*The Investigation of Error Propagation in Aerial Triangulation**†

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A STUDY of the photogrammetric literature published during the last thirty years would undoubtedly lead to the conclusion that the propagation of error in aerial triangulation has been both a greatly investigated and a well discussed subject. Such investigation and discussion has been largely stimulated by the need for a solution to the problem of adjusting triangulated strips and blocks in order to provide pass-point positions that are consistent amongst themselves and as accurate as possible under a given set of conditions.

It would seem, in view of the tremendous expenditure of time and talent on the problem of error propagation in aerial triangulation, that by now we would know all that is worth knowing about this area of photogrammetry. Nothing could be further from the truth; there are still many facets of this interesting subject which have not yet been thoroughly investigated and the problems associated with them resolved.

Of the more than 700 titles collected and filed by the author at the University of Illinois for a bibliography on aerial triangulation, a very large percentage deals either directly or indirectly with error propagation. To anyone familiar with the subject, this is not too surprising. This present paper is not intended to add any new basic knowledge to this vast reservoir of material already existing. Instead, it is simply intended to be a brief survey of a few of the more notable theoretical and experimental investigations catalogued in this bibliography.

Although the idea of aerial triangulation in space had undoubtedly been conceived in the very early years of this century by such pioneers as Sebastian Finsterwalder and Theodor Scheimpflug, it wasn't until the period following the First World War that the necessary instrumentation and methods were developed to make aerial triangulation in space a practical procedure. It seemed only natural during this period of vigorous growth in the science of photogrammetry that interest would be drawn toward those problems dealing with the accuracy of the results produced by the various methods of aerial triangulation. Indeed, the problem of analyzing the various sources of irregular and systematic errors in the triangulation procedure attracted the attention of many European photogrammetrists in the period 1930 to 1940. Thus, in those years immediately preceding World War Two, the investigation of errors in aerial triangulation became one of the big challenging problems in the theory and practice of photogrammetry. An interesting account of the work during this period was given by W. Schermerhorn in a publication in *Photogrammetria* entitled "Einleitung zur Fehlertheorie der räumlichen Aerotriangulation" (Introduction to the Theory of Errors of Aerotriangulation in Space) [15].

Of particular importance in these early years was the contribution made by the distinguished German photogrammetrist, Otto von Gruber. The experimental work of von Gruber and his associates in the field of aerial triangulation actually began in 1924 shortly after the Zeiss Stereoplanigraph was introduced. Much of this experimental work and the theoretical investigations that developed from it were documented by von Gruber in 1935 in a monumental article entitled "Beitrag zu Theorie und Praxis von Aeropolygonierung und Aeronivellement" (Contribution to the Theory and Practice of Aerotraversing and Aeroleveling). In this article, von Gruber demonstrated the effects of two specific systematic errors in a triangulated strip—a y-tilt common to all photographs and a uniform convergence error in the nadir directions of consecutive models. Assuming the x direction to be along the length of the strip, the constant y-tilt error produced a height error that varied linearly with x and an x-coordinate error that varied parabolically with x,

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each function containing a coefficient which depended on the magnitude of the y-tilt error, and which Professor Schermerhorn later proposed to designate the *Gruber Coefficient.* The uniform convergence error produced no x-coordinate error but it did produce a height error that varied parabolically with x.

The year 1935 also marked the beginning of the practical application of aerial triangulation to mapping projects. Literally hundreds of strips were triangulated in the Netherlands for large aerial survey projects all over the world during the period 1935 to 1940; a great deal of valuable experience was obtained. Gruber coefficients were computed for the triangulated strips and corrections were made accordingly. However, certain unusual things were revealed when the strips were triangulated forward and backward and the respective coordinates were compared, or when the coordinates of points common to adjacent strips were compared. For one thing, the coordinate discrepancy curves indicated much irregularity in the Gruber coefficients, and for another thing, decided breaks or jumps were occurring in the curves.

