contribute to the uncertainty of this location. Because of this, and the difficulty of statically determining when the c.g. is truly centered, displacements will certainly occur. To further minimize these displacements, it appears mandatory that in-flight film shifts must also be compensated to maintain a reasonable balance. From an examination of the derived equation of motion (8) it is also evident that the longer the isolator base, the smaller will be the rotation due to an unbalance. Structural rigidity, however, must be maintained. This assumption has been made throughout this discussion for both the camera and its mounting support.

It has also been assumed throughout this discussion that vibratory inputs to the camera platform are purely translational. This is the only type data usually available from airframe manufacturers. Aircraft structural flexure, however, must certainly have some rotational modes of vibration of which very little information is known.

Of further importance in the evaluation of the use of vibration isolators as a mounting method is the uniformity of each support. Identical isolators have been assumed here. Unless the spring constants and damping factors of these isolators are carefully matched to each other, additional rotation above that predicted by this analysis can result.

Although other factors, such as atmospherics, aircraft roll and pitch, and image motion exist which affect resolution, the discussion herein has been with reference to those effects caused only by vibration. In particular, the effect of c.g. location on resolution and the manner by which it may be evaluated has been outlined.

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*Simultaneous Three-Dimensional Transformation of Higher Degrees**

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Abstract: *Transformation equations of second and higher degrees are mainly used to improve the fit on given control points by correcting for deformations incidental to strips or blocks. The common practice that is almost always applied at the present time is to perform the transformation for the horizontal position, X and Y, separate from that for height, Z. The purpose of this paper is to briefly present some of the equations currently used and then give a detailed development of new equations that would allow the simultaneous adjustment of all three coordinates, X, Y, and Z, in one operation.*

Introduction

Transformation and adjustment of instrument coordinates to fit available ground-control became a problem with the development of instrumental triangulation on the first-order stereo-plotters. Owing to the absence of better techniques and/or facilities, the early solution to the problem was in the form of graphical

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methods supplemented by transformation computations on the desk calculator.[1] However, with the advent of electronic computers, the practice slowly diverted from the graphical methods to numerical solulutions. Consequently, different organizations published a variety of mathematical formulations for the numerical adjustment of instrument data on electronic computers. (See references 2 to 5.)

DR. EDWARD M. MIKHAIL

The common characteristic of the majority of these methods is the separation of the horizontal adjustment from that for the heights. Principal reasons for this practice are:

(1) ground-control is normally classified as horizontal and vertical-control;
(2) analytical formulae are derived from earlier graphical techniques which followed the same practice; and
(3) simultaneous three-dimensional equations suitable for photogrammetric adjustment are not available.

Therefore, no serious attempts were made to consider the simultaneous horizontal and vertical adjustments.

The present programs of adjustment almost exclusively make use of polynomials to account for different kinds of deformations incidental to strips or blocks. These polynomials may be in the form of conformal or non-conformal transformation equations of second and higher degrees in the model coordinates. The following are examples of such equations for horizontal and vertical adjustment:

horizontal adjustment:[5]

$$X = A_0 + Ax + By + E(x^2 - y^2) + F \cdot 2xy$$
$$Y = B_0 - Bx + Ay - F(x^2 - y^2) + E \cdot 2xy \quad (1)$$

vertical adjustment:[4]

$$Z = z + Gx^2 + Hxz + Ixy + Jy + Kx + L \quad (2)$$

where:

x, y, z: are the model coordinates in the strip,
X, Y, Z: are the ground coordinates,
$A_0, A, \cdots, L$: the unknown coefficients of transformation.

Because of the separation between the horizontal and vertical adjustments there are usually some requirements which the model coordinates must satisfy. For instance, for the horizontal adjustment the strips must be approximately level in order to ensure that the height differences have no appreciable effect on the plane coordinates. For the vertical adjustment, the model coordinates must be in the so-called axis-of-flight coordinate system and the model heights must be at the same scale of the ground system.

Current methods of coordinate transformation and adjustment do perform the task required especially with respect to instrument coordinates. One disadvantage of this method is the need for two separate programs to transform and adjust data. Also, present-day analytical methods of triangulation remove the restriction imposed by separate horizontal and vertical ground-control points. A simultaneous three-dimensional transformation and adjustment of coordinates would furnish a more compact and efficient method of dealing with the problem. In the following paragraphs, equations for a three-dimensional transformation of higher degrees are developed.

THREE-DIMENSIONAL TRANSFORMATION

The transformation equations of three-dimensions that are commonly known are those of the first degree, given by:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = s \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} \tag{3}$$

where:

s: a change of scale factor,
$a_{11}, a_{12}, \cdots, a_{33}$: the nine elements of the orthogonal matrix of rotation,
(A_0, B_0, C_0): the vector of the three elements of translation.

