*Use of Two-Directional Triplets in a Sub-Block Approach for Analytical Aerotriangulation**

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ABSTRACT: *A study of past and current methods of analytical aerotriangulation clearly indicates the close similarity in basic thinking between analytical treatments and instrumental techniques of triangulation. In this paper, the author attempts to reappraise the situation and introduces some new concepts and ideas that are more suitable for analytical photogrammetry. An equal all-around overlap, thus allowing for forming square units or sub-blocks, instead of strips, is advocated. For this sub-block approach, two methods of solution are presented: "Cantilever Sub-Block Extension" and "Sub-Block Best-Fit by Relaxation Procedure."*

INTRODUCTION

ELECTRONIC digital computers have been undergoing enormous improvements in the past few years. It is, therefore, expected that more research and development in the field of Analytical Photogrammetry will take place. By analytical photogrammetry we mean the purely digital solution of the perspective problem that yields ground coordinates of objects from their measured image coordinates.

Different survey organizations, both in this country and abroad, have been using the electronic computers for the computation of analytical aerial triangulation up to, but not including, the final block adjustment. The latter phase has been considered only in the more recent experiments. Even in these tests, the block considered is usually composed of only a few photographs in *the same strip* adjusted simultaneously. The only reason for this, in the authors opinion, is that most investigators have not been able to get away from the conventional approach of dividing the block of photography into strips. The concept of block adjustment, as commonly used nowadays, refers to the adjustment of all photographs in a strip or a block to fit the given ground control. When the word block is used it is often referred to as combining the different strips together after being partially adjusted whether instrumentally or mathematically.

The author believes that the basic unit in photogrammetry is the single photograph. In other words, all the photographs in a block of photography should be treated equally, and hence can be grouped in many different ways. In this respect, a group of strips happens to be one of these methods. To explain this generalized fact Figure 1 shows photos $(m-1)$, m, and $(m+1)$ in each of the strips $(n-1)$, n, and $(n+1)$. Consider the center photo (n, m). Since we are now concerned with the complete analytical solution, there will be no reason to give photos $(n, m-1)$ and $(n, m+1)$ preference over photos $(n-1, m)$ and $(n+1, m)$ in the amount of overlap with photo (n, m). Furthermore, it must be agreed that any one photograph in the block has its own orientation elements in space at the moment of exposure completely independent of all the others.

At this point, the reader is urged to consider whether there is really need to continue maintaining the conventional 2/3 forward

* In the 1963 competition for the Wild Heerbrugg Instruments Award to a graduate student, this paper was the winner.

‡ At the time of submitting this paper in the competition the author was a graduate student at Cornell University.

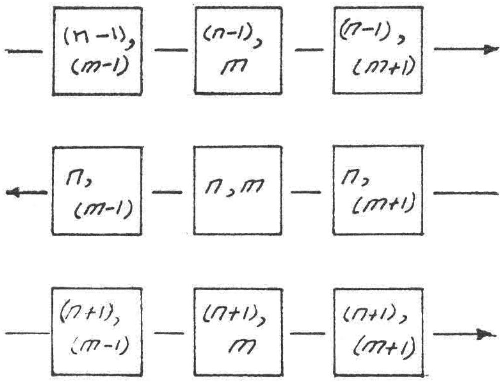

FIG. 1

overlap and 20% or so sidelap or not. It will soon appear to him that there is no need for such restriction. Actually he will be more inclined to provide for $\frac{2}{3}$ overlap in both directions, since there is no preference one way or the other. With the 67% overlap in all directions the geometric strength of the photogrammetric problem would be very great indeed.

The New Outlook

In reference (1), the new idea of the use of triplets was developed, and their advantages over conventional methods in analytical aerotriangulation were pointed out. In this paper, this idea will be continued much further and a treatment of the general problem of analytical photogrammetry will be tried.

The ultimate and most precise solution of the photogrammetric problem would result from the simultaneous solution of all the photographs in the block in one massive operation. This solution is expensive and time consuming, because of the large number of unknowns encountered. Accordingly ways are sought to reduce the computing time. These factors have been instrumental in leading researchers to resort to breaking the block into strips and the strips into models. The triplet approach was also developed along that line of thought.

A study of the strip approach yields the following comments:

(1) Within each strip, especially in case of cantilever extension, there exists unavoidable propagation of systematic errors. Furthermore, even if the absence of systematic errors is assumed, model warping still exists in long strips due to the presence of random errors[2].

(2) The use of triplets within the strip could help to decrease, but not to eliminate, the aforementioned warpage. Then, with the strip adequately adjusted, another difficulty is again faced, that is of the hinge effect between strips. At this point, one begins to realize how harmful the 20% sidelap is in causing the model discontinuities between adjacent strips particularly for analytical work.

With the above comments in mind, a turn can now be made to the block of photography to look for a better way of subdividing it into smaller units other than strips. The provision of $\frac{2}{3}$ overlap in all directions is the first step toward treating all photographs equally. A small unit, called the sub-block, composed of 2×2, 3×3, 4×4, $\cdots$, etc., photos will closely resemble the basic element (i.e. the single photo). The entire block will then consist of groups of these sub-blocks. Since the sub-block provides a much stronger element in itself, tying successive sub-blocks to a central one should help in reducing propagation of errors. Furthermore, with these sub-blocks as units there can be expected neither excessive model deformations (or warpage) nor model discontinuities between strips.

