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A Trigonometric Derivation of the Formulas for the Three-Dimensional Rotation Matrix

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THE formulas for a three-dimensional orthogonal rotation matrix have already been derived by two other methods: (1) algebraic substitution[1] in the formulas for the rotation of axes in plane analytic geometry; and (2) the multiplication[2] of the ordered series of matrices that represent the component plane rotations. Here a third derivation is presented using spherical trigonometry.

Figure 1 shows the ends of the "original" axes x^*, y^*, z^* piercing the surface of a sphere whose center is the origin. The axes form an orthogonal set, being mutually perpendicular. Regarding the x^*-axis as an axis of rotation, y^* is rotated counter clockwise (positive) through an angle ω to $y^{*\prime}$ while z^* is moved the same angular amount to $z^{*\prime}$, maintaining the mutually perpendicular orientation among the points x^*, $y^{*\prime}$, $z^{*\prime}$.

A second rotation about the $y^{*\prime}$-axis as the axis of rotation through an angle ϕ sends x^* into $x^{*\prime}$ and $z^{*\prime}$ into z, retaining the relative orthogonal orientation. A third rotation about z as an axis through an angle κ sends $x^{*\prime}$ into x, and $y^{*\prime}$ into y, again preserving the orthogonality.

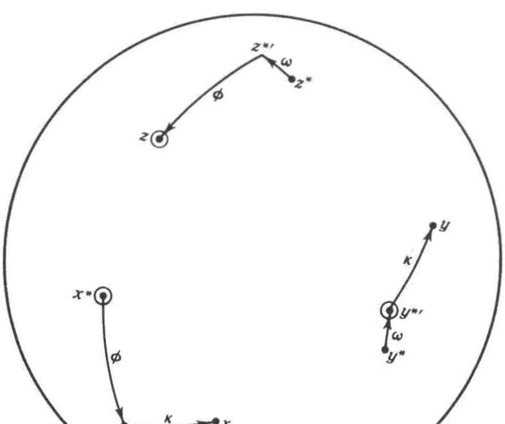

FIG. 1

It is desired to establish formulas for the nine direction cosines a_{ij} in terms of ω, ϕ, κ in the array:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix}. \quad (1)$$

[1] Harris, W. D., et al., "Analytic Aerotriangulation," Technical Bulletin No. 21, U. S. Coast & Geodetic Survey, Washington 25, D. C., p. 30, 1962.
[2] *Ibid.*, p. 9.
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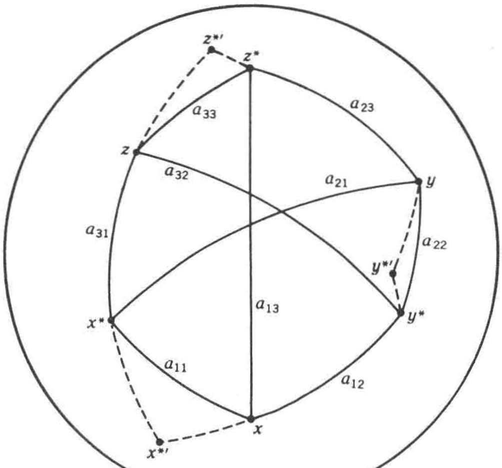

FIG. 2

The following table may assist one in identifying the specific direction angles:

	x^*	y^*	z^*
x	a_{11}	a_{12}	a_{13}
y	a_{21}	a_{22}	a_{23}
z	a_{31}	a_{32}	a_{33}.

(2)

The term a_{11} is the cosine of the angle between the x and x^* axes, namely, the angle xOx^* where O is the origin; a_{32} is the cosine of the angle between the z and y^* axes, etc.[3] Figure 2 illustrates these angles.

A few of the principles of spherical trigonometry are mentioned here as background inasmuch as they are seldom contained in elementary textbooks.[4] The radius of the sphere is considered to be unity. A great circle arc is the line of intersection of the surface of the sphere and a plane passing through the center of the sphere. In this presentation all *lines*, such as those forming the sides of spherical triangles, are great circle arcs. A *side* of a spherical triangle is expressed as an *angle* at the center of the sphere subtended by the vertices. An *angle* of a spherical triangle is the dihedral angle between the two great planes that form the angle. A *right* spherical triangle has one angle equal to 90°, as shown in Figure 4.

Any line on a sphere is a portion of an *equator* which has two *poles*. The length of a line from pole to equator is 90°. In Figure 3, x^*z is a line whose pole is at $y^{*\prime}$. A line from a pole to its equator is a *meridian*; a meridian intersects its equator at right angles. An *angle* at a pole between two meridians is equal to the *arc* length on the equator intersected by the meridians. In triangle $x^*y^{*\prime}z$, the angle at $y^{*\prime}$ is equal to the opposite side x^*z.

