# *Formula for Conversion of Stereoscopically Observed Apparent Depth of Water to True Depth, Numerical Examples and Discussion*

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ABSTRACT: *A formula is derived for the conversion of a photogrammetrically observed apparent depth of water to the true depth. The correction factor by which an apparent depth must be multiplied is a variable. Its magnitude is dependent on the apparent water depth itself, the index of refraction, and the location of the submerged feature on the stereo model. External parameters, such as flight elevation and photographic base length, are also factors. The influence of these variables is discussed using calculated examples, and it is shown how a digital computer can routinely perform the numerical operations.*

## INTRODUCTION

MEASUREMENT of submarine features is important to many and is becoming of greater importance to more. Shipping, fishing and recreational interests now mingle with those of mining and petroleum companies, which are investigating or constructing harbors or engaged in underwater geologic exploration itself. Furthermore, coastal engineers are investigating off-shore sediment movement, beach erosion or hurricane protection problems, and the armed services need accurate maps of the continually changing near-shore environment.

The observation that aerial photography can portray details of submerged topography, coupled with the axiom: "What can be photographed can be measured," raises a number of questions, one of which is: "by what factor must a photogrammetrically determined depth be multiplied, in order that the true depth may be determined?"

This contribution represents the partial results of a continuing investigation being made by Aero Service Corporation, in the field of underwater mapping.

In the following an equation, originally derived by Tewinkel [1], is re-derived and transformed into equations that are capable of yielding numerical solutions. Such solutions are given for a single set of external conditions and are discussed with regard to precision and computer applications.

## DERIVATION OF BASIC EQUATION

In Figure 1, two aerial cameras, located respectively at $L_1$ and $L_2$ at height $h$ above a horizontal, flat water surface $L_1'L_2'D'$ each record the image of object $D''$, located at a true depth $h'$ below the surface of the water.

Rays emanating from $D''$ can only arrive at $L_1$ and $L_2$ by travelling in planes normal to the water surface as these planes must contain the normal to the water/air interface, according to Snell's Law. This Law states further that refraction takes place at the water/air interface, in such a manner that:

$$n_1 \sin i = n_2 \sin r \tag{1a}$$

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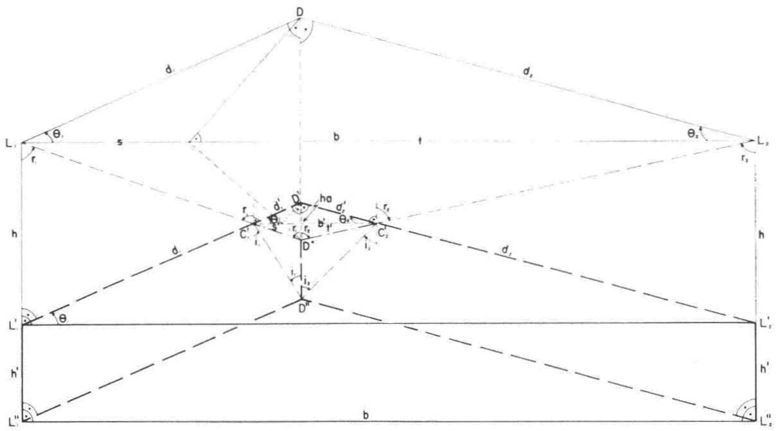

FIG. 1. Geometry of refraction.

where

$i$ = angle of incidence measured from the normal,
$r$ = angle of refraction measured from the normal,
$n_1$ = refractive index of water,
$n_2$ = refractive index of air,

or:

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1} = \frac{1}{n_1/n_2} = \frac{1}{n} \tag{1b}$$

where $n$ is a constant, the refractive index of water measured in air, approximately equal to 1.33.

For $D''$ to be seen from both $L_1$ and $L_2$, it is necessary that $L_1$, $C_1'$, $C_2'$ and $L_2$ lie in one plane. $D''$ is then observed as occurring at $D^*$ along the prolongation of lines $\overline{L_1C_1'}$ and $\overline{L_2C_2'}$. $D^*$ is located vertically above $D''$ because $L_1C_1'D''$ and $L_2C_2'D''$ lie in vertical planes, whose intersection is a vertical. As $D'$ is the vertical projection of both $D''$ and $D^*$, it follows that the stereoscopically plotted position of $D^*$, with regard to a horizontal coordinate system, is true in location. The depth of $D''$, however, is incorrectly plotted as being $\overline{D'D^*} = h_a$.

