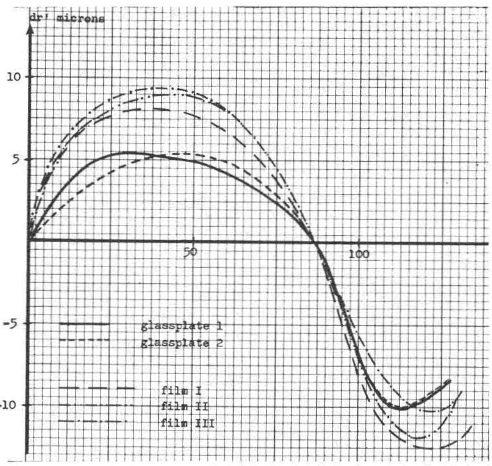

FIG. 8. Radial distortion curves from the same camera but determined from glass plates and films. The difference is significant and may be caused by lacking flatness of the films.

cameras and that this variation confirms the results of previous investigations of aerial photographs taken under operational conditions, Figure 7.

The geometrical quality of glass plate and film negatives was compared. Two types of film were used. In one of them there was a very pronounced affine shrinkage. After correction for the affinity, the averages of the standard errors of unit weight became practically equal, 3 to 4 microns, for the glass plates and the films. Further, a significant difference in the radial distortion curves was found from the glass plate negatives in comparison with the film negatives, Figure 8. This indicates that there are additional sources of this regular error in the film negatives, probably caused by lacking flatness in the supporting back of the magazine.

In summary, the application of the method of least squares to the adjustment of the single-point resection problem in connection with camera calibration in a multicollimator has proved to be of great value. In particular it has proved indispensable for the determination of unique values of the elements of the interior orientation, regular errors of the image coordinates and the standard errors of these data. The weight relation of the image coordinates can also be uniquely determined, which is of basic importance for analytical photogrammetry. The laboratory calibration should always be completed through grid tests under operational conditions.

*Elevations from Parallax Measurements**

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I. INTRODUCTION

IN THIS paper a survey is made of methods for determining absolute or differential elevations of points on the ground, by use of parallax measurements from a stereoscopic pair of vertical, or nearly vertical, aerial photographic prints. It also includes a proposal for improvement of the calculation of flying height, H, for use in the computation of elevations determined from parallax measurements.

"Parallax" is the term often used to denote displacement of one object with relation to another. In photogrammetry, parallax on aerial photographs is expressed in terms of rectangular coordinates X and Y, with the principal point of the photograph as the origin of the axes, and with the X-axis parallel to the line of flight. "Absolute stereoscopic parallax" then, or simply "parallax" is assumed to mean displacement along or parallel to the line of flight, Y-parallax being displacement at

* This paper was the winner in the Competition for the Bausch and Lomb Undergraduate Award of the American Society of Photogrammetry.

right angles to the line of flight. "Absolute stereoscopic parallax", p, is the algebraic difference, parallel to the line of flight, of the distances of the two images of a given object from their respective principal-points, assuming the two photographs are truly vertical and are taken from the same height. Hence, parallax of a point A on the ground, appearing as a and a' respectively on a pair of overlapping photographs:

$$P_a = X_a - X_a' \tag{1}$$

The basic formula for parallax in terms of elevation is easily derived from the geometry of the stereoscopic pair of photos:

$$h = H - \frac{Bf}{P} \tag{2}$$

where:

f = focal-length of the camera.
h = the height of the object above datum.
H = flying height of the plane above same datum.
P = absolute parallax of the object.
B = Air-base = Actual ground distance between exposure stations.

II. Determination of Absolute Elevations

Using the Formula (2) the absolute elevations (as opposed to "differential" elevations) of points on the ground can be determined. Depending on the data available and the accuracy derived we can distinguish the following cases:

A. GENERAL EXPRESSION

In order to determine the elevation of a point when we do not know the elevation of any control points, we will need to know the flying height H (from the plane altimeter readings), and the focal-length of the camera f. We can measure the air base B, from our map, and the parallax P_x of a point x, from the stereo-pair. Then substituting the above four factors into Formula (2), we can compute the elevation h_x of point x.