Any of the sub-blocks mentioned above (2×2, 3×3, 4×4, $\cdots$, etc.) is certainly a better unit than the strip. The choice between them depends—among other things—on the capacity of the computer at hand. Yet, the 3×3 sub-block has some advantages over the others. It is easily shown in Figure 2 that the 3×3 sub-block is mainly composed of

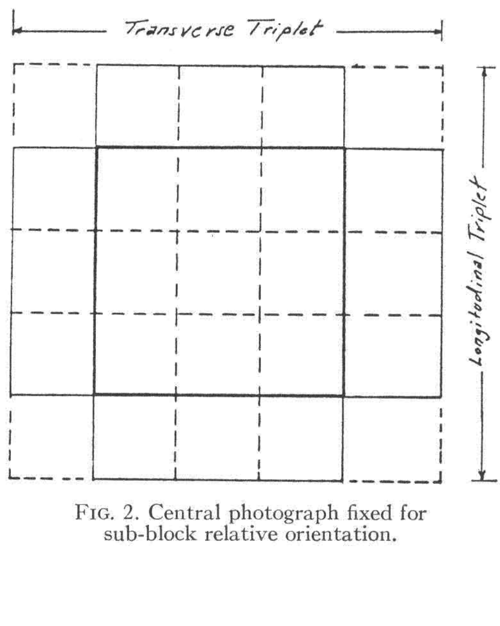

FIG. 2. Central photograph fixed for sub-block relative orientation.

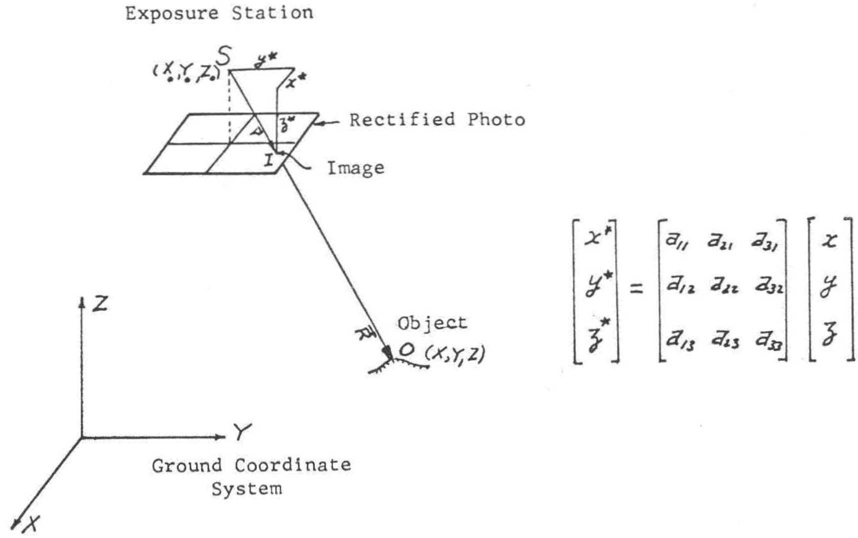

FIG. 3

two triplets, one in each direction and four photos rigidly connected to them at the four corners (shown by dotted lines in Figure 2). Therefore, all the advantages of the use of triplets in a strip, are also pertinent to the sub-block. Furthermore, with the existence of triple overlap in all directions (resulting in an area common to all nine photos) extension of control could be more effectively carried out by making best use of the existing control.

THE MATHEMATICAL FOUNDATION[3,4]

Figure 3 shows the basic geometric relationships between exposure station, image, and object.

The three point collineation condition is:

$$\vec{R} = \lambda \vec{r} \tag{1}$$

where λ is a scalar factor. Breaking the above equation into its three components

$$\begin{aligned} X &= X_0 + \lambda x^* \\ Y &= Y_0 + \lambda y^* \\ Z &= Z_0 + \lambda z^* \end{aligned} \tag{2}$$

in which x^*, y^*, z^* are the image coordinates rectified in a coordinate system parallel to the ground system through the application of the rotation A-matrix quite familiar to the reader.

Upon elimination of the scalar λ there comes the pair of equations expressing the line in space through points S, I, and O. It is obvious that for each image on every photo there can be written such a pair of equations. It must be noted though, that the unknown coordinates X, Y, Z of the object are included in the equations. Also, the rotational elements of the camera orientation are implicitly included in the elements of the rotation matrix.

To summarize, the unknown parameters can be divided into two categories: 1. Orientation: or exposure station parameters including three translations X_0, Y_0, Z_0 and three rotations ω, ϕ, κ, 2. Ground Coordinates X, Y, Z.

The total number of the unknown parameters for the 3×3 sub-block is not fixed, as it is clear that for every one ground point three new unknowns X, Y, Z are added. The orientation parameters, though, are fixed to:

$$3 \times 3 \times 6 = 54 \text{ unknowns.}$$

Figure 4 shows, in plan, the basic sub-block (3×3) with the $\frac{2}{3}$ overlap in all directions. The zones, designated by Roman numbers, indicate the number of photographs in which ground points in the specified area will appear. For example, ground points appearing in the area indicated by II will appear in two photos, in area IV will appear in four photos, etc. Table 1 shows these different zones and the number of independent equations every point yields.