The factor is now sought which transforms $h_a$, the apparent or observed depth into $h'$, the true depth. Thus:

$$h' = Fh_a \tag{2a}$$

For calculation of $F$, we proceed as follows:
From $\triangle C_1'D'D''$ and $\triangle C_2'D'D''$:

$$d_1' = h' \operatorname{tg} i_1 \quad \text{and} \quad d_2' = h' \operatorname{tg} i_2 \tag{3a, 3b}$$

From $\triangle C_1'C_2'D'$:

$$b' = d_1' \cos\theta_1 + d_2' \cos\theta_2$$

$$= h' \operatorname{tg} i_1 \cos\theta_1 + h' \operatorname{tg} i_2 \cos\theta_2$$

$$h' = b'/(\operatorname{tg} i_1 \cos\theta_1 + \operatorname{tg} i_2 \cos\theta_2) \tag{3c}$$

Similarly; from $\triangle C_1'D'D^*$ and $\triangle C_2'D'D^*$:

$$d_1' = h_a \operatorname{tg} r_1 \quad \text{and}$$

$$d_2' = h_a \operatorname{tg} r_2 \tag{4a, 4b}$$

From $\triangle C_1'C_2'D'$:

$$b' = h_a \operatorname{tg} r_1 \cos\theta_1 + h_a \operatorname{tg} r_2 \cos\theta_2$$

$$h_a = b'/(\operatorname{tg} r_1 \cos\theta_1 + \operatorname{tg} r_2 \cos\theta_2) \tag{4c}$$

Substitution and simplification, using (3c) and (4c), yields:

$$F = \frac{\operatorname{tg} r_1 \cos\theta_1 + \operatorname{tg} r_2 \cos\theta_2}{\operatorname{tg} i_1 \cos\theta_1 + \operatorname{tg} i_2 \cos\theta_2} \tag{5}$$

This is the basic equation; in this form, however, it is not suitable for the calculation of correction factors over the area of a stereo model. In the following section we shall transform the equation into equations that are capable of yielding numerical solutions.

## Transformation of Basic Equation

The numerator of Equation (5), may be simplified as follows:

$$\operatorname{tg} r_1 = \frac{d_1}{h + h_a}; \qquad \operatorname{tg} r_2 = \frac{d_2}{h + h_a} \tag{6a, 6b}$$

$$\cos\theta_1 = \frac{s}{d_1}; \qquad \cos\theta_2 = \frac{t}{d_2} \tag{7a, 7b}$$

Then:

$$\operatorname{tg} r_1 \cos\theta_1 + \operatorname{tg} r_2 \cos\theta_2 = \frac{s}{h + h_a} + \frac{t}{h + h_a} = \frac{b}{h + h_a} \tag{8}$$

The numerator is thus, for shallow depths, $(h \gg h_a)$ approximately equal to the base/height ratio; however, it is a variable that depends on the magnitude of the apparent depth. The significance of this variable is discussed below.

The denominator of equation (5) is simplified as follows for numerical analysis.

From Snell's Law:

$$\operatorname{tg} i_1 = \frac{\sin r_1}{\sqrt{n^2 - \sin^2 r_1}} \tag{9a}$$

and,

$$\operatorname{tg} i_2 = \frac{\sin r_2}{\sqrt{n^2 - \sin^2 r_2}} \tag{9b}$$

From Figure 1:

$$\sin r_1 = \frac{d_1}{\sqrt{d_1^2 + (h + h_a)^2}} \tag{10}$$

Therefore, substituting and simplifying:

$$\operatorname{tg} i_1 = \frac{d_1}{\sqrt{(n^2 - 1)d_1^2 + (h + h_a)^2 n^2}} \tag{11a}$$

and, by analogy:

$$\operatorname{tg} i_2 = \frac{d_2}{\sqrt{(n^2-1)d_1^2 + (h+h_a)^2 n^2}} \tag{11b}$$

