PROF. DR.-ING. HORST JOCHMANN
Technical University
Dresden, Germany

Number of Orientation Points

The number and distribution of points used in analytic relative orientation affect the accuracy of the elements of orientation.

THE FUNDAMENTAL TASK of aerotriangulation is the accurate extension of a control through successive images in a given strip of stereomodels. In order to achieve a high degree of accuracy in this aerotriangulation procedure, it is essential that the orientation errors contributed by each model in the strip be kept to an absolute minimum. The application of analytical methods provides an efficient means for increasing the number of observation points at which y-parallaxes are measured in each model. This is particularly true, if an automatic observation method is used in this procedure. From a theoretical standpoint a limited degree of improved accuracy can be achieved by increasing the number of y-parallax readings at each of five or six characteristic points distributed throughout the model. This accuracy limitation is caused by the influence of both image errors and observation errors.

In order to reduce all errors to an absolute minimum it is necessary to increase the number of observation points within the model. The number and distribution of these points will influence the elements of position and angular orientation. The subject of the following analysis will be the determination of the influence of these phenomena on the accuracy of the elements of orientation.

By a symmetrical distribution of the points it is possible to simplify the computations as derived in [2]. In the following considerations this possibility shall be used to obtain simple formulas for the mean square error of the elements of orientation.

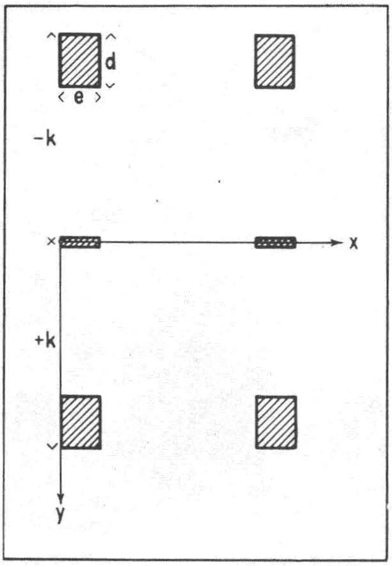

FIG. 1

1. MATHEMATICAL FOUNDATIONS

The foundation of the following derivations shall be the parallax-equation in the form

$$p_y = db_y - \frac{y}{z}\,db_z + (x - x_d)d\kappa + \frac{y(x - x_d)}{z}\,d\phi + \left((z - z_d) + \frac{y(y - y_d)}{z}\right)d\omega \quad (1)$$

derived in [2].

In this equation are the x, y, z space coordinates of the observed points. The origin of the coordinate-system is the projection-center of the left image (Figure 1). The z-axis is directed towards the nadir-point. x_d, y_d and z_d are the coordinates of the rotation-center of ϕ, ω and κ. The location of the rotation-center shall be determined by means of the following derivations.

The angular elements of orientation calculated by Formula (1) are the same as obtained in the usual way. Only the position elements db_y and db_z differ from corresponding elements, obtained in the usual way, for the projection-center of the right image does not coincide with the rotation-center. The relations between the elements db_y' and db_z', calculated by an equation with

$$x_d = b$$

$$y_d = z_d = 0$$

and the elements db_y, db_z, obtained by (1) are

$$\begin{aligned} db_y' &= db_y + x_d d\kappa - z_d d\omega \\ db_z' &= db_z - x_d d\phi. \end{aligned} \tag{2}$$

To obtain simple expressions it is advantageous to choose a symmetrical distribution of points for parallax measurements. There is also the possibility of simplifying these expressions by choosing a suitable location of the rotation center.

ABSTRACT: *Analytical solutions of aerotriangulation problems allow the economic possibility to increase the number of parallax-observation points on more than the conventional minimum of six points. This is especially true, if automatic observation-methods are used. The accuracy of the orientation-elements will depend on the distribution and number of points at which the y-parallaxes are measured. It is mathematically demonstrated that by a suitable choice of parallax-equations the matrix of the normal-equations become a diagonal-matrix, which simplifies the calculation of weight numbers.*