Through use of this table, one can check the solvability of the sub-block for a given number of points and their configuration. For example, in Figure 4, nine points per photograph, representing 25 points on the ground in the area covered by the sub-block, are chosen. The total number of unknown parameters

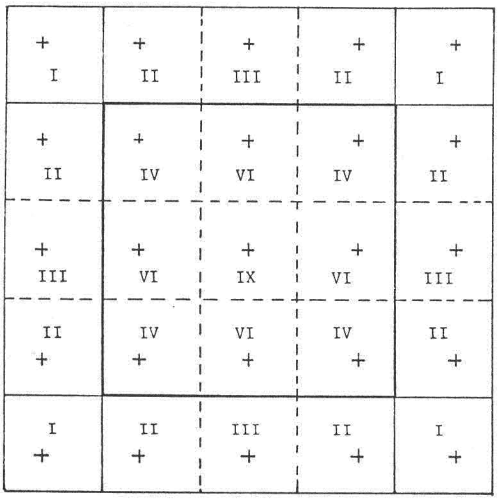

FIG. 4

will be

$$54 + 3 \times 25 = 129,$$

and the total number of equations is

$$9 \times 9 \times 2 = 162.$$

Therefore, for this particular example, 162 equations in 129 unknowns must be solved. This will involve calculations requiring the use of a computer of a fairly large size. However, as will be shown in the following sections, one can eliminate the 75 ground unknowns and possibly some of the 54 orientation unknowns. This elimination of unknowns will depend on two factors: (a) the manner in which the sub-blocks are tied together to agree with one another; and (b) the method by which the sub-blocks are fitted to the fixed ground control points present in the area covered by the whole block of photography.

Before discussing the details of solving the basic sub-blocks and the method of tying them together, there should be mentioned that, in general, one should not expect to have the total number of photographs to be a multiple of three in both directions. In other words, the number of photographs might have either longitudinally (in the line of flight direction) or transversely might be non-divisible by the number three. In such a case, incomplete sub-blocks might be encountered which could be composed of 3×2, or 2×2, photographs which will be referred to as edge sub-blocks. It will become obvious that solving these sub-blocks will be a straight-forward modification of the basic sub-block.

THE BASIC SUB-BLOCK

Two methods can be employed in solving the block problem through the method of sub-blocks: A. Cantilever Sub-Block Extension, and B. Adjustment of Sub-Blocks to the Best-Fit by Relaxation.

A. CANTILEVER SUB-BLOCK EXTENSION:[5]

This is a method directly adopted from the conventional cantilever strip extension. A basic sub block at the center of the area covered by photography is chosen as a starting point. After absolutely solving this central sub-block, and due to the presence of $\frac{2}{3}$ overlap in all directions, one can then proceed radially and tie the other sub-blocks to the central one. In the process of extension, one can enforce the orientation elements in two ways: (a) by using three photographs from the preceding sub-block (Figure 5(a)); or (b) by using the advantageous principle of triplets where six photographs are employed (Figure 5 (b)). Only the initial sub-block will involve 54 unknowns, while all succeeding sub-blocks will encounter 18, 12, or 6 unknown parameters for the method of Figure 5(b).

B. SUB-BLOCK BEST-FIT BY RELAXATION PROCEDURE:

In this method, the solution is basically divided into two operations; (1) sub-block

TABLE I

Zone	*Total Number of Independent Equations per Point (including X, Y & Z)*	*Total Number of Independent Equations per Point (without X, Y & Z)*
I. Single-Photo Zone	$2\times1=2$	
II. Two-Photo Zone	$2\times2=4$	$4-3=1$
III. Three-Photo Zone	$2\times3=6$	$6-3=3$
IV. Four-Photo Zone	$2\times4=8$	$8-3=5$
VI. Six-Photo Zone	$2\times6=12$	$12-3=9$
IX. Nine-Photo Zone	$2\times9=18$	$18-3=15$

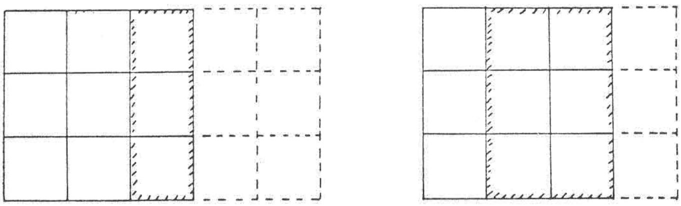

FIG. 5

relative orientation, or solving the constituent sub-blocks relatively and separately (without the use of any ground control); and (2) adjustment of the sub-blocks to the best-fit with regard to each other as well as to ground control. The second operation in this approach bears a resemblance to Jerie's analogue plan solution[6].

A comparison between the Cantilever Sub-Block Extension and the relaxation procedure leaves no doubt that the second is much preferable to the first. In the first method the old problem of error propagation is still not overcome. The carried-over parameters previously determined from preceding sub-blocks are always considered error-free, whereas this is not the case. Consequently, when the edge sub-blocks are reached, fairly large errors (in absolute sense) would have accumulated especially in extensive blocks of photography. Furthermore a careful examination of the sub-block extension procedure reveals the fact that an excessive number of sub-blocks must be solved for a varying number of unknowns. This is particularly true when the triplet principle is applied. As an example consider a block of 6×6 photographs. The operation of sub-block extension will require solving twelve sub-blocks as follows:

1 sub-block in 54 unknowns
6 sub-blocks in 18 unknowns each
4 sub-blocks in 12 unknowns each
and 1 sub-block in 6 unknowns.

with a total number of unknowns of 216 which is equal to $6\times6\times6$ (as a check). The same block, by the relaxation procedure will require the solution of four sub-blocks each in 48 unknowns; four sub-blocks each in 30 unknowns; one sub-block in 18 unknowns, and their adjustment to the best-fit as will be explained later.