Then,

$$\operatorname{tg} i_1 \cos \theta_1 = \frac{d_1}{\sqrt{(n^2-1)d_1^2 + (h+h_a)^2 n^2}} \times \frac{s}{d_1}$$

$$= \frac{s}{\sqrt{(n^2-1)d_1^2 + (h+h_a)^2 n^2}} \tag{12a}$$

and, by analogy:

$$\operatorname{tg} i_2 \cos \theta_2 = \frac{t}{\sqrt{(n^2-1)d_2^2 + (h+h_a)^2}}, \tag{12b}$$

The denominator of equation (5) is thus:

$$\operatorname{tg} i_1 \cos \theta_1 + \operatorname{tg} i_2 \cos \theta_2 = \frac{s}{\sqrt{(n^2-1)d_1^2 + (h+h_a)^2 n^2}} + \frac{t}{\sqrt{(n^2-1)d_1^2 + (h+h_a)^2 n^2}} \tag{12c}$$

In its transformed state, equation (5) becomes:

$$F = \frac{\dfrac{b}{h+h_a}}{\dfrac{s}{\sqrt{(n^2-1)d_1^2 + (h+h_a)^2 n^2}} + \dfrac{t}{\sqrt{(n^2-1)d_2^2 + (h+h_a)^2 n^2}}} \tag{13}$$

In this form, the basic equation (5) is amenable to numerical analysis. It is clear that the correction factor $F$, by which an apparent depth $h_a$ must be multiplied in order to obtain the true depth $h'$, is dependent on the photographic base length $b$, the flight altitude $h$, the index of refraction $n$, the location $(X, Y)$ of the object $D$ (expressed as $d_1$, $d_2$, $s$ and $t$) as well as the apparent depth $h_a$, itself. The correction factor $F$, which may be termed the *effective index of refraction* is small for a small base/height ratio. In the following we have calculated values for the effective index of refraction using a 70 per cent forward overlap, which corresponds to a base/height ratio of .45. A greater amount of forward overlap might result in an excessive degradation of the stereo model.

## Numerical Examples

Examples have been calculated for the following conditions:

| | |
|---|---|
| Flight altitude | h = 2,500′ |
| Photographic lens | $f$ = 6″ |
| Forward overlap | 70% |
| Negative size | 9″×9″ |
| Negative scale | 1/5,000 |
| Photographic base: | $b$ = 1,126′ |
| Side overlap | 40%, 50% |

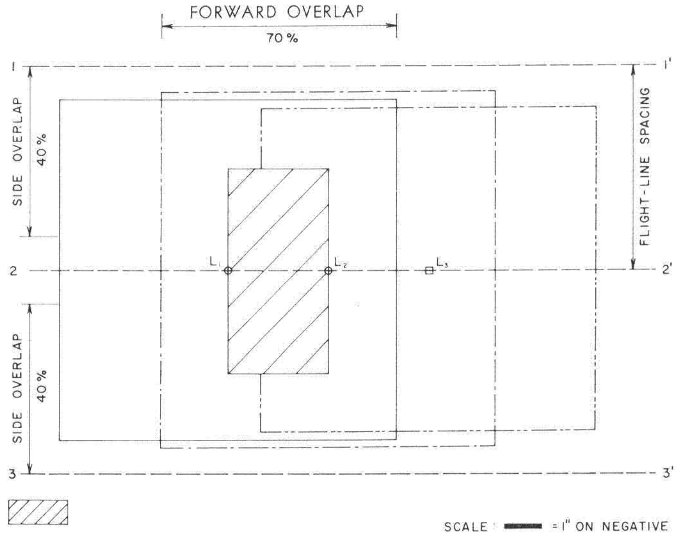


FIG. 2. Illustration of net gain for 70% forward, 40% side overlap.

| | |
|---|---|
| Flight-line interval | 2,250′, 1,875′ |
| Index of refraction | n = 1.35* |

Apparent depths $h_a$: 0, 10′, 25′, 50′, and 100′

Figure 2 shows the overlapping relationship between successive photographs. It is to be noted that the area of net stereo gain per model using successive photographs can also be covered by using alternate photographs; however, this practice would necessitate the calculation of a different set of effective index of refraction figures. In the following, the figures have been calculated for successive photographs only.