As shown in Figure 1 four areas are chosen, in which the points for parallax observations are equally distributed. Further points are distributed along the base line. The spacing of these points is the same as for other points set in an x-direction within each given area. The distances of the points in the x direction are e' and in the y direction are d'. In a single area the number of points is

$$n = \left(\frac{d}{d'} + 1\right)\left(\frac{e}{e'} + 1\right)$$

and the total number in the model is

$$N = 4n + 2\left(\frac{e}{e'} + 1\right).$$

When the number of points is larger than five, the elements of orientation will be obtained by adjustment. Formula (1) is the observation equation. It can be written in the form

$$v = -p_y + a_1 db_y + a_z db_z + a_3 d\kappa + a_4 d\phi + a_5 d\omega \tag{3}$$

in which the coefficients a_i are equal to corresponding coefficients in (1). By the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{N,1} & a_{N,2} & a_{N,3} & a_{N,4} & a_{N,5} \end{pmatrix}$$

and the vectors

$$x = \begin{pmatrix} db_y \\ db_z \\ d\kappa \\ d\phi \\ d\omega \end{pmatrix} \qquad P = \begin{pmatrix} p_{y1} \\ p_{y2} \\ \cdot \\ \cdot \\ \cdot \\ p_{yN} \end{pmatrix}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{pmatrix}.$$

One obtains the following system of observation equations

$$Ax - P = v \tag{4}$$

from which the normal-equations can be obtained by an adjustment procedure. The normal equations are:

$$A^T A x - A^T P = 0 \tag{5}$$

and

$$A^T A = \begin{pmatrix} \sum a_1a_1 & \sum a_1a_2 & \sum a_1a_3 & \sum a_1a_4 & \sum a_1a_5 \\ \sum a_1a_2 & \sum a_2a_2 & \sum a_2a_3 & \sum a_2a_4 & \sum a_2a_5 \\ \sum a_1a_3 & \sum a_2a_3 & \sum a_3a_3 & \sum a_3a_4 & \sum a_3a_5 \\ \sum a_1a_4 & \sum a_2a_4 & \sum a_3a_4 & \sum a_4a_4 & \sum a_4a_5 \\ \sum a_1a_5 & \sum a_2a_5 & \sum a_3a_5 & \sum a_4a_5 & \sum a_5a_5 \end{pmatrix}. \tag{6}$$

From (5) the unknown elements of orientation

$$x = (A^T A)^{-1} A^T P \tag{7}$$

are obtained.

From (6) we obtain

$$Q = (A^T A)^{-1} \tag{8}$$

the matrix of the weight and correlation numbers. All elements in the diagonal of this matrix are weight numbers and one obtains the mean square errors of the unknowns by

$$m_i = \sqrt{Q_{ii}} \cdot m_0.$$

It is necessary to calculate the matrix Q for determining the accuracy of orientation. One obtains the matrix Q by inversion of the matrix $A^T A$. This operation is made easier, if $\boldsymbol{A}^T\boldsymbol{A}$ is a diagonal-matrix. The suitable choice of the rotation-center provides the establishment of a diagonal-matrix.

All observation equations have the coefficient $a_1=1$ and all coefficients of the normal equations, which include a_1, can be written as follows:

$$\begin{aligned}
\sum a_1 &= N \\
\sum a_1a_2 &= \sum a_2 \\
\sum a_1a_3 &= \sum a_3 \\
\sum a_1a_4 &= \sum a_4 \\
\sum a_1a_5 &= \sum a_5.
\end{aligned}$$

These coefficients must be zero if one wants to obtain a diagonal matrix.

The symmetrical distribution of the observed points requires that:

$$\sum a_2 = 0.$$

Then the term a_3 is obtained by

$$\begin{aligned}
\sum a_3 = \sum (x - x_d) &= 2\left\{\left(\frac{d}{d'}+1\right)+1\right\}\left(1+2+3+\cdots\left(\frac{e}{e'}+1\right)\right)e' \\
&+ \left\{2\left(\frac{d}{d'}+1\right)+1\right\}\left(b-e'+b-2e'+\cdots b-\left(\frac{e}{e'}+1\right)e'\right) \\
&- 4\left\{\left(\frac{d}{d'}+1\right)\left(\frac{e}{e'}+1\right)+2\left(\frac{e}{e'}+1\right)\right\}x_d = 0.
\end{aligned}$$