It is obvious from Equation (3) that this kind of transformation accounts for a shift of the origin parallel to the three axes, three rotations about the three axes, and a change in scale. However, since this equation is linear in the coordinates, no allowance can be made for the deformations common to the models represented by a strip or a block and formed either instrumentally, or analytically. Therefore, the result we are seeking now is to develop three-dimensional transformation equations of second and higher degrees to account for such deformations.

Let us first consider the transformation equations that are conformal.

CONFORMAL THREE-DIMENSIONAL TRANSFORMATION OF HIGHER DEGREES

The three-dimensional transformation equations of the second degree can be written in general form as follows:*

$$\begin{aligned} X &= A_0 + A_1x + A_2y + A_3z + A_4x^2 + A_5y^2 + A_6z^2 + A_7xy + A_8yz + A_9zx \\ Y &= B_0 + B_1x + B_2y + B_3z + B_4x^2 + B_5y^2 + B_6z^2 + B_7xy + B_8yz + B_9zx \\ Z &= C_0 + C_1x + C_2y + C_3z + C_4x^2 + C_5y^2 + C_6z^2 + C_7xy + C_8yz + C_9zx \end{aligned} \tag{4}$$

In order that Equations (4) may represent conformal transformation, the following condition must be satisfied:

$$\begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{bmatrix} \equiv \text{orthogonal matrix} \tag{5}$$

Once the partial derivatives are performed and substituted in the matrix of (5), the conditions of orthogonality can then be applied to obtain the relationships between the thirty coefficients of Equations (4) to make the transformation conformal. However, since this derivative is quite complicated for second-degree and more so

* Second degree equations are considered only as an example, since Equations (4) can be easily extended to include third, fourth and higher degree terms.

for third and higher degree equations, it has not been attempted. Consequently, a different approach is investigated next.

USE OF QUATERNIONS

Quaternion algebra can be used to develop the transformation equations sought, in the same fashion the complex numbers are used to derive the two-dimensional conformal Equations (1). A real quaternion is a hypercomplex number composed of a quadruple of real numbers written in the definite order:

$$q = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \tag{6}$$

which shows that it is composed of the sum of a real number, a_0, and *a* vector $(a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_2\boldsymbol{k})$. The rules of addition, subtraction and multiplication by a scalar for quaternions are the same as those applied in vector and matrix algebra. Multiplication of quaternions, however, is performed in a manner similar to complex numbers applying the following rules

$$\begin{aligned} \boldsymbol{i}^2 &= \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1 \\ \boldsymbol{ij} &= -\boldsymbol{ji} = \boldsymbol{k} \\ \boldsymbol{jk} &= -\boldsymbol{kj} = \boldsymbol{i} \\ \boldsymbol{ki} &= -\boldsymbol{ik} = \boldsymbol{j} \end{aligned} \tag{7}$$

In applying quaternions to the problem at hand, one encounters the difficulty of using a four-dimensional element (the quaternion) to obtain a three-dimensional relationship. To illustrate the use of quaternions, the following equation expresses the conformal transformation between the four-dimensional systems (h, x, y, z) and (H, X, Y, Z):

$$\begin{aligned} (H + X_i + Yj + Zk) = {} & (a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) + (b_0 + b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \\ & \cdot (h + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) + (c_0 + c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}) \\ & \cdot (h + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})^2 + \cdots + \cdots \end{aligned} \tag{8}$$

Equation (8) has two undetermined parameters h and H which cannot be easily interpreted in the present problem that encounters only transformation problems of the third dimension. Therefore, in an attempt to adapt Equation (8) to our purpose, the four-dimensional transformation system is projected onto a three-dimensional system by enforcing the condition that

$$h = H = 0 \tag{9}$$

and hence Equation (8), after manipulating its right-hand side, becomes:

$$\begin{aligned} Xi + Yj + Zk = {} & [a_0 - b_1x - b_2y - b_3z - c_0(x^2 + y^2 + z^2) + \cdots] \\ & + [a_1 + b_0x - b_3y + b_2z - c_1(x^2 + y^2 + z^2) + \cdots]\boldsymbol{i} \\ & + [a_2 + b_3x + b_0y - b_1z - c_2(x^2 + y^2 + z^2) + \cdots]\boldsymbol{j} \\ & + [a_3 - b_2x + b_1y + b_0z - c_3(x^2 + y^2 + z^2) + \cdots]\boldsymbol{k} \end{aligned} \tag{10}$$

Equating the coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, on both sides of Equation (10) and assembling the results in matrix form, we get:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} b_0 & -b_3 & b_2 \\ b_3 & b_0 & -b_1 \\ -b_2 & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - (x^2 + y^2 + z^2) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{11}$$

with the additional condition that

$$a_0 - b_1x - b_2y - b_3z - c_0(x^2 + y^2 + z^2) = 0 \tag{12}$$

The result of this investigation shows clearly the following two disadvantages:

(a) Although the four-dimensional transformation of Equation (8) is in itself conformal, when it is reduced to three-dimensional system it no longer remains conformal as shown by the matrix of coefficients of Equation (11).