For the above mentioned reasons the second method will be considered for more detailed investigation.

(1) SUB-BLOCK RELATIVE ORIENTATION

The absolute solution of the basic sub-block involves 54 unknown parameters as mentioned before. However, the sub-blocks will not be rigidly oriented in the geodetic system. In other words, orientation will not take place with respect to the ground control points in the project area. Consequently it is necessary only to relatively tie the photographs in the sub-block together. This step is quite similar to the step of relative orientation and hence it is called Sub-Block Relative Orientation.

The central photograph of the sub-block is held fixed (see Figure 2) and the surrounding eight photographs are simultaneously tied to it. (Notice the complete resemblance with the triplet solution in reference 1, and hence the name two-directional triplet.) Therefore, the six parameters of the central photo are zero and then one can solve for 48 unknown parameters. It should be mentioned that this step is taken in any arbitrary coordinate system.

Since this step is basically relative orientation, then only image points that appear in the overlap area are useable, i.e. those in zones II to IX in Figure 4.

A point appearing in two photos (Figure 6): will yield two of equations (1), i.e.

$$\vec{R_1} = \lambda_1\vec{r_1} \quad \text{and} \quad \vec{R_2} = \lambda_2\vec{r_2} \tag{3}$$

or in component form

$$\begin{aligned} X &= X_{01} + \lambda_1 x_1^* = X_{02} + \lambda_2 x_2^* \\ Y &= Y_{01} + \lambda_1 y_1^* = Y_{02} + \lambda_2 y_2^* \\ Z &= Z_{01} + \lambda_1 z_1^* = Z_{02} + \lambda_2 z_2^* \end{aligned} \tag{4}$$

These are six equations in terms of both the orientation and ground unknown parameters from which X, Y, and Z are easily eliminated. Then upon elimination of the scalars λ_1 and λ_2 one obtains the following equation expressing the Intersection-Condition:

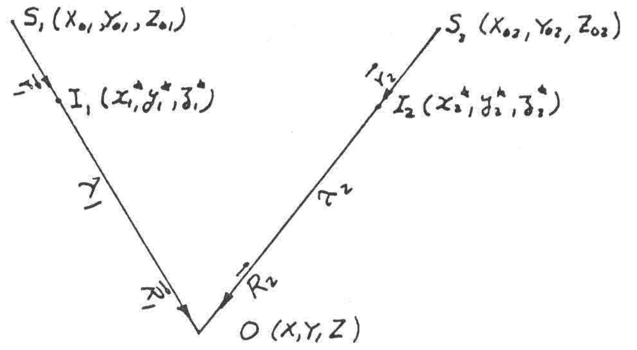

FIG. 6

$$(X_{02} - X_{01})(y_2^*z_1^* - y_1^*z_2^*) + (Y_{02} - Y_{01})(x_1^*z_2^* - x_2^*z_1^*) + (Z_{02} - Z_{01})(x_2^*y_1^* - x_1^*y_2^*) = 0 \quad (5)$$

So, any point that is common to two photographs yields one and only one observation equation that expresses the condition that the two rays must intersect.

Table 1 shows other different situations when a point appears in more than two photographs. The case of a point appearing in four photos is considered next just to help arrive at a general scheme that is applicable to any case. (Figure 7)

Four triplet Equations (2) are written involving X, Y, Z and λ_1, λ_2, λ_3, and λ_4 as unknowns besides the orientation parameters. If these seven undesired unknowns are eliminated from the twelve equations, five completely independent observation equations are obtained. It is now interesting to note that if one considers the type of Equation (3) to be necessary and sufficient, one will be in error. The reason is there can be written 4!/2!2! or 6 Intersection-Condition equations that will enforce every two rays to intersect, but there is no guarantee that all the four rays will intersect in the same point. (This is more so when three or more rays lie on or near a common plane.)[3]

Now, start with rays 1, 2, and 3 in Figure 7. There can be written an equation of type (3) for the intersection of rays (1) and (2) IC1 and another one for rays (2) and (3) IC2.

