IV in the location grid line up, are denoted by the subscripts UI, UII, etc., and if the regular errors are denoted with the subscript c for the comparator and g for the grid, the following expressions can be derived.

$$dm_{xc} - dm_{yc} = \frac{dm_{xU\text{I}} - dm_{yU\text{I}} + dm_{xU\text{II}} - dm_{yU\text{II}} + dm_{xU\text{III}} - dm_{yU\text{III}}{}^{+} + dm_{xU\text{IV}} - dm_{yU\text{IV}}}{4}$$

$$dm_{yg} - dm_{xg} = \frac{-(dm_{xU\text{I}} - dm_{yU\text{I}}) + dm_{xU\text{II}} - dm_{yU\text{II}} - (dm_{xU\text{III}} - dm_{yU\text{III}}) + dm_{xU\text{IV}} - dm_{yU\text{IV}}}{4}$$

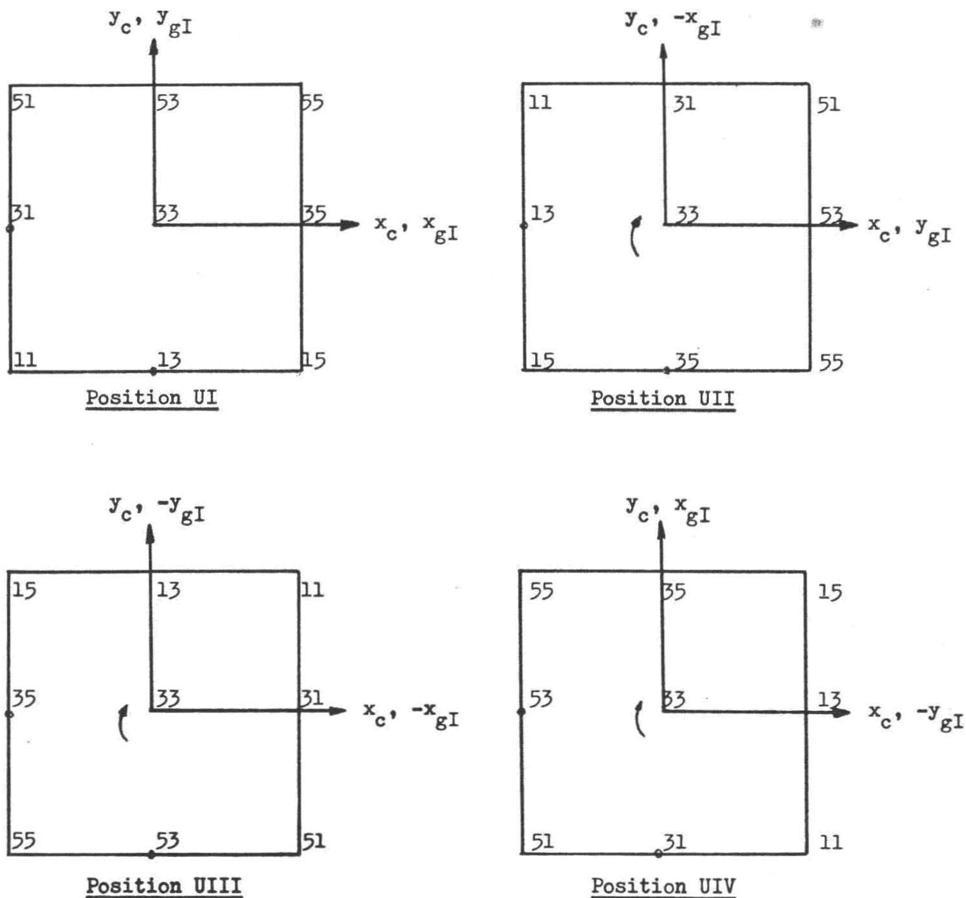

FIG. 5. The four positions of the grid (*lines up*) for comparator tests. x_c, y_c = the coordinate system of the comparator. x_g, y_g = the coordinate system of the grid.

$$d\beta_c = -\frac{d\beta_{UI} + d\beta_{UII} + d\beta_{UIII} + d\beta_{UIV}}{4}$$

$$d\beta_g = -\frac{d\beta_{UI} - d\beta_{UII} + d\beta_{UIII} - d\beta_{UIV}}{4}$$

It is also possible to adjust the results of the measurements in the four positions simultaneously. There are 18 unknowns and consequently 18 equations. General solutions of these equations have been made Ref. [3].

It is also possible to combine the measurements in the two locations up and down of the grid.

In these cases no absolute values of the scale can be determined, only scale differences. Therefore, no standard error of unit weight can be determined in the same sense as if the grid coordinates are regarded as given errorless values.

The absolute scale can be determined with the aid of measurements in a glass scale of sufficient geometrical quality. Some examples of such test measurements will be shown.