B. MOFFITT'S METHOD

In case the elevation of a control point is known, we again will need to measure the parallaxes of the points together with the air-base and the focal-length of the camera. By using the parallax and known elevation of the control point, together with B and f, the flying height above the datum is first determined:

$$H = h_c + \frac{B}{P} f \tag{3}$$

where the subscript c denotes "control point". By applying Equation (2) to the parallax of each point, the elevations of the points are obtained. To facilitate computations, the following could be used:

Point	P mm.	$\frac{B}{P} f_{(ft)}$	$h = H - \frac{B}{P} f_{ft}$

(4)

At the given flying height, the computed elevations of the points may be in error by 4 percent or more just because of tilts and unequal flying heights. Errors in the air-base, together with observational errors and differential paper shrinkage, will also affect the computed elevations, either favorably or unfavorably.

In case that the elevations of more than one well placed control points are known, the errors due to unequal flying heights can be reduced. Moffitt[1] proposes the following method: A flying height is determined from the parallax and elevation of each control point using Equation (3). The average of the above flying heights is computed and this H is used to compute the theoretical parallaxes of the control points.

The difference between the theoretical parallax and the measured parallax of each point, represents the correction to be applied to the parallax of the point. A transparent overlay is placed over the overlay area and the values of the corrections to the parallax readings are entered at the control points. By interpolating between the control points, lines of equal correction may be drawn. In this way we construct a "parallax-correction graph" which can be used to obtain the correction to the parallax of any point in the overlay.

III. Determination of Differential Elevations

The elevation of a point, as we have previously discussed, may be determined from parallax measurements by using Equation (2) and using the known quantities p, f, B, and H. The difference in elevation between two points can be obtained by measuring the parallax of each point, applying Equation (2) to get the elevation for each measured parallax, and finally subtracting one elevation from the other. This difference in elevation can also be determined directly by measuring the difference in parallax between the two points. Then if the elevation of one of the points is known, the elevation of the second point can be obtained.

A. Desjardin's Method

For determining the difference in elevation between two points, Desjardin[3] proposed the following formula:

$$dh = \frac{fS_r b_r dp}{b_s b_m} \tag{5}$$

where:

dh = difference in feet in elevation between two points on the ground whose images are recognized on the stereo pair.

f = focal-length (feet).

$1:S_r$ = scale of a radial or templet assembly.

b_r = distance between photo principal-points in the assembly (mm.).

dp = difference in parallax between the two points in question (mm.).

b_s&b_n = photo bases as corrected for the two elevations in question (mm.).

With reference to Figure 1, we see that a simpler form of Equation (5) would be

$$dh = \frac{fBdp}{b_1 b_2} \tag{6}$$

The derivation of the above follows: Starting with the standard parallax formula

$$dp = b_1 \frac{dh}{H_2} \tag{7}$$

we know that

$$\frac{H_n}{f} = \frac{B}{b_n} \tag{8}$$

where

H_n = flying height above any level

b_n = photo-base for that level

Therefore

$$\frac{H_2}{f} = \frac{B}{b_2} \tag{8a}$$

and

$$H_2 = f\frac{B}{b_2} \tag{8b}$$

Therefore

$$dp = \frac{b_1 dh b_2}{fB} \quad \text{or} \quad dh = \frac{fBdp}{b_1 b_2} \tag{6}$$

To understand the applicability of the Formula (6), we must clarify the relationship between "parallax" and "photo-base". These terms in their strict meaning are synonymous, and it would have been quite accurate to have used the letter p each time in place of b. Figure 1 illustrates the two exposure stations M and N, air base B, photo plane, and two ground levels 1 and 2. The directions NR and RH in which point R, on level 1, is seen from the two stations, make an angle r(NS parallel to MR). Point R' is located vertically below M (angle $R'MN$ is a right angle) on level 1. The true parallax angle is $R'NT=r'$. For practical purposes $r=r'$ since their photo intercepts $p_r=p_1$ (parallax = photo intercept of the parallax angle). It has been proven (4, p. 480) that on truly vertical photos all ground points at the same height above a given datum will have the same parallax. But p_1 and p_2 are actually b_1 and b_2. Hence the parallax or effective photo base for any point in the stereo-pair is found by measuring the photo-base on either of the photos provided that this is corrected to the same elevation by adding or subtracting the difference in parallax between the point in question and the transferred principal-point end of the photo-base.