One of the standard formulas for a spherical right triangle will be used here:

$$\cos c = \cos a \cos b$$

where c is the hypotenuse and b are the other two sides. Another standard formula from spherical trigonometry that will be used is the Law of Cosines for the sides of an oblique triangle:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

where A is the angle opposite the side a.

Figure 3 shows the side x^*z whose cosine is a_{31}. By virtue of the orthogonality relation, the arc $x^*zz^{*\prime}$ is an equator with $y^{*\prime}$ as a pole, whence the angles at $y^{*\prime}$ are equal to the arcs on $x^*zz^{*\prime}$. Also, $x^*z^{*\prime}$ is 90°, $zz^{*\prime}$ is ϕ, whence x^*z is $(90° - \phi)$, and

$$\cos x^*z = a_{31} = \cos (90° - \phi) = \sin \phi. \tag{3}$$

In Figure 4, by construction and definition z is a pole to the equator $x^{*\prime}xy^{*\prime}y$. Therefore the angle at $x^{*\prime}$ is 90°, triangle $xx^{*\prime}x^*$ is a right triangle, the hypotenuse is xx^* whose cosine is desired, and the other two sides have the lengths ϕ and κ; then

$$\cos xx^* = a_{11} = \cos \phi \cos \kappa. \tag{4}$$

Similarly, in Figure 5, $y^{*\prime}$ is a pole of arc $z^{*\prime}zx^*x^{*\prime}$, triangle $zz^{*\prime}z^*$ is a right triangle with legs ϕ and ω, and hypotenuse zz^*:

$$\cos zz^* = a_{33} = \cos \phi \cos \omega. \tag{5}$$

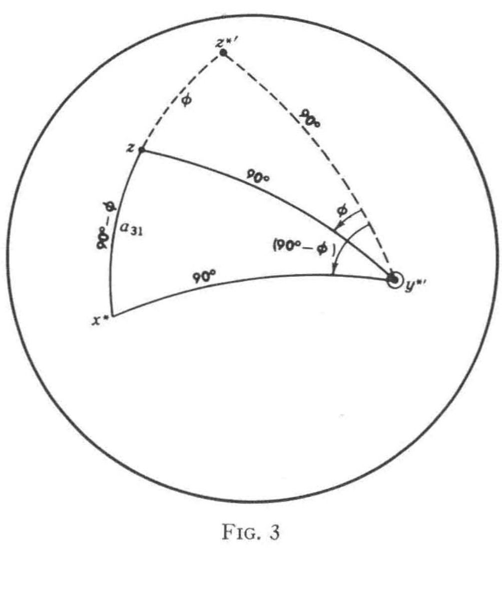

FIG. 3

[3] Bell, Robert J. T., "An Elementary Treatise on Coordinate Geometry of Three Dimensions," Chapter IV, MacMillan & Co., Third Edition, 1959.

[4] Eshbach, Ovid W., "Handbook of Engineering Fundamentals," p. 2–75, John Wiley & Sons, 1957.