Figure 3 shows the location of points for which the effective index of refraction $F$ has been calculated using Equation (13). Because of symmetry, only one-quarter of the model needs to be calculated. The calculated values are shown in Table 1.

## DISCUSSION

Upon inspection of the values shown in Table 1, it is apparent that the influence of the apparent depth $h_a$ on the magnitude of the correction factor $F$ is small. The maximum departure for a single location on the model shown in Figure 3 is in the corners of that model for $d_1 = 1{,}126'$ and $d_2 = 1{,}600'$ (40% side lap). Table 2 shows the distribution of maximum departures when the effective index of refraction is normalized for an apparent depth of 25′.

Assuming that measured apparent depths do not exceed 50′ (true depth of approximately 73′), the maximum error introduced by normalizing all effective indices of refraction to a value calculated for an apparent depth of 25′ is less than ±.16% for the particular model under consideration. For a greater flight elevation $h$, under otherwise similar conditions of forward and side overlap, the influence of errors from this source alone becomes very small.

Figure 4 shows the contoured values of the correction factor $F_{25}$, calculated for the points shown in Figure 3. It is clear that the curves of equal effective index of refraction are elliptically disposed about the center of the model. Table 3 shows the

* This value was used for the present calculations although a value of $n = 1.340$ is probably closer for ocean water. See Utterback, C. L., et al. [2].

TABLE 1

EFFECTIVE INDICES OF REFRACTION $F$ CALCULATED FOR VARIOUS APPARENT DEPTHS $h_a$ AND LOCATIONS ON STEREO MODEL ($d_1$, $d_2$, $s$, $t$)

| *Location on Stereo Model* | | | | *Effective Index of Refraction F Apparent Depth $h_a$* | | | | |
|---|---|---|---|---|---|---|---|---|
| $d_1$ | $d_2$ | $s$ | $t$ | *0′* | *10′* | *25′* | *50′* | *100′* |
| 563′ | 563′ | 563′ | 563′ | 1.3652 | 1.3653 | 1.3651 | 1.3647 | 1.3642 |
| 600 | 600 | 563 | 563 | 1.3676 | 1.3673 | 1.3671 | 1.3667 | 1.3661 |
| 800 | 800 | 563 | 563 | 1.3808 | 1.3804 | 1.3802 | 1.3796 | 1.3785 |
| 1,000 | 1,000 | 563 | 563 | 1.3980 | 1.3976 | 1.3968 | 1.3961 | 1.3942 |
| 1,200 | 1,200 | 563 | 563 | 1.4184 | 1.4179 | 1.4170 | 1.4157 | 1.4135 |
| 1,251 | 1,251 | 563 | 563 | 1.4244 | 1.4235 | 1.4226 | 1.4212 | 1.4188 |
| 0 | 1,126 | 0 | 1,126 | 1.4104 | 1.4100 | 1.4091 | 1.4082 | 1.4062 |
| 400 | 1,200 | 0 | 1,126 | 1.4181 | 1.4179 | 1.4170 | 1.4157 | 1.4135 |
| 800 | 1,400 | 0 | 1,126 | 1.4424 | 1.4418 | 1.4408 | 1.4388 | 1.4358 |
| 1,126 | 1,600 | 0 | 1,126 | 1.4696 | 1.4685 | 1.4673 | 1.4651 | 1.4608 |
| 200 | 1,000 | 144 | 982 | 1.3920 | 1.3916 | 1.3908 | 1.3904 | 1.3887 |
| 400 | 1,000 | 192 | 934 | 1.3909 | 1.3906 | 1.3899 | 1.3895 | 1.3879 |
| 400 | 800 | 346 | 780 | 1.3736 | 1.3734 | 1.3731 | 1.3729 | 1.3720 |
| 600 | 1,200 | 83 | 1,043 | 1.4145 | 1.4141 | 1.4131 | 1.4120 | 1.4099 |
| 600 | 1,000 | 284 | 842 | 1.3903 | 1.3898 | 1.3891 | 1.3886 | 1.3872 |
| 600 | 800 | 438 | 688 | 1.3756 | 1.3753 | 1.3750 | 1.3746 | 1.3738 |
| 800 | 1,200 | 213 | 913 | 1.4112 | 1.4107 | 1.4098 | 1.4088 | 1.4068 |
| 800 | 1,000 | 409 | 717 | 1.3917 | 1.3913 | 1.3906 | 1.3902 | 1.3885 |
| 1,000 | 1,400 | 142 | 984 | 1.4366 | 1.4361 | 1.4350 | 1.4334 | 1.4305 |
| 1,000 | 1,200 | 367 | 759 | 1.4117 | 1.4112 | 1.4103 | 1.4093 | 1.4072 |
| 326 | 800 | 326 | 800 | 1.3730 | 1.3730 | 1.3728 | 1.3725 | 1.3716 |
| 1,159 | 1,400 | 300 | 826 | 1.4348 | 1.4342 | 1.4331 | 1.4315 | 1.4287 |