The main point in using Formula (6) is the fact that

$$b_2 = b_1 + dp \tag{9}$$

We can state Formula (6) in a different form as

$$dh_{cn} = \frac{fBdp_{cn}}{b_c b_n} \tag{6a}$$

where:

c = refers to any control point of known elevation
n = refers to any point whose elevation is sought
dp_{cn} can be measured
b_c is derived from $b_c = b_p \pm dp_{pc}$ (9a)

In (9a) p = refers to one of the principal-points where b_p is measurable.

b_n is derived from $b_n = b_c \pm dp_{cn}$ (9b)

The above relationships also disclose that a workable formula for dp when dh is known, is impossible. For when dp is unknown we cannot determine the factors b. So, although the formula might be mathematically correct, it would be of no use.

The advantage of the Desjardin Formula (6) is that all its factors are exactly determinable. Hence, it gives a precise and workable parallax—elevation relationship, something that is not achieved by any standard formula for dp.

B. MOFFITT'S METHOD

Various authors have proposed approximate parallax—elevation relationships. Among them Moffitt[1] begins with the basic formula for the elevations of two points

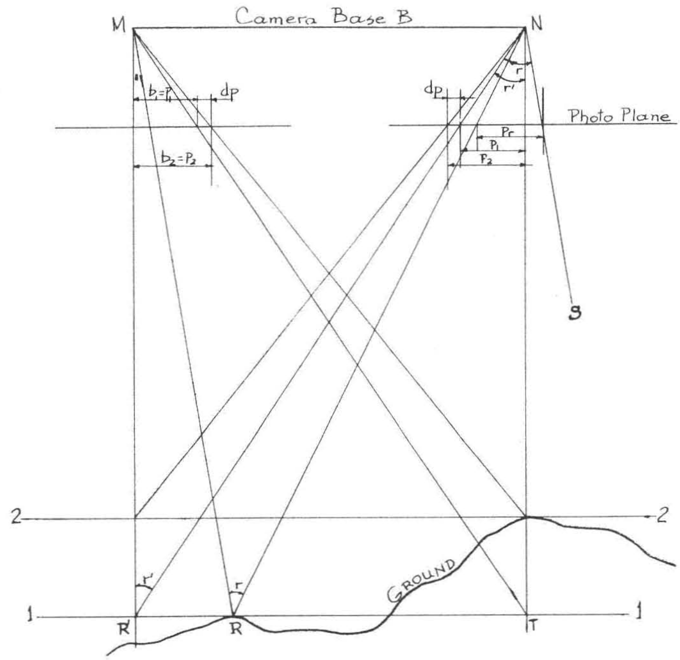

FIG. 1

A and X.

$$h_a = H - \frac{B}{P_a}f, \qquad h_x = H - \frac{B}{P_x}f \tag{2}$$

The difference $(h_a - h_x)$ is

$$dh_{ax} = \frac{Bfdp_{ax}}{P_a(P_a + dP_{ax})}$$

To obtain a simpler approximate relationship Moffitt makes two assumptions:

(1) The vertical control point used, and the two ground principal points all lie in the same elevation.

(2) The flying height is measured above the elevation of the control point.

Then, from the geometry of the stereo pair he derives

$$B = \frac{HaPa}{f} \tag{11}$$

and

$$dh = \frac{H_a dp}{dp_{ax} + Pa} \tag{12}$$

C. THOMPSON'S METHOD

Another approximation for the computation of differential heights, introduced by A. R. Robbins[10] and modified by Professor E. H. Thompson[5] makes use of the simplified formula

$$h_{ab} = (H - h_a)\frac{\Delta Pab}{Pb} \tag{13}$$

where ΔPab = difference in parallax between points a and b.

This formula is rigorous, in the sense that it gives the correct result if the photographs are untilted. Equally important, it is easily evaluated with a slide rule.

D. R. Crone[6] comments on the applicability of the above formula, saying that in order to obtain acceptable results from overlaps with a great height range, the flying height H should be calculated from the map length of the photo-base and the relative tilts of the two photos, determined from measurements of "want of correspondence". These relative tilts, can be obtained from paper prints, and being approximate only they can be of no use in evaluating the corrections to heights; but they are, in general, sufficient to give a value of H adequate for relief up to 20 percent with 4 degree tilts.