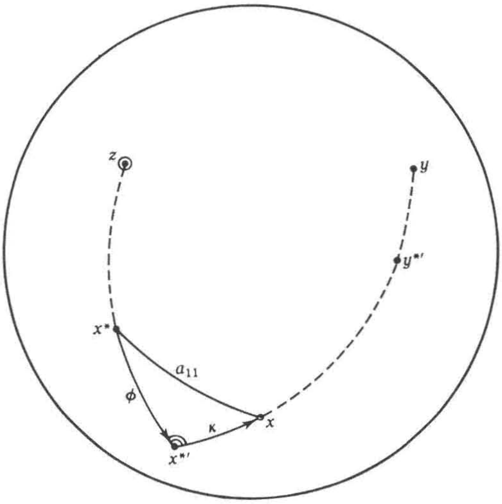

FIG. 4

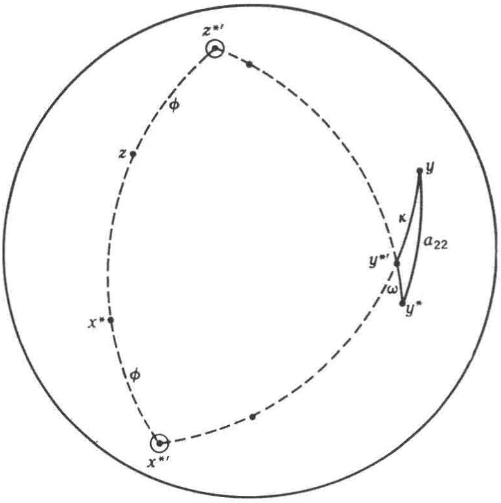

FIG. 6

The rotational elements ω and κ together with yy^* form an oblique spherical triangle wherein the angle $yy^{*\prime}y^*$ is known, as follows (Figure 6). Consider the exterior triangle $x^{*\prime}y^{*\prime}z^{*\prime}$. By construction, $y^{*\prime}$ is a pole of the equator $z^{*\prime}zx^*x^{*\prime}$. The length of the equatorial arc is $(90°+\phi)$ by construction, and this is also the polar angle $x^{*\prime}y^{*\prime}z^{*\prime}$ at $y^{*\prime}$. The angle $yy^{*\prime}y^*$ of the small triangle is equal to the opposite angle $x^{*\prime}y^{*\prime}z^{*\prime}$ or $(90°+\phi)$. Applying the Law of Cosines for sides to triangle $yy^{*\prime}y^*$,

$$\cos yy^* = a_{22} = \cos\omega\cos\kappa + \sin\omega\sin\kappa\cos(90° + \phi)$$
$$= \cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa. \qquad (6)$$

Consider triangle $x^*x^{*\prime}y$ in Figure 7. The angle at $x^{*\prime}$ is a right angle by construction. One side length is ϕ and the other is $(90°+\kappa)$. Then the third side x^*y is the hypotenuse and

$$\cos x^*y = a_{21} = \cos\phi\cos(90° + \kappa)$$
$$= -\cos\phi\sin\kappa. \qquad (7)$$

In triangle $y^*zz^{*\prime}$ shown in Figure 8, the angle at $z^{*\prime}$ is a right angle by construction and definition, the two legs have the lengths ϕ and $(90°+\omega)$, and the hypotenuse is therefore given by

$$\cos y^*z = a_{32} = \cos\phi\cos(90° + \omega)$$
$$= -\cos\phi\sin\omega. \qquad (8)$$

In triangle $yy^{*\prime}z^*$ of Figure 9, the side $yy^{*\prime}$ has an arc length κ, the side $y^{*\prime}z^*$ is $(90°-\omega)$, the angle at $y^{*\prime}$ is the complement $(90°-\phi)$ of the outer angle $(90°+\phi)$. Applying the law

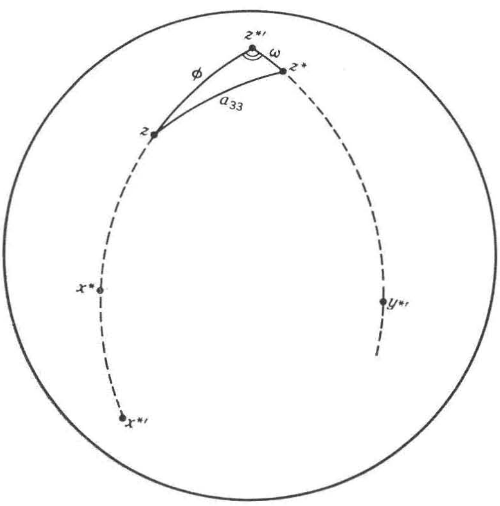

FIG. 5

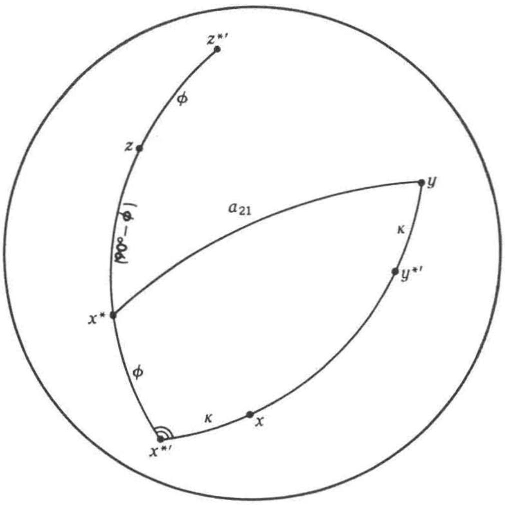

FIG. 7

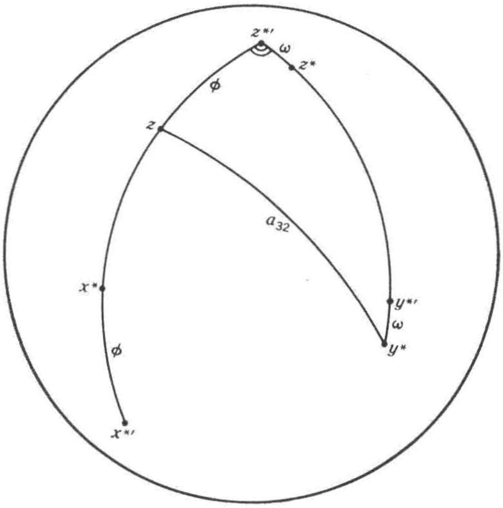

FIG. 8

of cosines,

$$\begin{aligned}\cos yz^* = a_{23} &= \cos\kappa\cos(90^\circ-\omega)\\ &\quad+\sin\kappa\sin(90^\circ-\omega)\cos(90^\circ-\phi)\\ &=\sin\omega\cos\kappa+\cos\omega\sin\phi\sin\kappa.\end{aligned}\tag{9}$$

