The Absolute Orientation of Near Verticals

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ITERATIVE methods of computation are well suited to many problems in photogrammetry and surveying. It is often necessary to compute accurately the values of certain unknowns whose approximate values are already known. For example the elements of relative orientation of two photographs with a common overlap, the elements of absolute orientation of a stereoscopic model, or the absolute orientation elements of a single photograph.

The approximate values of the unknowns are used as initial values in the computation, and the number of iterations required will depend on the desired accuracy and the rate at which the process converges. For the determination of the six elements of orientation of a single near-vertical photograph, initial values of the elements are adopted on the assumption that the photograph was taken with the camera axis truly vertical, and that the height of the perspective center of the camera is known. The data required for the computation consists of the co-ordinates and heights of three ground control points, and the co-ordinates of their images as measured in the photograph.

The solution of the unknowns requires the determination of the space co-ordinates of the image-points at the instant of exposure, the perspective center, and the ground control points in some arbitrary system of co-ordinate axes. At that instant each image-point was located on the straight line defined by the perspective center and its respective ground point.

The condition that three points, i, p and I, lie on one straight line is given by;

$$\frac{X_i - X_p}{X_p - X_I} = \frac{Y_i - Y_p}{Y_p - Y_I} = \frac{Z_i - Z_p}{Z_p - Z_I} \tag{1}$$

therefore each control and image-point pair gives rise to two condition equations:

$$\frac{X_i - X_p}{X_p - X_I} = \frac{Z_i - Z_p}{Z_p - Z_I} \quad \text{and} \quad \frac{Y_i - Y_p}{Y_p - Y_I} = \frac{Z_i - Z_p}{Z_p - Z_I} \tag{2}$$

We shall regard the arbitrary system of co-ordinates as having its XY plane parallel to the XY plane of the ground system, and its origin at the perspective center of the camera. The Equations (2) then reduce to:

$$\begin{aligned} X_iZ_I - X_IZ_i &= 0 \\ Y_iZ_I - Y_IZ_i &= 0 \end{aligned} \tag{3}$$

If P is the orthogonal matrix that transforms the photo co-ordinates into the system chosen above, then, as has already been shown, P can be expressed as the sum of the unit matrix and a skew symmetric matrix, on condition that the rotations, which we call ω, ϕ and κ, are each small. Thus:

$$P \simeq \begin{pmatrix} 1 & -p_{21} & -p_{31} \\ p_{21} & 1 & -p_{32} \\ p_{31} & p_{32} & 1 \end{pmatrix}$$

and the matrix P is orthogonal if we neglect powers of the rotations higher than the first.

The space co-ordinates of the photo points are given by:

$$\begin{pmatrix} X_i \\ Y_i \\ Z_i' \end{pmatrix} = P \begin{pmatrix} X_i' \\ Y_i' \\ Z_i' \end{pmatrix} \tag{4}$$

where X_i' and Y_i' are the measured photo co-ordinates, and f is the principal-distance of the camera. As a first approximation we take $P=I$ the unit matrix.

If κ is the rotation about a vertical axis (or nearly vertical in the event of it not being the primary axis), then the rotations ω and ϕ will always be small for nearly vertical photographs. However κ will not be small, for in the general case, the direction of flight will not be such that the X and Y co-ordinate axes in the photograph are approximately parallel to the corresponding axes in the ground system. It is therefore necessary, in order that κ be small, to perform a preliminary rotational transformation of the control points about a vertical axis. While only a rotation is necessary, it is convenient for purposes of computation to change the scale and origin as well. This will give control points which are of the same magnitude as the co-ordinates of the image points.

If X_I' and Y_I' are the given co-ordinates of a control point, and X_I Y_I the co-ordinates after the preliminary transformation, we have:

$$\begin{pmatrix} X_I \\ Y_I \end{pmatrix} = \begin{pmatrix} a & b & C_1 \\ -b & a & C_2 \end{pmatrix} \begin{pmatrix} X_I' \\ Y_I' \\ 1 \end{pmatrix} \tag{5}$$

where the elements a, b, C_1 and C_2 are computed in the usual way from any two of the three controls, such that after transformation the "terrain" co-ordinates of these two points will be identical to their measured photo co-ordinates, and the computations can then be carried out in the photo scale.