The following expression can be written by the application of well known sum-formulas:

$$\begin{aligned}
\sum a_3 = &\left\{2\left(\frac{d}{d'}+1\right)\left(\frac{e}{e'}+1\right)+\left(\frac{e}{e'}+1\right)\right\} \\
&b - 2\left\{2\left(\frac{d}{d'}+1\right)\left(\frac{e}{e'}+1\right)+\left(\frac{e}{e'}+1\right)\right\}x_d.
\end{aligned}$$

From the condition $a_3=0$, it follows that:

$$x_d = \frac{b}{2}\cdot \tag{9}$$

By the symmetrical distribution of the points the condition

$$\sum a_1a_4 = \frac{\sum a_2 \sum a_3}{N} \tag{10}$$

is caused. This means that from (9) follows also

$$\sum a_1a_4 = \sum a_4 = 0.$$

The coordinate z_d is obtained from the condition

$$\sum a_1a_5 = \sum a_5 = 0.$$

If $y_d = 0$ is chosen, we obtain from this condition by a similar derivation as for x_d:

$$z_d = z + \frac{2nk(k-d) + \frac{1}{3}ndd'\left(2\frac{d}{d'} + 1\right)}{\left(2n + \frac{e}{e'} + 1\right)z}. \tag{11}$$

The choice of x_d and z_d by the Formulas (9) and (11) provides that all elements of the matrix (5), besides the diagonal-elements, become zero. The condition that the elements of the first row of the matrix shall become zero, sets the condition that all other elements, which are not diagonal-elements, become zero. The symmetrical distribution of the observation points requires that all elements of the matrix are established by condition (10).

By considering the previously derived coordinates of the rotation-center, one obtains the following diagonal-elements of the matrix:

$$\begin{aligned}
\sum a_1a_1 &= N \\
\sum a_2a_2 &= \frac{4}{z^2}\, n(k(k-d) + \frac{d}{6}(2d + d')) \\
\sum a_3a_3 &= \frac{N}{2}\left(\frac{e}{3}(2e - e') - be + \frac{b^2}{2}\right) \\
\sum a_4a_4 &= \frac{2}{z^2}\, n(k(k-d) + \frac{d}{6}(2d + d'))\left(\frac{e}{3}(2e + e') - be + \frac{b^2}{2}\right) \\
\sum a_5a_5 &= \frac{4n}{z^2}(k^4 - 2dk^3 + dk^2(2d + d') - d^2k(d + d') \\
&\quad + \frac{1}{30}d(2d + d')(3d^2 + 3dd' - d'^2) \\
&\quad - \frac{n}{N}(2k(k-d) + \frac{1}{3}d(2d + d')^2).
\end{aligned} \tag{12}$$

These expressions allow the computation of the elements of orientation by means of (6). With formula (7) the following weight numbers are obtained

$$\begin{aligned}
Q_{b_yb_y} &= \frac{1}{\sum a_1a_1} \\
Q_{b_zb_z} &= \frac{1}{\sum a_2a_2} \\
Q_{\kappa\kappa} &= \frac{1}{\sum a_3a_3} \\
Q_{\phi\phi} &= \frac{1}{\sum a_4a_4} \\
Q_{\omega\omega} &= \frac{1}{\sum a_5a_5}.
\end{aligned} \tag{13}$$

These expressions can then be used to analyse the accuracy of the procedure.

2. Computation of Weight Numbers

The following is a computation of weight numbers for several configurations of y-parallax observation points within a single model.

The subject of the following considerations shall be the determination of the influence of the number and distribution of y-parallax observation points on the mean square error of orientation-elements. This is obtained by computation of weight numbers for several selected numbers of parallax observation points.