Thompson proceeds to correct the computed heights from distortions due to tilt, using the expression:

$$h' - h = a_0 + a_1 + a_2y + a_3xy + a_4x^2 \tag{14}$$

Where:

h' = true height

h = calculated height

The formula implies that we assume:

(1) The tilts are small so that powers above the first may be neglected.

(2) $h' - h$ is independent of h' which would be reasonable only if the variation in h' on the overlap is small compared with H. In order to be able to solve for the coefficients $a_0 \cdots a_4$, we need five control points, so that we can set up five simultaneous equations. The distribution of the control points on the overlap is important. The ideal distribution is one point at each corner of the overlap and the fifth point midway on the base line. When this is not possible the following rules should be borne in mind:

1. The controls should be well situated on the overlap so that extrapolation is avoided.
2. Extrapolation on one overlap may often be dealt with as interpolation on the adjacent overlap.
3. No four points must lie on or near a straight line.
4. No three points must lie on or near a straight line perpendicular, or nearly perpendicular to the base.

D. CHITTENDEN'S METHOD

Finally, H. M. Chittenden[7] proposes a formula to convert parallax measurements into difference in elevation. This varies somewhat from the conventional form in that the photo-base, employing the average measurement between the principal and conjugate points of each photo, is dispensed with and the total separation distance between the principal points of the stereo pair is employed. With reference to Figure 2.

$$\Delta h = \frac{(H - h_a)(D_a - D_x)}{D - D_x} \tag{14}$$

$$\therefore \; Da - Dx = \Delta p = \frac{(D - Dx)\Delta h}{H - ha} \tag{14a}$$

Where

D_a = parallax measurement between images of point a (elevation known) on photos 1 and 2.

D_x = parallax measurement between images of x (elevation sought) on photos 1 and 2.

D = separation distance principal-points of photos 1 and 2.

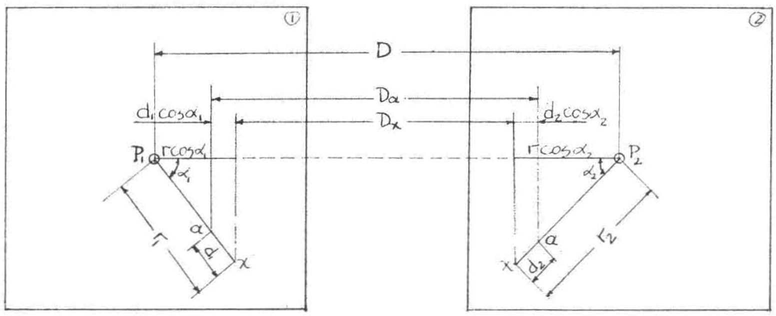

FIG. 2

According to Chittenden these equations are based on the assumption that the photographs are free from distortion from any cause. In order to adjust the computed heights for distortions due to any source, the following method could be used:

Given several points of known elevation, compute the parallax difference for two points, say A and B, using Equation (14a)

$$\Delta P_a' = \frac{(D - D_a)h_a}{H}$$

$$\Delta P_b' = \frac{(D - D_b)h_b}{H}$$

If the datum positions of points A and B could be observed, the parallax readings for the points would be

$$D_{d_a} = D_a + \Delta P_a'$$

$$D_{d_b} = D_b + \Delta P_b'$$

Since for points of equal elevation (A and B at datum) the parallax measurement is constant, D_{d_a} must be equal to D_{d_b}. If we find that $D_{d_a} \neq D_{d_b}$, there is a distortion in the measurements of A and B. Hence a correction made to the observed parallax measurements of points A and B, would have the effect of restoring these points to their correct relative position on the photo. Therefore, select an arbitrary value of D_d, and

$$D_d - D_{d_a} = \text{Correction to parallax measurement of point } A = C_a \qquad (15)$$

$$D_d - D_{d_b} = C_b \qquad (15a)$$

Then

$$P_a = P_a' + C_a$$

$$P_b = P_b' + C_b$$

Finally

$$\text{Adjusted } \Delta P_{ab} = P_a - P_b.$$

If we have enough control points of known elevation, we can construct a correction graph (Figure 3) which can be applied to the whole overlay.