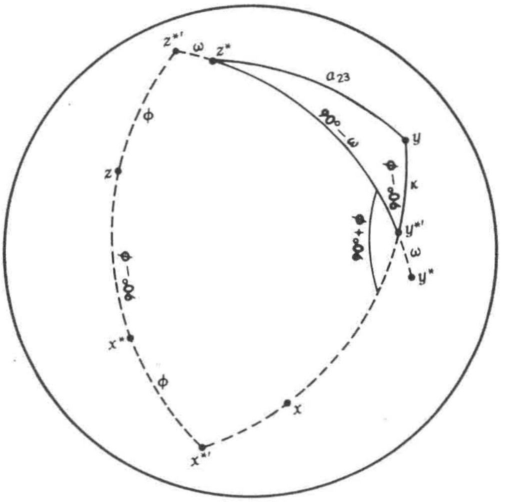

FIG. 9

A similar situation exists for the term a_{12} as shown in Figure 10. In the triangle $xy^*y^{*\prime}$, the length of side $y^*y^{*\prime}$ is the angle ω, that for the side $xy^{*\prime}$ is $(90^\circ-\kappa)$, and the angle at $y^{*\prime}$ is $(90^\circ-\phi)$. From the Law of Cosines for sides,

$$\begin{aligned}\cos xy^* = a_{12} &= \cos\omega\cos(90^\circ-\kappa)\\ &\quad+\sin\omega\sin(90^\circ-\kappa)\cos(90^\circ-\phi)\\ &=\cos\omega\sin\kappa+\sin\omega\sin\phi\cos\kappa.\end{aligned}\tag{10}$$

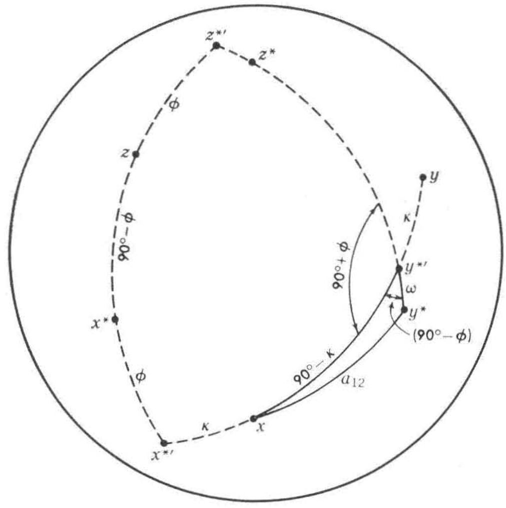

FIG. 10

Figure 11 illustrates the unknown side xz^* of the triangle $xy^{*\prime}z^*$. The side $xy^{*\prime}$ is $(90^\circ-\kappa)$, side $y^{*\prime}z^*$ is $(90^\circ-\omega)$, and the angle at $y^{*\prime}$ is $(90^\circ+\phi)$. Applying again the Law of Cosines for the sides,

$$\begin{aligned}\cos xz^* = a_{13} &= \cos(90^\circ-\omega)\cos(90^\circ-\kappa)\\ &\quad+\sin(90^\circ-\omega)\sin(90^\circ-\kappa)\cos(90^\circ+\phi)\\ &=\sin\omega\sin\kappa-\cos\omega\sin\phi\cos\kappa.\end{aligned}\tag{11}$$

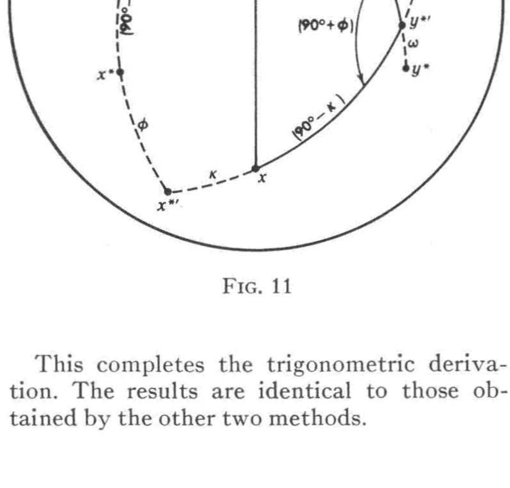

FIG. 11

This completes the trigonometric derivation. The results are identical to those obtained by the other two methods.