distribution of the departures which these contours represent with respect to a median value (1.410) for an apparent depth of 25′.

For certain purposes, such a single value might be used, providing suitable forward and side overlap conditions are chosen. Such a value might be used if the outlines of the stereo models used for the compilation of the contour maps are shown and a correction plate showing the departures, accompanies the map.

Where alternate photographs are used for the construction of a stereo model, in order to avoid specular interference or to inciease stereo perception, the range of the departures becomes larger. However, they may be calculated, for any given set of conditions, with equal facility.

For precise topographic submarine bottom contouring, the most accurate procedure appears to be to plot apparent depths, apply the correct correction factor $F$ to obtain the true depth and contour the resulting values using apparent depth contours as a guide. While such an operation may appear laborious, the use of a digital

TABLE 2

MAXIMUM DEPARTURES FROM TRUE DEPTH, ($\Delta$) WITH NORMALIZATION OF $F$ to $F$ AT 25 FEET APPARENT DEPTH

| | $F$ | $h'$ | $F_{25}\cdot h_a$ | $\Delta$(*feet*) | $\Delta$(%) |
|---|---|---|---|---|---|
| $F_0$ | 1.4696 | $1.4696\cdot h_a$ | $1.4673\cdot h_a$ | $-.0023\cdot h_a$ | −.16 |
| $F_{10}$ | 1.4685 | 14.685 | 14.673 | −.012 | −.08 |
| $F_{25}$ | 1.4673 | 36.68 | 36.68 | .000 | .00 |
| $F_{50}$ | 1.4651 | 73.255 | 73.365 | +.110 | +.15 |
| $F_{100}$ | 1.4608 | 146.08 | 146.73 | +.65 | +.44 |

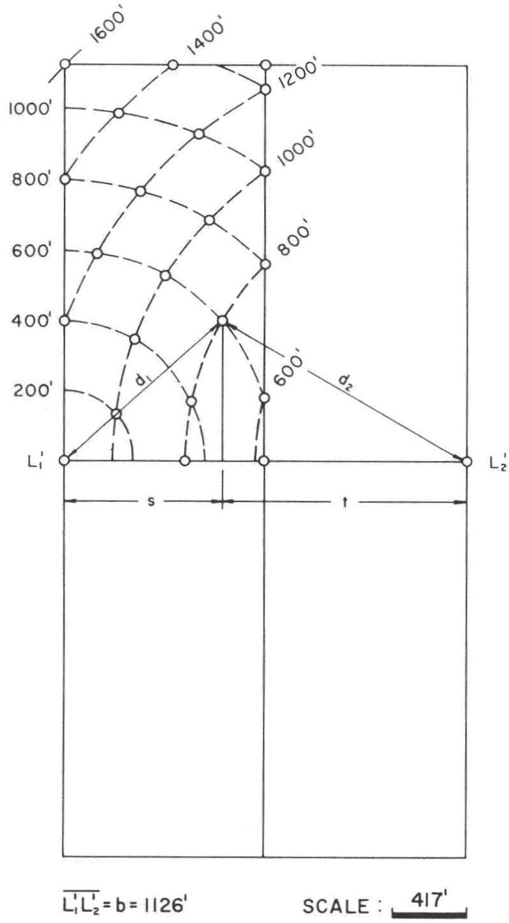


FIG. 3. Location of points on stereo model for which effective index of refraction $F$ is calculated.