This unusual problem was studied during the years of the Second World War and some rather revealing experimental data resulted, principally that of E. Gotthardt of Germany and R. Roelofs of the Netherlands. Accounts of these experiments were published by Gotthardt in 1944 and by Roelofs in 1949 [12]. These two independent studies showed that purely random errors, propagating through a strip of triangulation as a double summation, could lead to coordinate error curves that exhibited a deceivingly systematic appearance. Furthermore, it was observed that the breaks or jumps in the coordinate error curves, witnessed previously by the Dutch photogrammetrists in their coordinate discrepancy curves, could be also generated from a double summation of purely random errors. Hence a breed of error was obtained that had a random origin but a systematic looking effect. The term "quasi-systematic" has been used by some photogrammetrists to describe this kind of error.

These experimental investigations of Gotthardt and Roelofs were followed later by the theoretical treatment of Professor E. H. Thompson of England, the results of which were published in 1953 in the very first issue of the *Photogrammetric Record* [19]. By using a simple mathematical model resembling that of a ground traverse between two fixed control points, Thompson showed that the systematic appearance of doubly summed random errors was attributable to the high degree of correlation existing between the coordinates of the successive stations. Indeed, it was established that the correlation coefficient between the coordinates of two adjacent stations was only slightly less than unity, and that the correlation coefficient between the coordinates of the most distant stations was about 0.5, noting that a value of zero implies no correlation while a value of unity implies full correlation.

The actual problem that interested Professor Thompson was that of detecting the presence of any real systematic error lying underneath the cover of deception manufactured by the doubly summed random errors. This prompted him to develop a statistical test for the detection of systematic error, using the station coordinate errors as the observed data.

Recently there has been some renewed activity, both theoretical and experimental, in the study of the systematic effect of random errors. In the October 1959 issue of the *Photogrammetric Record* [6] Professor B. Hallert of Sweden drew attention to the arc sine law of probability theory, taking a particular experimental example from William Feller's textbook "An Introduction to Probability Theory and Its Applications." The example presented the results of 10,000 tosses of a coin as a plot of cumulative gain versus toss number, and Professor Hallert pointed out that there was a striking resemblance between the errors found in an aerial triangulation and the results of the 10,000 tosses of the coin. Here again, although the result of an individual toss of the coin was purely random, the cumulative gain exhibited a very systematic looking appearance. The arc sine law reflects the fact that frequently an enormous number of tosses is required before the cumulative gain returns to zero.

At the Ninth International Congress of Photogrammetry held in 1960 in London, Rudolf Förstner reported an experiment carried out at the German Institute for Applied Geodesy [4], in which 1,200 random values were obtained by tossing four dice. These values were doubly summed, thus imitating the random errors of an aerial triangulation. When plotted, these doubly-summed values produced a very systematic looking curve, as expected in view of the previous experimental work of Gotthardt and Roelofs. In order to show the irregular course of the errors more clearly, polynomials up to the tenth degree were fitted to the results, and the residual deviations were plotted. It was revealed that these residuals exhibited distinct oscillations. It was also found that the magnitude of the mean residual deviation decreased rapidly at first with an increase in the degree of polynomial used, but then leveled off, showing that additional terms in the polynomial provided little improvement.

The latest contribution to this phase in the study of error propagation was made by H. Moritz of Austria [10]. The work of Moritz is entirely theoretical, using the concept of conditional expectation to explain the relative smoothness of the doubly-summed error curve and its tendency to change direction very slowly.

Concurrent with the investigations aimed specifically toward the problem of doubly-summed random errors were the more formidable studies into the general propagation of accidental errors using more complex mathematical models to portray the triangulation procedure. The pioneering work of Roelofs should be mentioned in this respect, an extensive account of it appearing in the 1941 volume of *Photogrammetria* [11]. Unfortunately, war conditions did not permit Professor Roelofs to publish his work in its entirety.

Of considerable significance is the work by W. K. Bachmann published in 1946 [1]. In this rather lengthy treatise, Bachmann developed expressions for the mean square errors in the nadir-point coordinates as functions of the mean square error in the observation of y-parallax, the photo-scale and the distance along the strip. He, of course, did not generalize the problem completely in that he considered only his own particular method of relative orientation and he assumed specific simple relationships in the geometry of his mathematical model.

Further theoretical work into the general propagation of random error was done in 1954 by P. A. Vermeir [20], the work resulting in expressions similar to those of Bachmann. Vermeir's treatment was characterized by the variables that were chosen as quasi-observations, these variables being the scale, azimuthal rotation, longitudinal tilt and lateral tilt of each model. Vermeir proceeded to show that these variables were uncorrelated, thus permitting him to simplify his treatment and arrive at the final expression in a relatively efficient manner.

