For further theoretical considerations, the above ideas could be extended to nine, fifteen or more number of points in the model in case a more precise relative orientation is aimed at.

REFERENCES

1. Jerie, H. G., "Numerische Orientierung gebirgigen Geländes." *Photogrammetria*, 1953–54, No. 1.
2. Kasper, H., "Relative Orientation of Vertical Photographs." PHOTOGRAMMETRIC ENGINEERING, 1956, No. 2.
3. Brandenberger, A. J., "Zur Ausgleichung von trigonometrisch bestimmten Passpunkten für luftphotogrammetrische Kartierungen." *Schweizerischen Zeitschrift für Vermessung und Kulturtechnik*, 1950, Hefte Nr. 9 & 10.
4. Jordan-Eggert, "Handbuch der Vermessungskunde."

Projective Nets Corrected for Radial Distortion for Graphical Rectification in Aerial Photogrammetry

MOHEB AZIZ ABDEL-MESSIH,
Faculty of Engineering, Alexandria Univ., U. A. R.

I. INTRODUCTION

RECTIFICATION in aerial photogrammetry, as defined by Hallert [1], implies that an image, photographed by a non-vertical camera over a plane terrain (not necessarily horizontal), may be transferred onto a map, i.e. an orthogonal projection of the terrain on a certain scale in a horizontal plane. Rectification may be performed with numerical, graphical, or optical-mechanical methods.

Methods of graphical rectification are based on constructing two projective nets in map and image. In the map the net is, generally, chosen of regular figures, e.g. squares [1, p. 107], or parallelograms [2, p. 357]. This net is then transferred to the image, which is assumed to be a strict central projection, i.e. free from distortion. More accurate results may, therefore, be obtained if the above process is corrected for distortion, which is mainly radial.

In this paper, two projective nets in map and image are so selected, that they can be corrected for the radial distortion. In the image, the net is taken to consist of rays passing through the principal point H', and concentric circles about H'. The corresponding net in the map consists then of rays through the point H (which corresponds to H') and of ellipses. If the image is considered as a strict central projection, these ellipses will then be the orthogonal projection, on the horizontal plane of the map, of the ellipses of intersection of the plane terrain with the coaxial cones of revolution, whose common axis is the camera axis and whose bases are the concentric circles in the image. This net is corrected for the radial distortion by displacing these ellipses in the map accordingly, as will be shown in IV. The method is here applied to the simple case of an oblique image and a horizontal plane terrain. Numerical data are assumed and the results are represented graphically. The method is, however, applicable to more general cases.

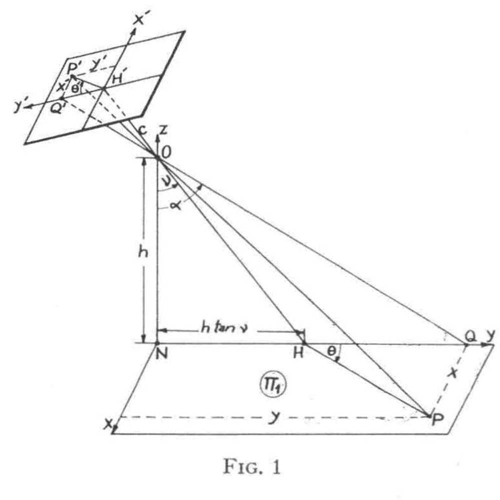

FIG. 1

II. Equations of Representation

The simple case of an oblique image and a horizontal plane terrain π_1 is assumed, Figure 1. We take:

- the vertical through the center of projection $0 = z$-axis,
- the nadir point N in π_1 = origin of the ground coordinate system,
- the plane through z-axis and camera axis $= z$-y vertical plane,
- the normal through 0 to z-y plane $= x$-axis in π_1,
- the image of y-axis $= y'$-axis in the picture plane,
- the normal through the principal point H' to y'-axis $= x'$-axis in the picture plane.

Let:

- h = flying height NO,
- c = camera constant (or principal distance),
- ν = nadir distance, (tilt)
- P = a point on the terrain, whose coordinates are x and y,
- P' = central projection of P, its image coordinates are x' and y'.

The following relations can easily be derived from the geometrical properties of Figure 1:

$$y = h \tan \alpha, \qquad y' = c \tan (\alpha - \nu),$$

hence

$$y' = \frac{c(y - h \tan \nu)}{h + y \tan \nu} \tag{1}$$

The straight lines parallel to the direction of the x-axis are parallel to the picture plane and therefore, are projected on this plane in straight lines parallel to the same direction. The triangles PQO and $P'Q'O'$ are therefore similar. This gives:

$$\frac{x'}{x} = \frac{c \sec (\alpha - \nu)}{h \sec \alpha},$$

or

$$x' = \frac{cx}{h \cos \nu + y \sin \nu} \tag{2}$$

From (1) and (2) we obtain

$$y = \frac{h(c \tan \nu + y')}{c - y' \tan \nu}, \tag{3}$$

$$x = \frac{x'}{c}(h \cos \nu + y \sin \nu) \tag{4}$$

(In equation (4), y may be expressed in terms of y' from Equation (3)).

Rays in π_1 passing through H are projected in rays passing through H', the correspondence between θ and θ' (cf. Figure 1) may be deduced by substituting from the above equations in the relations

$$\tan \theta = \frac{x}{y - h \tan \nu} \quad \text{and} \quad \tan \theta' = \frac{x}{y'}$$

This gives

$$\tan \theta' = \sec \nu \tan \theta \tag{5}$$

III. Projective Nets

In the image, a net of rays through H' and concentric circles about H' is chosen. The corresponding net in the map consists of rays through H and eccentric ellipses.