2.1. Six points

For this case it is assumed that $n=1$, $d'=d=0$, and $e'=e=0$. By application of these values to (12) one obtains

$$\begin{aligned} \sum a_1a_1 &= 6 \\ \sum a_2a_2 &= 4\frac{k^2}{z^2} \\ \sum a_3a_3 &= \frac{3}{2}b^2 \\ \sum a_4a_4 &= \frac{k^2b^2}{z^2} \\ \sum a_5a_5 &= \frac{4}{3}\frac{k^4}{z^2} \\ z_d &= z + \frac{2}{3}\frac{k^2}{z}. \end{aligned} \tag{14}$$

2.2. Twelve points

The formulas for this case are established by setting $d=d'=0$, $e'=e$, $n=2$:

$$\begin{aligned} \sum a_1a_1 &= 12 \\ \sum a_2a_2 &= 8\frac{k^2}{z^2} \\ \sum a_3a_3 &= 6\left(\frac{b^2}{2} - e(b-e)\right) \\ \sum a_4a_4 &= 4\frac{k^2}{z^2}\left(\frac{b^2}{2} - e(b-e)\right) \\ \sum a_5a_5 &= \frac{8}{3}\frac{k^4}{z^2} \\ z_d &= z + \frac{2}{3}\frac{k^2}{z}. \end{aligned} \tag{15}$$

2.3. 20 POINTS

Here are set $d=d'$, $e=e'$, $n=4$:

$$\sum a_1a_1 = 20$$

$$\sum a_2a_2 = \frac{16}{z^2}\left(k(k-d) + \frac{d^2}{2}\right)$$

$$\sum a_3a_3 = 10\left(\frac{b^2}{2} - e(b-e)\right)$$

$$\sum a_4a_4 = \frac{8}{z^2}\left(k(k-d) + \frac{d^2}{2}\right)\left(\frac{b^2}{2} - e(b-e)\right) \tag{16}$$

$$\sum a_5a_5 = \frac{8}{5z^2}\left\{k^2(3k^2 - 2(k-d)^2) + (k-d)^2(3(k-d)^2 - 2k^2)\right\}$$

$$z_d = z + \frac{1}{5z}\left\{4k(k-d) + 2d^2\right\}.$$

2.4. 42 POINTS

$$d' = \frac{d}{z}; \qquad e' = \frac{e}{z}; \qquad n = 9.$$

$$\sum a_1a_1 = 42$$

$$\sum a_2a_2 = \frac{36}{z^2}\left\{k(k-d) + \frac{5}{12}d^2\right\}$$

$$\sum a_3a_3 = 21\left\{\frac{b^2}{2} - e\left(b - \frac{5}{6}e\right)\right\}$$

$$\sum a_4a_4 = \frac{18}{z^2}\left(k(k-d) + \frac{5}{12}d^2\right)\left(\frac{b^2}{2} - e\left(b - \frac{5}{6}e\right)\right) \tag{17}$$

$$\sum a_5a_5 = \frac{36}{z^2}\left(\frac{k^4}{7} + \frac{26}{7}k^3d + \frac{13}{14}k^2d^2 - \frac{11}{14}kd^2 = \frac{25}{148}d^4\right)$$

$$z_d = z + \frac{1}{7z}\left(6k(k+d) + \frac{5}{2}d^2\right).$$

2.5. 110 POINTS

$$d' = \frac{d}{4}; \qquad e' = \frac{e}{4}; \qquad n = 25.$$

$$\begin{aligned}
\sum a_1a_1 &= 110 \\
\sum a_2a_2 &= \frac{100}{z^2}\left(k(k-d) + \frac{3}{8}\,d^2\right) \\
\sum a_3a_3 &= 55\left(\frac{b^2}{2} - e\left(b - \frac{3}{4}\,e\right)\right) \\
\sum a_4a_4 &= \frac{50}{z^2}\left(k(k-d) + \frac{3}{8}\,d^2\right)\left(\frac{b^2}{2} - e\left(b - \frac{3}{4}\,e\right)\right) \\
\sum a_5a_5 &= \frac{100}{z^2}\,(0{,}091\,k^4 - 0{,}182\,k^3d + 0{,}887\,k^2d^2 - 0{,}910\,d^3k + 0{,}144\,d^4)
\end{aligned} \tag{18}$$