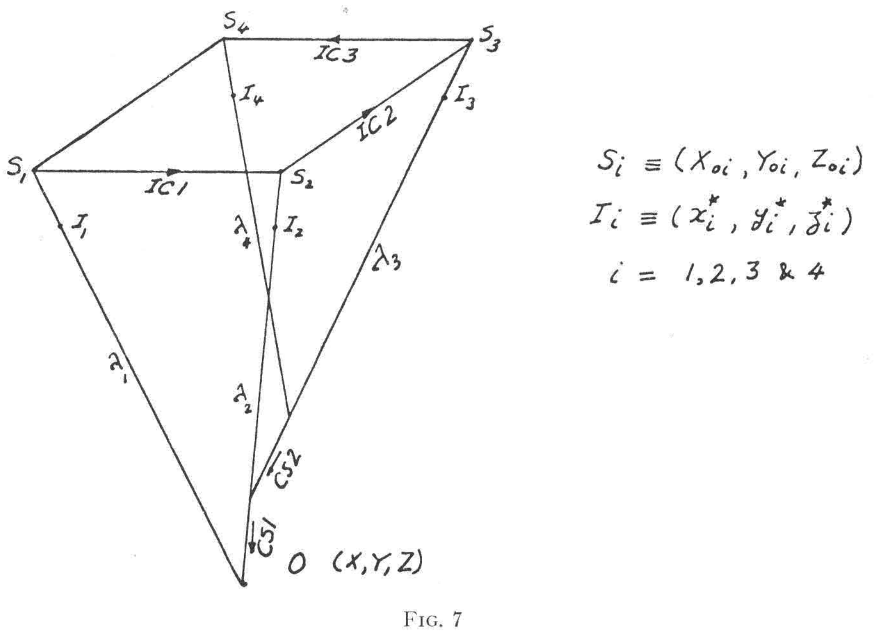

FIG. 7

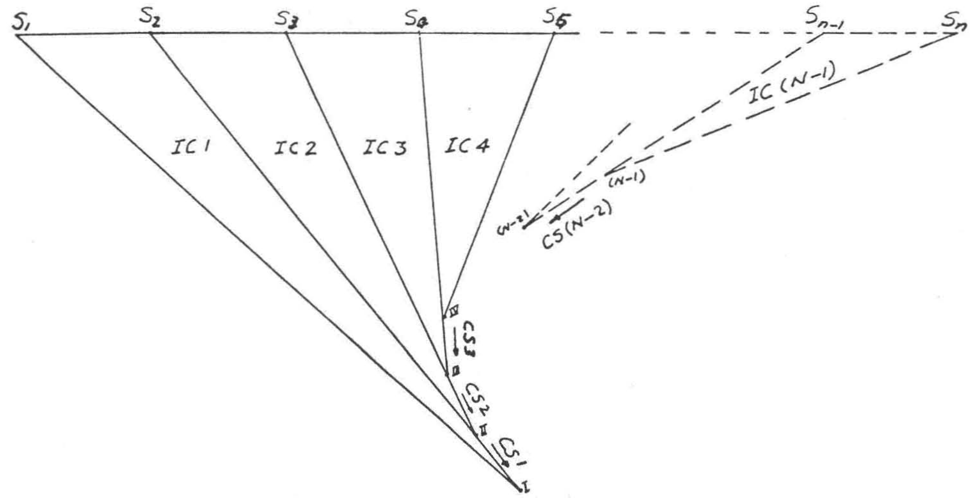

FIG. 8

Instead of writing a third one for rays (1) and (3), we force the condition that the points of intersection of rays (1) and (2) and (2) and (3) are the same point. This is shown schematically in Figure 7 by eliminating the mismatch along ray (2). To show the form of this equation—which will be called Common-Scale-Condition—one writes the following equations for the first three stations

$$
\begin{aligned}
X &= X_{01}+\lambda_1 x_1^* = X_{02}+\lambda_2 x_2^* = X_{03}+\lambda_3 x_3^* \quad \text{(a)}\\
Y &= Y_{01}+\lambda_1 y_1^* = Y_{02}+\lambda_2 y_2^* = Y_{03}+\lambda_3 y_3^* \quad \text{(b)}\\
Z &= Z_{01}+\lambda_1 z_1^* = Z_{02}+\lambda_2 z_2^* = Z_{03}+\lambda_3 z_3^* \quad \text{(c)}
\end{aligned}
\tag{6}
$$

Incidentally, one can arrive at independent equations that vary in appearance according to the scheme used in eliminating the λ's. The author found that the ones mentioned hereafter are easier to handle and to standardize. The scheme is to obtain the value for λ_2 from two equations from 6(a) and 6(b) and another value from two equations from 6(b) and 6(c), then equate the two. Eliminating the manipulation steps, the following are the two Common-Scale-Condition equations along rays (2) and (3)

$$
\begin{aligned}
&[(Y_{02}-Y_{01})x_1^*-(X_{02}-X_{01})y_1^*](x_3^*y_2^*-x_2^*y_3^*)\\
&\quad -[(Y_{03}-Y_{02})x_3^*-(X_{03}-X_{02})y_3^*] \quad \text{CS1}\\
&\quad \cdot(x_2^*y_1^*-x_1^*y_2^*)=0
\end{aligned}
$$

$$
\begin{aligned}
&[(Y_{03}-Y_{02})x_2^*-(X_{03}-X_{02})y_2^*](x_4^*y_3^*-x_3^*y_4^*)\\
&\quad -[(Y_{04}-Y_{03})x_4^*-(X_{04}-X_{03})y_4^*] \quad \text{CS2}\\
&\quad \cdot(x_3^*y_2^*-x_2^*y_3^*)=0
\end{aligned}
$$

Probably at this point the reader starts to visualize the general scheme. Figure 8 shows clearly the number of independent Intersection-Condition and Common-Scale-Condition equations for the case of a point in N photos. These are

$$
\begin{aligned}
&\text{No. of Intersection-Condition equations} = N-1\\
&\text{No. of Common-Scale-Condition equations} = \underline{N-2}\\
&\text{The total number of independent equations} = 2N-3
\end{aligned}
\tag{7}
$$

As a check, the total number of $(2N-3)$ is consistent with what was mentioned previously, that is twice the number of rays minus the three unknown parameters X, Y, Z of the point on the ground which are eliminated.