IV. Proposal of a Method Based on Weighted Flight Heights

One of the main factors causing discrepancies in elevations computed from parallax measurements is the use of unequal flying heights. As earlier indicated, Mof-

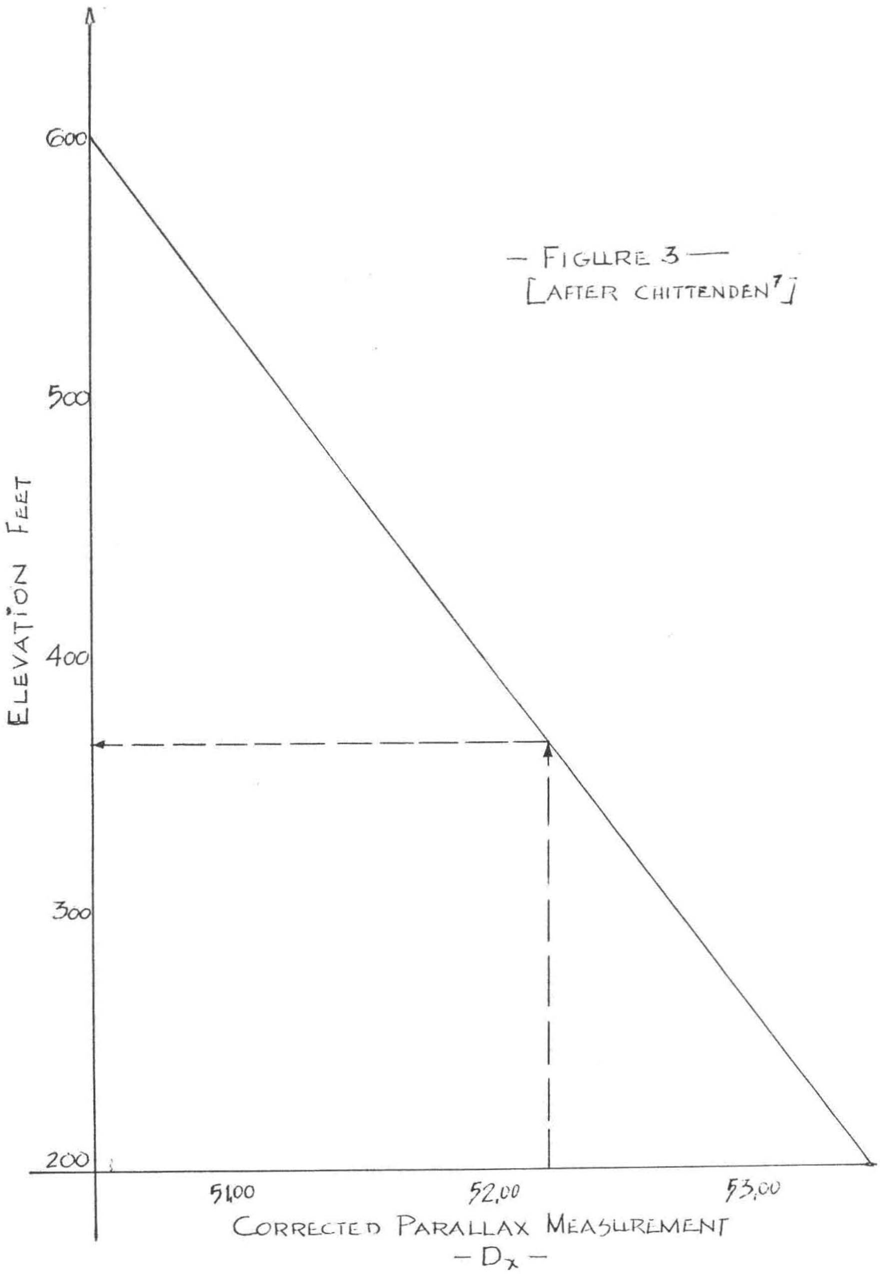

FIG. 3

fitt proposes a method of correcting the errors due to flying height differences, by using the average of the control point flying heights to construct a parallax correction graph. The flaw in the above method, in my opinion, is that the computed average H, when used to determine the elevation of a point x (or the correction to p_x) may be as much in error, if not more, than the flying height determined by either one control point, or the flying height as determined by altimeter readings.

The following procedure is proposed for adjusting H. This procedure should give better results in computing elevations.