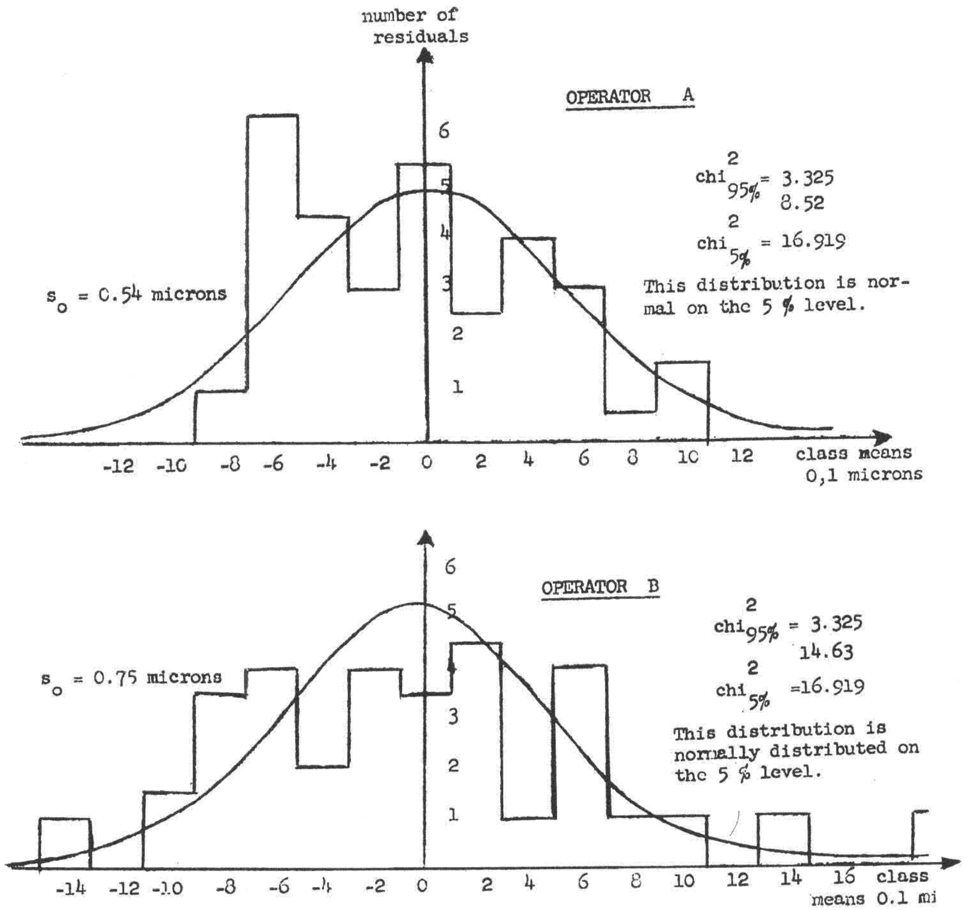

FIG. 6. Histogram and normal distribution curves for the residuals after adjustments of comparator measurements in glass scales. Two operators A and B.
The chi²-test indicates that the upper histogram is more "normal" than the lower one.

4. Test Measurements with Glass Scales

Some series of test measurements of glass scales in comparators have been received from Dr. Markowitz, U. S. Naval Observatory and from Dr. Washer, National Bureau of Standards.

Adjustments of the discrepancies between given and measured data have been made according to least squares in order to determine the basic geometrical quality of such measurements. The adjustments were made with two parameters, one translation and one scale change.

Only the final results of the test computations will be given here.

Glass scale 1. Length 170 mm.
Scale correction:
0.0129 micron/mm.
Standard error of unit weight: 0.2 micron

Glass scale 2. Length 340 mm.
Average scale correction:
−0.0182 micron/mm.

Standard errors of unit weight (two operators): 0.5 and 0.7 micron. In this case there were so many residuals that normal distribution tests could be done. The results are shown in Figure 6.

The results of these glass scale tests show that a very high geometrical quality can be obtained in the determination of absolute scales of a comparator.

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Anaglyphic Calculating Apparatus

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Abstract: *In this paper is described a simple apparatus that may be used instead of a normal slide-rule. It is composed of a special nomogram printed in two complementary colors, of anaglyph viewers and a movable index. With this apparatus it is possible to multiply and to divide some numbers and also to add and to subtract them. This accuracy is equal to that of a normal slide-rule.*

Introduction

As is known, one of the methods of obtaining stereoscopic vision is the anaglyphic process. For this, two photographs of an object, taken from two stations A and B, are printed one on top of the other on the same sheet of paper. The distance from A to B is parallel to the plane of reference of the object. It is called "the Stereoscopic Base." Each photograph is colored in one of two complementary colors e.g. red and green.

The picture is observed through anaglyph viewers with glasses of the same complementary colors as the photographs. Because the glasses are of transparent colored matter, the eye which looks through the red glass perceives the background as red, and the red-printed details merge with this red color. Against that the green-printed details are much more visible and their color is almost black. In a like manner, the eye which looks through the green glass perceives only the red-printed details because the green-printed details merge with the color of the background.

The result is that each eye sees only one of the two photographs; this gives rise to stereoscopic vision—namely the formation of the *"Optical Model."*

Now if a needle is introduced between the optical model which is being observed and the anaglyph viewers, this single needle is seen by both eyes. Hence, by reason of the stereoscopic property of the binocular view, the needle is seen in its true place above the sheet of paper.

It follows that when the needle point is seen as touching a point of the optical model, the height of this needle point above the picture is equal to the height of the point of the model above the same picture.

As known, the heights of the various points