Figure 9 illustrates schematically a simple method for determining the number and type of independent equations for a particular case. The equations for a point lying in a zone of type IX (9 photo) are shown. The exposure stations are shown in plan with lines starting at one station and progressing in a sequential manner to the other end. Two stations are connected with one line only (the first and last) and the rest $(n-2)$ with two lines. Figures 9(a) and (b) show different ways of accomplishing the same thing depending on the numbering system of the stations.

Reference is now made to Figure 4 for the display of points to be used in the solution. The four points at the corners (or those in zone I) are excluded. Then, by virtue of Equation (7) above, points in the remaining zones will yield the number of equations indicated in column 3 of Table 1. For the 21 points shown in Figure 4 (4 points excluded), the number of observation equations will be:

$$8\times 1+4\times 3+4\times 5+4\times 9+1\times 15 = 91 \text{ equations.}$$

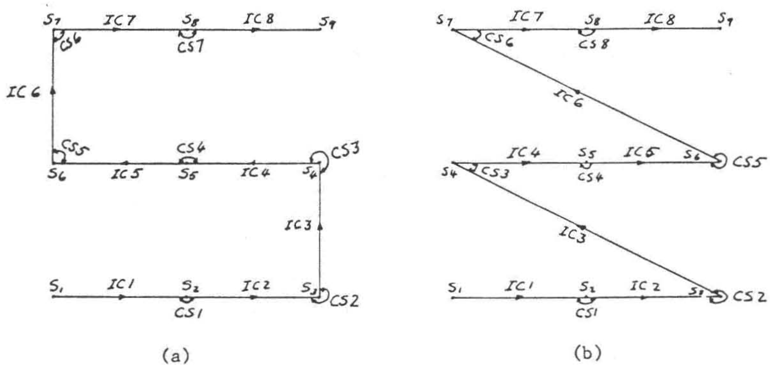

FIG. 9. IC Intersection-Condition; CS Common-Scale-Condition. No. of IC's$=9-1=8$, No. of CS's$=9-2=7$. (Write CS for lines between exposure stations in sequential manner with two stations connected with only one line—S_1 & S_9—and the rest with only two lines.)

These are then reduced through the principle of least squares to 48 normal equations to be solved in 48 unknowns.

(2) ADJUSTMENT OF THE SUB-BLOCKS TO THE BEST-FIT BY RELAXATION

After Sub-block Relative Orientation has been completed, each sub-block will be a strong independent unit in itself. This factor will permit thinking of the sub-block as a large photograph that can be translated or rotated in any direction as one unit. Then, at this point, methods are sought to connect these units together to fit each other and the given ground control in the project area. It is interesting to note that the existence of $\frac{2}{3}$ overlap in all directions is a twofold advantage. Mr. Schut[7] mentions as one of the reasons for preferring strip adjustment over adjustment by sections (as he calls it) is that "the connection between adjoining models in a strip can be made more accurately than connection between adjoining models in overlapping strips." In our case this is not so, since the sub-block is a unique section that overlaps equally with all surrounding sub-blocks. Secondly, no weighting system is necessary in tying the sub-blocks together, since they exhibit the same strength when connected to each other.

The Relaxation Procedure involves two steps: (a) All the sub blocks, that were treated independently are roughly brought into one coordinate system, preferably the ground system, by employing linear transformations. Owing to the advantage of having $\frac{2}{3}$ overlap in all directions, three air stations at each end of any sub-block are common to adjacent sub-blocks. These air stations are used in the approximate preliminary transformations to roughly tie sub-blocks together. Their use will improve considerably the strength of the geometrical construction of the block and particularly will help to reduce the discrepancy in the Z-coordinates and will expedite the solution of heights in the final adjustment. Furthermore, there will be no necessity for the intermediate step of computing coordinates of pass points at this stage of the game.

(b) The second step is the final adjustment of the sub-blocks to the best-fit with one another and with the present geodetic control. Such an adjustment consists of a solution for seven transformation coefficients for every sub-block. This linear conformal transformation will include three translations, three rotations and a scale-change for each sub-block. There are two ways of solving for these unknown coefficients. One is by the simultaneous solution of all the unknowns whose number is 7 times the number of sub-blocks[8]. This procedure is neither practical nor economical since it demands solution of a great many equations requiring a large computer for long periods of time. The second is a much simpler and faster approach requiring only the solution for 7 unknowns at a time. This approach is based on a relaxation method of successive approximations which is similar, but not identical, to Jerie's[6] horizontal block adjustment by analogue computer.

Figure 10 shows four sub-blocks, having a common point i, after their approximate transformation in a general block coordinate system X, Y, Z. The centroid of every sub-block is computed from the coordinates of all tie points in it; for example, for sub-block (j, k) in Figure 10, (which has points l, m, n and i as its corners)

$$\begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_{j,k} = \frac{1}{N} \begin{bmatrix} \Sigma X \\ \Sigma Y \\ \Sigma Z \end{bmatrix}^{s}_{j,k}$$

where

$$\begin{aligned} s &\equiv 1, m, n, \& \ i \\ N &= 4 \ \text{(for this particular case)} \end{aligned} \tag{8}$$

Local sub-block coordinate systems u, v, w are then passed through the centroids of the sub-blocks and parallel to X, Y, Z. Therefore

$$\begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}_{j,k} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}^{i} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}^{i}_{j,k}$$

where

$$\begin{bmatrix} \Sigma u \\ \Sigma v \\ \Sigma w \end{bmatrix}^{s}_{j,k} = 0 \qquad s \equiv l, m, n \ \& \ i \tag{9}$$