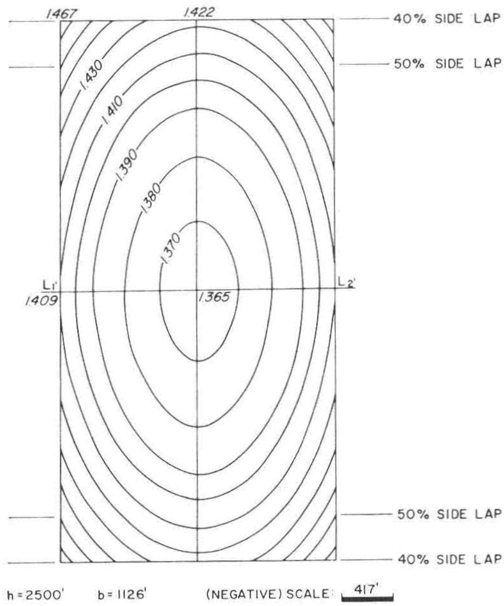


FIG. 4. Variation of effective index of refraction $F_{25}$ over stereo model.

computer can materially reduce the manual effort which such a procedure occasions.

The index of refraction for water is dependent on temperature and dissolved constituents (salinity or chlorinity). Utterback, et. al. [2] have measured variations in the index of refraction of ocean water with respect to temperatures and chlorinity. Their values were obtained using sodium $D$ light; selected values are given in Table 4.

Inspection of the values in Table 4 shows that the range of indices of refraction is small for major variations in chlorinity and temperature. The index of refraction decreases with increasing temperature and increases with increasing chlorinity. The maximum departures are between pure water at 0°C. and ocean water with a chlorinity of 21.381°/₀₀ at 0°C., the indices of refraction are respectively: 1.33402 and 1.34158. Calculated values of $F_{25}$, using equation (13) for the most unfavorable location on the stereo model ($d_1=1,126'$, $d_2=1,600'$, $s=0$, $t=1,126'$) are: 1.4466 (0°C., 0°/₀₀ Cl) and 1.4564 (0°C., 21.378°/₀₀ Cl). The maximum possible error, introduced by using values derived for pure water in lieu of values derived for ocean water, for the particular case under consideration, is calculated to be .7%. Minor variations in the chlorinity of the sea water may be disregarded for photogrammetric purposes; the

TABLE 3

DISTRIBUTION OF DEPARTURES OVER AREA OF STEREO MODEL

| *Contour* | *h'(feet)* | $\Delta$*(feet)* | $\Delta$(%) |
|---|---|---|---|
| 1.467 | 36.68 | 1.43 | 4.1 |
| 1.460 | 36.50 | 1.25 | 3.5 |
| 1.450 | 36.25 | 1.00 | 2.8 |
| 1.440 | 36.00 | 0.75 | 2.1 |
| 1.430 | 35.75 | 0.50 | 1.4 |
| 1.420 | 35.50 | 0.25 | 0.7 |
| 1.410 | 35.25 | 0.00 | 0.0 |
| 1.400 | 35.00 | −0.25 | −0.7 |
| 1.390 | 34.75 | −0.50 | −1.4 |
| 1.380 | 34.50 | −0.75 | −2.1 |
| 1.370 | 34.25 | −1.00 | −2.8 |
| 1.365 | 34.13 | −1.22 | −3.5 |

Systematic departures of up to ±4% may be incurred if a single, median value for the effect. index of refraction is used for the whole stereo model.

TABLE 4

SELECTED INDICES OF REFRACTION OF PURE WATER AND OCEAN WATER

| | *0°C.* | *15°C.* | *25°C.* |
|---|---|---|---|
| pure water | 1.33402 | 1.33340 | 1.33250 |
| 1.477% Cl | 1.33453 | 1.33388 | 1.33299 |
| 10.476% Cl | 1.33774 | 1.33692 | 1.33595 |
| 19.227% Cl* | 1.34082 | 1.33985 | 1.33881 |
| 21.381% | 1.34158 | 1.34055 | 1.33949 |

* Average ocean water.

most suitable value for calculations is an index of refraction $n=1.340$ for ocean water, and $n=1.333$ for inland water.

For special applications, e.g. salt lakes, it would appear advisable to obtain samples of the water in order that the index of refraction may be determined either experimentally or calculated from the Gladstone and Dale relationship upon chemical analysis of the dissolved constituents.