It should be noted that with the origin of the arbitrary space system at the perspective center of the camera, the co-ordinates X_I Y_I will always be opposite in sign to those of the corresponding image-points. However, it is not necessary to change the signs of the X_I Y_I if Equations (3) are altered to:

$$\begin{aligned} X_i Z_I + X_I Z_i &= 0 \\ Y_i Z_I + Y_I Z_i &= 0 \end{aligned} \tag{6}$$

The initial values of the Z_I are computed from:

$$Z_I = (h_I' - H)\lambda \tag{7}$$

where the h_I' are the given heights of the ground control points, H is the flying height of the aircraft as recorded by the camera's altimeter, and λ is the scale factor of the transformation given in Equation (5)

$$\lambda = + (a^2 + b^2)^{1/2} \tag{8}$$

Having computed $X_I\ Y_I\ Z_I$ for the three control points and any additional controls which may be present, the given data are in a form suitable for the improvement of the preliminary values of the unknowns by iteration.

For the sake of convenience, the perspective center will be kept as the origin of the arbitrary space system at each stage of the iteration. As any displacement of the perspective center relative to the terrain in X, Y and Z can be regarded as a displacement of the terrain relative to the perspective center, we shall adopt as unknowns the co-ordinates of any one of the terrain points, and not those of the perspective center.

THE ITERATION PROCEDURE

From Equation (4) we have that:

$$\begin{aligned} X_i &= X_i' - p_{21}Y_i' - p_{31}f \\ Y_i &= p_{21}X_i' + Y_i' - p_{32}f \\ Z_i &= p_{31}X_i' + p_{32}Y_i' + f \end{aligned} \tag{9}$$

Differentiating with respect to $p_{21}\ p_{31}$ and p_{32} we obtain the derivatives of $X_i\ Y_i$ and Z_i

$$\begin{aligned} dX_i &= -\ Y_i'dp_{21} - f\ dp_{31} \\ dY_i &= X_i'dp_{21} - f\ dp_{32} \\ dZ_i &= X_i'dp_{31} + Y_i'dp_{32} \end{aligned} \tag{10}$$

If we rewrite Equations (6) thus:

$$\begin{aligned} X_iZ_I + X_IZ_i &= U \\ Y_iZ_I + Y_IZ_i &= V \end{aligned}$$

then differentiating U with respect to $X_iZ_I\ X_I$ and Z_i, and V with respect to $Y_i\ Z_I\ Y_I$ and Z_i we have:

$$\begin{aligned} dU &= Z_IdX_i + X_idZ_I + X_IdZ_i + Z_idX_I \\ dV &= Z_IdY_i + Y_idZ_i + Y_IdZ_i + Z_idY_I \end{aligned}$$

and substituting from Equations (10):

$$\begin{aligned} dU &= -\ Y_i'Z_Idp_{21} + (X_IX_i' - fZ_I)dp_{31} + X_IY_i'dp_{32} + Z_idX_I + X_idZ_I \\ dV &= X_i'Z_Idp_{21} + Y_IX_i'dp_{31} + (Y_IY_i' - fZ_I)dp_{32} + Z_idY_I + Y_idZ_I \end{aligned} \tag{11}$$

When the initial values of $X_i\ Y_i\ Z_i$ and $X_I\ Y_I\ Z_I$ are substituted in the left-hand sides of Equations (6), the right-hand sides will not be zero. Let $K_1K_2 \cdots, K_6$ be the values of the right-hand sides. These six values are substituted with signs reversed for dU and dV in Equations (11), and the set of six equations is solved for the unknowns dp_{21}, dp_{31}, dp_{32}, dX_I, dY_I and dZ_I.