Seven unknown coefficients are then introduced to every sub-block that will allow its rotation, translation and scale-change to minimize all mismatches for both tie and control points within the sub-block. For the sake of explanation, consider the sub-block (j, k) with a scale-change $\mu_{j,k}$ rotation angles (α, β and γ) about $(u, v, \text{and } w)_{j,k}$ and translation $(\Delta\overline{X}, \Delta\overline{Y}, \text{and } \Delta\overline{Z})_{j,k}$ of its centroid. The effect of rotation of axes is commonly expressed as a 3×3 matrix whose elements contain trigonometric functions of the three angles (the well known A-matrix in Figure 3). If there is assumed that the rotation angles are small, then we could substitute the angle for its sine and 1 for its cosine and the rotation A-matrix can then be simplified as:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}^{i}_{j,k} = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix}_{j,k} \begin{bmatrix} u \\ v \\ w \end{bmatrix}^{i}_{j,k} \tag{10}$$

The effect of the seven coefficients, will then be in matrix notation[9]

$$\begin{bmatrix} \overline{X} + \Delta\overline{X} \\ \overline{Y} + \Delta\overline{Y} \\ \overline{Z} + \Delta\overline{Z} \end{bmatrix}_{j,k} + (1 + \mu_{j,k}) \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix}_{j,k} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix}^{i}_{j,k} = \begin{bmatrix} X + \Delta X \\ Y + \Delta Y \\ Z + \Delta Z \end{bmatrix}^{i}_{j,k} \tag{11}$$

Considering $\mu_{j,k}$ and $(\alpha, \beta, \gamma)_{j,k}$ to be small, their products are neglected. Then by matrix subtraction:

$$\begin{bmatrix} \Delta\overline{X} \\ \Delta\overline{Y} \\ \Delta\overline{Z} \end{bmatrix}_{j,k} + \begin{bmatrix} \mu & \gamma & -\beta \\ -\gamma & \mu & \alpha \\ \beta & -\alpha & \mu \end{bmatrix}_{j,k} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix}^{i}_{j,k} = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}^{i}_{j,k} \tag{12}$$

in which

$(\Delta\overline{X}, \Delta\overline{Y}, \Delta\overline{Z}, \mu, \alpha, \beta, \text{and } \gamma)_{j,k}$ are the 7 unknowns

and

$(u, v, w)_{j,k}{}^{i}$ are coordinates of point i in the local sub-block coordinate system.

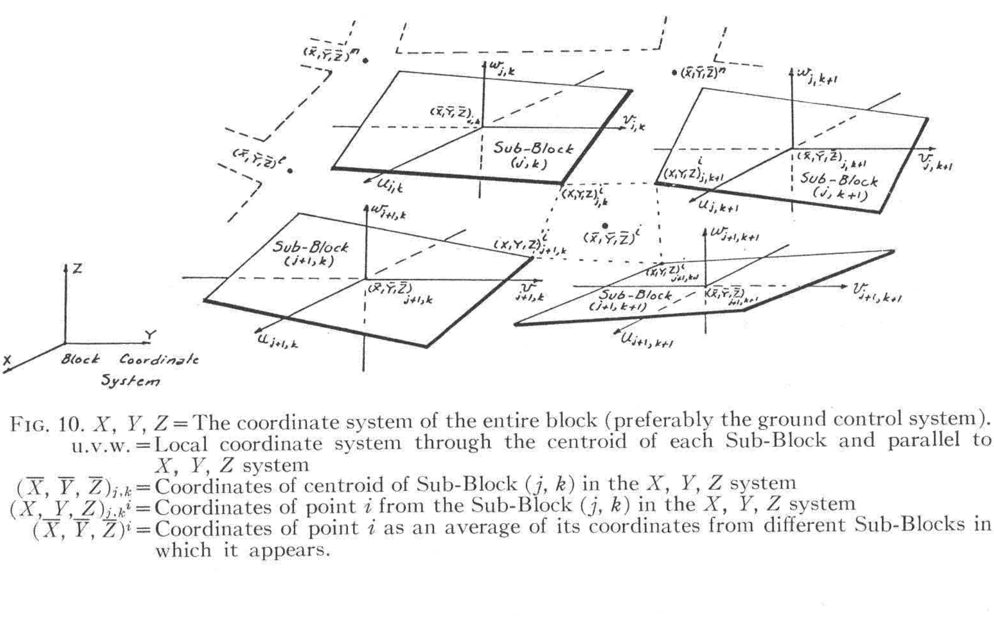

FIG. 10. X, Y, Z = The coordinate system of the entire block (preferably the ground control system).
u.v.w. = Local coordinate system through the centroid of each Sub-Block and parallel to X, Y, Z system
$(\overline{X}, \overline{Y}, \overline{Z})_{j,k}$ = Coordinates of centroid of Sub-Block (j, k) in the X, Y, Z system
$(X, Y, Z)_{j,k}{}^{i}$ = Coordinates of point i from the Sub-Block (j, k) in the X, Y, Z system
$(\overline{X}, \overline{Y}, \overline{Z})^{i}$ = Coordinates of point i as an average of its coordinates from different Sub-Blocks in which it appears.