(b) The presence of an additional condition (Equation 12) with two more undetermined coefficients, a_0 and c_0, poses a considerable complication on the application of these equations, particularly when the method of least squares is used in the solution.

It may seem, at this point, that the application of quaternion algebra is not very useful. However, this mathematical technique still possesses an attractive advantage of the great simplicity in manipulation as compared to the approach of partial derivatives given above. Furthermore, this quaternion treatment casts a shadow of doubt as to whether three-dimensional conformal transformations of second and higher degrees even exist. This fact is left for future investigation to prove its definite validity.

THREE-DIMENSIONAL TRANSFORMATION OF HIGHER DEGREES, CONFORMAL IN THE THREE PROJECTION PLANES

If conformal transformation in two dimensions is defined as that which preserves *planar* angles, the three-dimensional conformal transformation can physically be defined as that which preserves *solid* angles. Since difficulty and complexity have been encountered in developing conformal transformation, a new idea which yields as nearly conformal transformation as possible, is introduced. The three-dimensional model is transformed in such a manner that its projections on the three planes of the system are transformed conformally. In other words, while the solid angles in the model itself are not absolutely preserved in the transformation, the three projection angles in the X-Y, Y-Z and Z-X planes are preserved. This condition is expressed by the following relations as referred to the general Equations (4).

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} = \frac{\partial Z}{\partial z}$$

$$\frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}$$

$$\frac{\partial Y}{\partial z} = -\frac{\partial Z}{\partial y}$$

$$\frac{\partial Z}{\partial x} = -\frac{\partial X}{\partial z} \tag{13}$$

Applying the conditions given by (13) to Equations (4), we obtain the following relations:

$$\begin{aligned} A_1 &= B_2 = C_3 &&= A \\ A_2 &= -B_1 &&= B \\ A_3 &= -C_1 &&= -C \\ B_3 &= -C_2 &&= D \end{aligned}$$

$$
\begin{aligned}
2A_4 &= -2A_5 = -2A_6 = B_7 = C_9 = 2E \\
A_7 &= 2B_5 = -2B_4 = -2B_6 = C_8 = 2F \\
A_9 &= B_8 = -2C_4 = -2C_5 = 2C_6 = 2G \\
A_8 &= B_9 = C_7 = 0 \qquad (14)
\end{aligned}
$$

Substituting the relations of (14) in the general Equations (4) we get the following transformation equations:

$$
\begin{aligned}
X &= A_0 + Ax + By - Cz + E(x^2 - y^2 - z^2) + 0 + 2Gzx + 2Fxy + \cdots \\
Y &= B_0 - Bx + Ay + Dz + F(-x^2 + y^2 - z^2) + 2Gyz + 0 + 2Exy + \cdots \\
Z &= C_0 + Cx - Dy + Az + G(-x^2 - y^2 + z^2) + 2Fyz + 2Ezx + 0 + \cdots \qquad (\ \ )
\end{aligned}
$$

which are the final form of the transformation equations sought.

APPLICATION AND CONCLUSION

Obviously, Equations (15) are not conformal and therefore should be used only in cases where the rotation angles are small. This can easily be realized by the fact that the three-dimensional linear conformal transformation Equation (3) normally precedes the application of higher degree equations, to allow for large rotation angles. Therefore, the scheme of the adjustment is simply composed of two steps:

(a) a simple linear transformation using Equation (3) to eliminate large angles between the two coordinate systems; and
(b) the final adjustment of the results obtained from (a) using Equations (15).

It should be emphasized that in order to obtain a high mathematical accuracy in the computations, differences between coordinates, and not the coordinates themselves, should be used in the solution.

The equations presented, to the best of my knowledge, are the first attempt for a simultaneous three-dimensional transformation of higher degree. They were developed for almost an ideal situation of adjusting square sub-blocks by a relaxation procedure.[6] However, the application of Equations (15) may not be limited to sub-blocks and can be used for strip adjustment, either directly or after minor modifications. For example, when applied to strips, these equations allow for the correction of: scale variation in the X-direction by the E-term, scale variation in the Y-direction and hence for X-azimuth by the F-term, and change of scale in the Z-coordinate and therefore the X- and Y-bends by the G-term.

In conclusion, it is hoped that the foregoing development may open a new direction in the adjustment of photogrammetric strips or blocks in which simultaneous three-dimensional transformation is considered. The further development and/or testing of the equations presented is therefore urged of those interested in the subject.

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