The corrections dX_I, dY_I and dZ_I are applied to the co-ordinates of each of the control points, and the corrections dp_{21}, dp_{31} and dp_{32} are applied to the previous values of the respective elements in the matrix P. Using the corrected values of p_{21}, p_{31} and p_{32}, the remaining nine elements of the matrix are computed such that P is strictly orthogonal. The transformation by P of the coordinates of the measured photo points yields the values of their space co-ordinates which are used in the next iteration.

The iterations are continued until sufficiently accurate values of ω and ϕ, as determined from the matrix P, are obtained. A transformation of the control points back into the ground system gives the X and Y co-ordinates in the ground system of the perspective center. These will simply be the constant terms of the transformation formula.

The height of the perspective center above any one of the control points is computed from:

Height above control point $= (-Z_I/\lambda)$

where Z_I is the value from the last iteration.

EXAMPLE

A fictitious example with exaggerated tilts $\omega = \phi = 10°$ was computed. The principal distance of the camera was taken as 100 mm. It was assumed that the altimeter was inaccurate by 50 meters, the initial value for the height of the perspective center being taken as 2,250 meters. The data used and the results of the computation are summarised by the tables below.

CO-ORDINATES IN THE GROUND SYSTEM

Point	*X meters*	*Y meters*	*Z meters*
1	162 214.6	35 714.9	600
2	162 281.8	37 640.4	400
3	164 221.0	35 299.2	200
Pers. center	163 200.4	36 531.4	2,200

MEASURED PHOTO CO-ORDINATES

Point	*X mm.*	*Y mm.*
1	55.6327	55.6614
2	−17.6327	−55.5004
3	−17.6327	115.4203

VALUES OF THE GROUND CONTROL POINTS AFTER THE PRELIMINARY TRANSFORMATION

Point	*X mm.*	*Y mm.*
1	55.6327	15.6614
2	−17.6327	−55.5004
3	− 5.8272	105.2233

VALUES OF THE UNKNOWNS AFTER SUCCESSIVE ITERATIONS

	ω	ϕ	*X meters*	*Y meters*	*Z meters*
First approx.	0°00′00″	0°00′00″	162 767.0	36 654.8	2,250.0
1st iter.	9°05′30″	9°36′50″	163 161.0	36 583.0	2,265.8
2nd iter.	10°15′10″	10°04′10″	163 210.5	36 521.5	2,205.0
3rd iter.	9°54′40″	9°55′50″	163 195.9	36 530.2	2,199.4
4th iter.	10°02′00″	10°00′30″	163 201.6	36 531.3	2,200.5

NOTE: The sixth element of orientation κ has been omitted from the above table but can be deduced from P in the same way as ω and ϕ.

CONCLUSION

The method outlined above has the advantage that, while the computations are considerable, they can conveniently be carried out on a desk calculator when an electronic computer is not available. The example given was calculated on a manually operated machine.

The convergence of the unknowns towards a specific value would be more rapid in a normal case where the tilts do not exceed 3°. Only four iterations were carried out in the example. However, at least one more should have been computed. The coefficients in Equations (11) were left unchanged after the second iteration. To a small extent this will also affect the rapidity of convergence adversely. With photographs which are more nearly vertical, one set of coefficients computed for the first iteration should be sufficient for the entire computation.

Appendix

When the tilt of the camera axis is large, as in convergent or oblique photography, it is still possible to use the above method for computing the elements of orientation by first carrying out a preliminary rectification of the photo co-ordinates. This is done by using the nominal tilt of the axis which is always known to within a few degrees. The rectified photograph, which Thompson calls the equivalent vertical, is then treated in the normal way.

For convergent photographs:

$$X' = \frac{f(X'' \cos \alpha - f \sin \alpha)}{X'' \sin \alpha + f \cos \alpha} \qquad Y' = \frac{fY''}{f \cos \alpha + X'' \sin \alpha}$$

where α is the angle that the camera axis makes with the vertical, X'', Y'' are the co-ordinates of the points in the tilted photograph, and X', Y' the co-ordinates in the equivalent vertical.

References

1) Thompson, E. H. "A Method of Relative Orientation in Analytical Aerial Triangulation," *Photogrammetric Record*, Oct. 1956, Vol. 2, No. 8, p. 145.
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