## COMPUTER CALCULATION OF EFFECTIVE INDEX OF REFRACTION

Equation (13) may, upon further simplification, be computed using a digital computer.

If the origin of a horizontal coordinate system is placed at $L_1'$ and $x$ is positive in the direction of $\overline{L_1'L_2'}$, and $y$ is positive in the direction normal to $\overline{L_1'L_2'}$ (see Figure 3), the following substitutions may be made in Equation (13):

$$s = x \tag{14a}$$

$$t = b - x \tag{14b}$$

$$d_1 = \sqrt{x^2 + y^2} \tag{14c}$$

$$d_2 = \sqrt{(b - x)^2 + y^2} \tag{14d}$$

The resulting equation may be programmed and tables run off for given flight altitudes $h$, base lengths $b$, refractive indices $n$, apparent depths $h_a$, and variable $x$, $y$ positions. With this information a coordinatograph* ($X$, $Y$ Plotter) can be used, to contour the output of the computer in a form similar to Figure 4.

## CONCLUSIONS

The following conclusions have been arrived at with the assumption of a horizontal, flat water/air interface.

1. The basic equation derived by Tewinkel [1] has been transformed into equations that may be used for obtaining numerical solutions using either a desk calculator or a digital computer.

* Such as the Aero/Digitork Automated Coordinatograph.

2. The correction factor by which a stereoscopically determined apparent depth must be multiplied in order to obtain the true depth or the effective index of refraction, is, for a given set of external conditions (flight elevation and photographic base length), dependent on the index of refraction at the water/air interface, the location of the point on the stereo model and the apparent depth itself.
3. There is no displacement of the submerged feature in the horizontal, $X$, $Y$ plane.
4. Within the range of 0–50′ of apparent depth, application of an average effective index of refraction at a given location on the stereo model results in errors which normally may be neglected if flight elevation and amounts of forward and side overlap are judiciously chosen.
5. For given conditions of flight elevation and photographic base-length (forward-overlap), equal values for the effective index of refraction are regularly disposed in an elliptical manner about the geometric center of the stereo model.
6. The range of values for the effective index of refraction may, for certain types of mapping, be averaged for use over the whole stereo model, if the external conditions of flight altitude and photographic base length are judiciously chosen. (Under such conditions, maximum errors in the calculated true depths may be less than $\pm 4\%$ of the actual depths.)
7. For precise work, or where alternate photographs are used for construction and measurement of a stereo model, or where normal over-land conditions of forward and side overlap exist (60% and 30% respectively), true depths need to be individually calculated. Thereafter, they may be contoured using a plan of apparent depth contours as a guide.
8. Ambient variations in temperature and dissolved solids content may be tolerated without jeopardizing the accuracy of true depths if these are calculated using an average index of refraction of 1.333 for sweet water and 1.340 for ocean water.
9. It is evident that extensive use can be made of a digital computer and an $X$, $Y$ Plotter (coordinatograph) in the preparation of submarine contour maps.

## Acknowledgments

Mr. E. Nowicki and Dr. W. Barth, both of Aero Service Corporation, first demonstrated to the author the feasibility of submarine photogrammetry using aerial photography. Their statement of the problems and stimulating interest is gratefully acknowledged. The author is also indebted to Mr. G. C. Tewinkel, research photogrammetrist, the United States Coast and Geodetic Survey, for an advance copy of his paper [1] which in part forms the basis for this contribution. These gentlemen are herewith thanked also for reviewing the manuscript and making pertinent suggestions. (The author did not have an opportunity to comment on the remarks of Van Wijk on page 647 in the July issue.—*Editor*.)

## References

[1] Tewinkel, G. C., "Water Depths from Aerial Photographs," Photogrammetric Engineering, v. XXIX, v. 6, pp. 1037–1042, 1963.
[2] Utterback, C. L., Thompson, T. G. and Thomas, B. D., "Refractivity—Chlorinity—Temperature Relationships of Ocean Waters," *Conseil Permanent International pour l'Exploration de la Mer Journal*, v. 9, pp. 35–38, 1934.

# ERRATA

1964 Yearbook, Page 373, Top of Right Hand Column.

Change Dr. Samuel W. Leving to Dr. Daniel Levine, the winner of the Talbert Abrams Award.