Professor Hallert has continued the general investigation of the propagation of random error in aerial triangulation using an approach built upon the work of Bachmann [8]. In contrast to Bachmann's treatment, however, Professor Hallert assumed the numerical method of relative orientation, and he additionally took account of the effect of errors in the first model of the triangulation. The cofactors for the x, y and z coordinates of the nadir-points were derived, and additionally, the cofactors for the coordinates of points along the strip edges were derived.

Parallel with the several investigations into the propagation of random errors were those chiefly concerned with the propagation of systematic errors in triangulated strips. The early work of von Gruber has, of course, been mentioned. Considerable additional activity in this aspect of aerial triangulation was generated at Zürich, Switzerland under the direction of Professor M. Zeller.

A notable contribution from the Swiss school was made in 1952 in a doctoral thesis by J. Zarzycki [22]. In this work, Zarzycki set forth the various sources of systematic error which could be present in a triangulation, and then proceeded to show how they developed into errors in the heights and planimetric coordinates of the nadir-points. Two distinct methods of aerial triangulation were considered—aeropolygons and aeroleveling. For aeropolygons, it was shown that the height and y-coordinate errors were parabolic functions of the strip length, while the x-coordinate error was a cubic function of the strip length. For aeroleveling, it was found that the height error was a linear function of the strip length, while the planimetric coordinate errors were parabolic functions of the strip length.

The possibility of improving the accuracy of the pass-point coordinates, by triangulating a strip of photographs several times and taking average values, has undoubtedly appealed to a number of photogrammetrists. Certainly there should be some improvement of the accuracy if such a procedure is followed but there is apparently little significant improvement. This circumstance has been largely attributed to the residual errors in the photograph image-points themselves due to such things as variations in atmospheric refraction, lens distortion and irregular film shrinkage. When the triangulation is repeated, these residual errors are repeated, and their effect on the pass-point coordinates is duplicated, thus affording no improvement in the accuracy.

An interesting series of experiments concerning this feature of aerial triangulation were conducted by P. Wiser for Commission A of the European Organization for Experimental Photogrammetric Research (O.E.E.P.E.). In 1960, Professor Wiser published some of the results of these experiments in the journal of the Belgian Society of Photogrammetry [21]. A strip of 102 photographs with a base-height ratio of 0.2 was used. From this single strip, three separate strips of 34 photographs, each with a base-height ratio of 0.6, were obtained. Strip *A* consisted of photographs 1, 4, 7, etc., strip *B* consisted of photographs 2, 5, 8, etc., and strip *C* consisted of photographs 3, 6, 9, etc. The investigation showed that if strip *A* were triangulated three times and average values taken, there was little improvement in the accuracy of the pass-point positions. Indeed, the residual error curves for each of the three runs exhibited a striking similarity. On the other hand, if strips *A*, *B* and *C* were each triangulated once and average values taken, then there was significant improvement in the accuracy of the pass-point positions. The residual error curves for the three separate strips were quite different. The term "pseudo-accidental" has been applied to those irregular errors which are reproduced when triangulation of the same strip is repeated.

In recent years a new approach to the study of errors in measuring systems has been made practicable because of the introduction of the electronic digital computer. This approach is that of simulation of the measuring process in the computer using fictitious data. The various sources of error in an aerial triangulation can be investigated separately using this technique, provided that the mathematical model used for the simulation is a faithful representation of the physical reality.

Mention has been made of the simulation technique by at least two photogrammetrists, Arthur H. Faulds of the United States [3] and U. Bartorelli of Italy [2]. This approach would seem to have great promise, since it can provide large statistical samples from which reliable information can be extracted.

This capsule account should serve to emphasize that the problem of error propagation in aerial triangulation has not yet been satisfactorily resolved. Certainly a great deal more is known now about error propagation than was known some three decades ago when investigation of the problem was initiated. But there still remains considerable ignorance surrounding the very basic sources of error in the photogrammetric triangulation process. Knowledge of the character of the various errors inherent in a system is fundamental to the design of methods aimed at improving the accuracy of the system. There is little doubt that the study of the propagation of error in aerial triangulation, far from being a thing of the past, will continue to challenge photogrammetrists for many years to come.

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