To proceed, we need to know or measure the following:

a) The elevations of more than one control point (h_a, h_b, $\cdots$, h_n). These control points should embrace the overlap, and should, in general, conform to the four requirements stated on the preceding page;
b) Measure the parallax of the various control points;
c) Obtain the focal-length f, of the camera used;
d) Determine the air-base B of the stereo-pair by one of two methods.
   1) Determine B by measuring the map distance between the two principal points of a radial line plot of an overlapping pair of photographs and applying the scale of the radial-line plot.
   2) Using the following formula given by Moffitt[1] (p. 181)

$$B = \left[\frac{D^2_{AB}}{\left(\frac{x_a}{P_a} - \frac{x_b}{P_b}\right)^2 + \left(\frac{y_a}{P_a} - \frac{y_b}{P_b}\right)^2} \right]^{1/2} \tag{16}$$

where

D_{AB} = a line of known ground length

AB: end points of above line, the images of which, ab, appear in the overlap of the stereo-pair.

With the above information available, one could follow the following procedure:

1) Determine a flying height H_i from the parallax (P_j) and elevation (h_i) of each control point, using Equation (3)

$$H_i = h_j + \frac{B}{P_j} f \tag{3}$$

2) Determine the distance (d_j) of any point x, the elevation of which is sought, to each of the control points, from a map of the overlap (Figure 4).
3) Compute a *weighted* flying height H_x, at x, as follows:

Let

$$\text{Weight Factor } W_j = \frac{1}{d_j}$$

Then

$$H_x = \frac{\frac{1}{d_1} H_a + \frac{1}{d_2} H_b + \cdots + \frac{1}{d_n} H_n}{\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_1}} = \frac{\Sigma w_j H_i}{\Sigma w_j} \tag{17}$$

4) Finally, using Equation (2) compute the elevation of the point x

$$h_x = H_x - \frac{Bf}{p_x} \tag{2}$$

In the above method a weighted flying height H_x is computed at every point for which an elevation is desired. On the other hand, there are no corrections applied to any of the parallax measurements. I believe that more accurate results can thus be obtained because of the fact that the farther the points lie from a given control point, the greater are the chances for elevation errors (Moffitt[1], p. 177). In other words, the accuracy of the flying height used in equation (2) is inversely proportional to the distance from point x to the control point. Hence the proposed use of the weight factors $1/d_j$. Finally, the proposed method may require a little more time than the

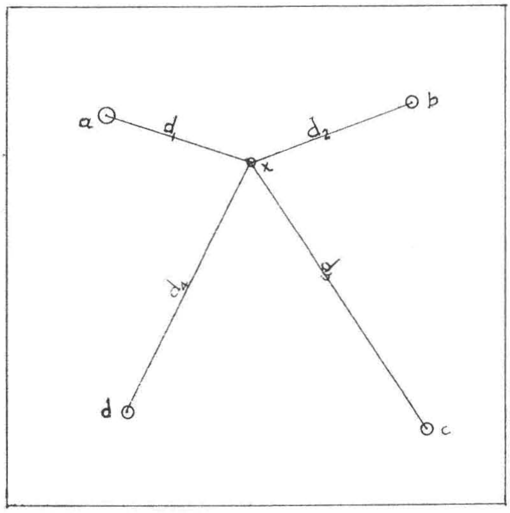

FIG. 4

method using a parallax correction graph. Yet, there is the possibility of developing an electronic computer solution for the proposed method, in which case "time" will not be of the essence.

V. CONCLUSIONS

The methods and formulae presented in this paper adequately summarize the original and essential contributions to the problem of the computation of elevations from parallax measurements on aerial photographic prints. Also a method has been suggested for determining elevations using weighted flying heights and parallax measurements.

The one point that I hope to have brought out is that elevations computed from parallax measurements are inherently erroneous, due to factors such as unequal flying heights, tilt, differential paper shrinkage, etc. Accurate vertical and horizontal position can be determined photogrammetrically by using precise stereo-plotting instruments or by applying complicated analytical procedures. Where the scope of the job requires these applications, such methods are well developed and are used. However, elevation and position determinations by relatively simple methods using inexpensive equipment are still in demand by many users of photogrammetric methods. It is the task of the photogrammetrist to develop workable methods of correcting for these errors, so that more accurate results can be obtained using parallax measurements from aerial photographic prints.

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