$(\Delta X, \Delta Y, \Delta Z)_{j,k}{}^i$ are taken equal and opposite to the discrepancy or mismatch between the coordinates of point i from sub-block (j, k) and its average position; or from Figure 10

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_{j,k}^i = -\left[\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{j,k}^i - \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix}^i\right] = -\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{bmatrix}_{j,k}^i \tag{13}$$

Therefore, the three observation equations yielded by point i are

$$\begin{aligned} \Delta\overline{X}_{j,k} + u_{j,k}{}^i\mu_{j,k} - w_{j,k}{}^i\beta_{j,k} + v_{j,k}{}^i\gamma_{j,k} &= -(\epsilon_x)_{j,k}{}^i \\ \Delta\overline{Y}_{j,k} + v_{j,k}{}^i\mu_{j,k} + w_{j,k}{}^i\alpha_{j,k} - u_{j,k}{}^i\gamma_{j,k} &= -(\epsilon_y)_{j,k}{}^i \\ \Delta\overline{Z}_{j,k} + w_{j,k}{}^i\mu_{j,k} - v_{j,k}{}^i\alpha_{j,k} + u_{j,k}{}^i\beta_{j,k} &= -(\epsilon_z)_{j,k}{}^i \end{aligned} \tag{14}$$

Three similar equations are written for each point in the same sub-block, resulting in a total of $3N$ observation equations to be solved for 7 unknowns (N= No. of points in the sub-block). In forming the seven normal equations by least squares, it is interesting to mention that the first four would contain only one unknown each[9]. Consequently $\Delta\overline{X}$, $\Delta\overline{Y}$, $\Delta\overline{Z}$, and μ are immediately obtained and only three simultaneous equations need to be solved for α, β, and γ.

The above computations explained for sub-block (j, k) are actually performed for all the sub-blocks in the project. Values of corrections are determined after each iteration, and then applied to previous values to obtain new values for the coordinates of the sub-block centroid, coordinates of tie points in adjacent sub-blocks, and their average values. Iterations are repeated till the entire block settles down and a Best-Fit within specified limits is gained.

Conclusion

An honest opinion with a few new ideas, free from ties and restrictions of the conventional line of thinking, is presented. This should make possible visualizing the following advantages of using $\frac{2}{3}$ overlap in all directions:

(1) The geometric strength of the photogrammetric problem is the maximum possible.

(2) The subdivision of a block of photography into sub-blocks of 3×3 photos more closely resembles the basic unit (the single photograph). The sub-blocks also provide a much stronger solution than strips in many respects.

(3) It allows the application of the triplet approach[1], with all its advantages, in all directions, thus making the best use of the existing control through more efficient ways of control extension.

(4) The difficulties of warpage, discontinuities, and hinge effects normally found in strip triangulation, are eliminated by the presence of the $\frac{2}{3}$ overlap in all directions. Furthermore the common $\frac{2}{3}$ overlap allows the use of air stations as tie points which give depth to the adjustment and therefore improves and hastens the solution of heights.

One disadvantage for this approach might be the cost of photography. With the required sidelap, about twice the number of photographs would be required. Even if the cost of photography is twice the normal amount, still it remains as a small proportion of the total cost of the mapping project. Furthermore, the compilation time is not affected by the excessive photographic coverage, since only every other flight line will be needed for compilation.

Another apparent disadvantage is that this optimization in analytical solutions might be hampered by the accuracy of the input data. In other words, this approach may require more accurate photo-coordinates than the present comparators can provide. The author, personally, considers this to be an advantage and not a drawback. The capacity of the present instrument should not prevent or even restrict any original photogrammetric thinking. Besides, new principles, such as fringe theories and interferometry, are already being applied in designing more accurate comparators. For example, Mr. Rosenfield states that high-accuracy 1-micron interferometer comparator has been under development at the Air Force Missile Test Center[10].

In conclusion one can draw the analogy between geodetic and aerial triangulation. Mr. Whitten of the U. S. Coast and Geodetic Survey in speaking of geodetic triangulation made the following remark: "It has been our experience that *area networks* have *greater* strength than *arcs*[11]."

Aerial triangulation is so similar to geodetic triangulation that the above remark could be adopted to read: The *sub-block tri-*

angulation has *greater* strength than *strips*. As a matter of fact, the two-directional triplets are probably the strongest sub-block units in analytical aerotriangulation.

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*Definition and Determination of Weights of Fundamental Photogrammetric Data and Results**

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THE concept *weight* is used in measuring sciences to indicate the relative importance of basic data or results. Weights are usually defined as inversely proportional to the squares of the standard errors (standard deviations) or the variances of the actual data or results. Distinction is usually made between *á priori* and *á posteriori* weights. The á priori weights are assigned to measurements before they are used in computations of other data, and refer to factors or relations which for some reason introduce different standard errors or standard deviations in the measured data. The á posteriori weights refer to the geometrical quality of data which are determined through computation from measured values, in particular through some kind of adjustment. In order to illustrate the weight concepts, two examples from photogrammetry will be shown.

It has been found from empirical experiments and least square adjustments that the standard errors of unit weight of image-coordinates increase significantly with the radius from the principal point, Figures 1 and 2. This is a quite natural consequence of the facts that the photographic image is a central projection and that the actual image coordinate measurements are of orthogonal nature, Figure 3. All deviations of the image from a plane must therefore cause errors in the orthogonal positions. There are and must always be deviations of the image from a mathematical plane because

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