Equation (5) expresses the relation between the above pencils of rays in image and map.

In the picture plane, assume a circle k' about H' as center. k' is the central projection of the ellipse k of intersection of π_1 with the cone of revolution, whose axis is the camera axis and whose base is k'. The axes of k may be constructed as follows (cf., e.g. [2]):

i) The major axis A_1A_2 lies on the y-axis, its ends correspond to the ends A_1' and A_2' of the diameter $A_1'A_2'$ in k', which lies on the y'-axis (Figure 2).

ii) The center of k is the middle point M of A_1A_2.

iii) Find the image M' of M (M' lies on the y'-axis).

iv) Determine an end B' of that chord in k', which passes through M' and is parallel to the x'-axis.

v) In the map determine the point B, which corresponds to B'. B is then an end of the minor axis of k.

Denoting the radius of k' by r', then the above steps may be expressed by the following equations:

$$\text{i)}\quad y_{A_1} = \frac{h(c \tan \nu - r')}{c + r' \tan \nu}, \qquad y_{A_2} = \frac{h(c \tan \nu + r')}{c - r' \tan \nu}$$

$$\text{ii)}\quad y_M = \frac{1}{2}(y_{A_1} + y_{A_2})$$

$$\text{iii)}\quad y'_{M'} = \frac{c(y_M - h \tan \nu)}{h + y_M \tan \nu}$$

$$\text{iv)}\quad x'_{B'} = \sqrt{r'^2 - y'_{M'}{}^2}$$

$$\text{v)}\quad x_B = x'_{B'}\left(\frac{h \cos \nu + y_M \sin \nu}{c}\right)$$

In numerical computations, if we consider:

$$\text{the unit of length} = 1 \text{ cm.},$$

$$\text{scale of image} = 1{:}1,$$

$$\text{scale of map} = n\frac{c}{h} = 1{:}\frac{h}{nc},$$

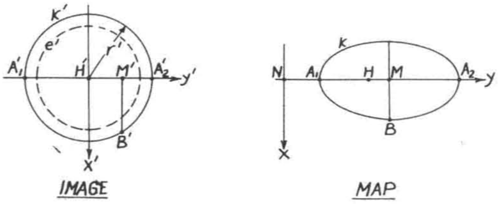

FIG. 2

where n is a constant, then the above equations may be converted to give the map and image coordinates directly in cms. The resulting equations are free from h; they are:

i) $y_{A_1} = \dfrac{nc(c \tan \nu - r')}{c + r' \tan \nu}, \qquad y_{A_2} = \dfrac{nc(c \tan \nu + r')}{c - r' \tan \nu}$

ii) $y_M = \dfrac{1}{2}(y_{A_1} + y_{A_2})$

iii) $y'_{M'} = \dfrac{c(y_M - nc \tan \nu)}{nc + y_M \tan \nu}$

iv) $x'_{B'} = \sqrt{r'^2 - y'_{M'}{}^2}$

v) $x_B = x'_{B'} \left(n \cos \nu + y_M \dfrac{\sin \nu}{c} \right)$

IV. Correction for Radial Distortion

In the image a circle k', of radius r' about H', is the position, to which a circle e', obtained according to the central projection, is displaced due to the radial distortion. The radius of e' is r'-$\Delta r'$, where $\Delta r'$ as function of r' is the radial distortion of the camera used. The above nets may be corrected for this distortion thus; to the circle k' of radius r' in the image let correspond in the map an ellipse, whose axes are constructed by replacing r' by (r'-$\Delta r'$) in the previous equations.

The nets drawn in Figure 3 are constructed for the case assumed above, when we put: $c = 20$ cms., $h = 100{,}000$ cms., $\nu = 45°$, $n = \frac{1}{2}$ (i.e. scale of map $= 1/10{,}000 \times$ scale of image). The density of the nets may be so much in-

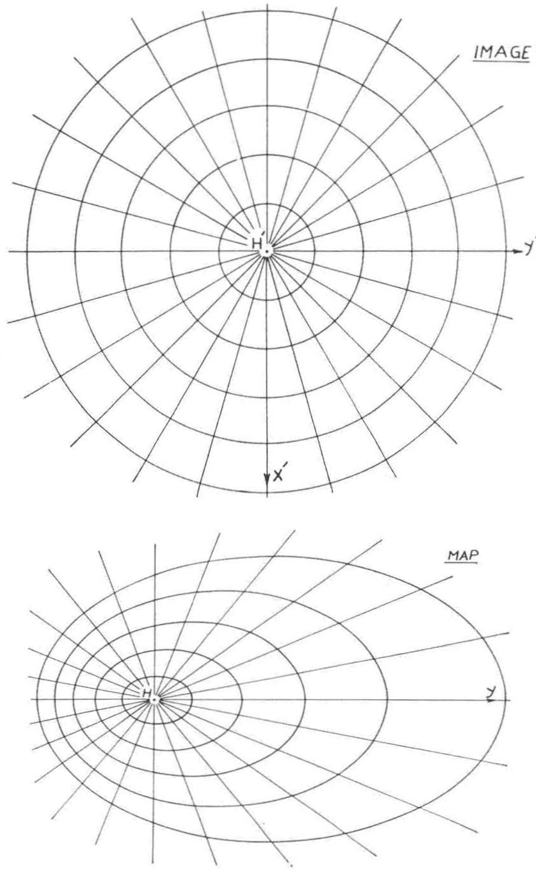

FIG. 3

creased to meet the accuracy requirements of the problem.

References

1. Hallert, B., "Photogrammetry," McGraw-Hill, 1960.
2. Müller, E. und E. Kruppa, "Lehrbuch der darstellenden Geometrie," Springer-Verlag, Wien 1948.