$$z_d = z + \frac{1}{11z}\left(10k(k-d) + \frac{15}{4}\,d^2\right).$$

2.6. $N \to \infty$ POINTS

$$d' \approx e' \approx 0$$

$$\begin{aligned}
\sum a_1a_1 &= N \\
\sum a_2a_2 &= \frac{4n}{z^2}\left(k(k-d) + \frac{d^2}{3}\right) \\
\sum a_3a_3 &= \frac{N}{z}\left(\frac{b^2}{2} - e\left(b - \frac{2}{3}\,e^2\right)\right) \\
\sum a_4a_4 &= \frac{2n}{z^2}\left(k(k-d) + \frac{d^2}{3}\right)\left(\frac{b^2}{2} - e\left(b - \frac{2}{3}\,e^2\right)\right) \\
\sum a_5a_5 &= \frac{4n}{z^2}\left(k^4 - 2dk^3 + 2d^2k^2 - d^3k + \frac{1}{5}\,d^4 - \frac{n}{N}\left(2k(k-d) + \frac{2}{3}\,d^2\right)^2\right)
\end{aligned} \tag{19}$$

$$z_d = z + \frac{2nk(k-d)}{(2n + \sqrt{n})z}.$$

These terms (19) are for point numbers larger than 100. The term z_d is an approximate formula, which is only exact for the case $d = e$.

2.7. THE NUMERICAL COMPUTATION OF WEIGHT NUMBERS

By the previous derived formulas it is possible to compute the weight numbers of the orientation-elements. In order to determine how the accuracy is influenced by the distribution of parallax observation points, two extreme cases shall be computed for each of the previous derived formulas (16) to (19). These two cases are:

1. The areas of the observed points are reduced to a small size, such that all observed points have nearly the same location (Figure 1).
2. The observed points cover the model area in equal spacing.

This way of consideration is a simplification, which is only allowed if one can assume that no other distribution exists which leads to an extreme value of the weight numbers. For testing this assumption several distributions were computed for the case $N=20$. The results are shown in a diagram (Figure 2). One sees that only the weight numbers $Q_{\omega\omega}$ have an extreme value between the presumed two cases.

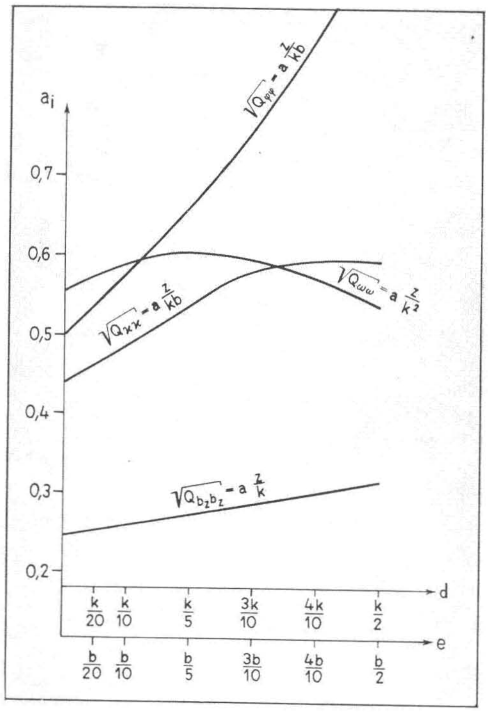

FIG. 2

It is seen, that $Q_{\omega\omega}$ has only a small variation resulting from the distribution of points. Therefore it is possible to neglect this extreme value for the following considerations.

The first mentioned case is obtained if in the Formulas (16) to (19) it is assumed that $d'=e'=0$.

For the second case one obtains the weight numbers if the following assumptions are made:

$$N = 12 \qquad d = 0 \qquad e = \frac{b}{3}$$

$$N = 20 \qquad d = \frac{k}{2} \qquad e = \frac{b}{3}$$

$$N = 42 \qquad d = \frac{2}{3}k \qquad e = \frac{2}{5}b$$

$$N = 110 \qquad d = \frac{3}{4}k \qquad e = \frac{4}{9}b$$

$$N \rightarrow \infty \qquad d = k \qquad e = \frac{b}{2}\,.$$

The radicals of the weight numbers are written in the following form:

$$
\begin{aligned}
Q_{b_3b_3} &= a_{b_3} \times \frac{z}{k} \\
Q_{\kappa\kappa} &= a_{\kappa} \times \frac{1}{b} \\
Q_{\phi\phi} &= a_{\phi} \times \frac{z}{kb} \\
Q_{\omega\omega} &= a_{\omega} \times \frac{z}{k^2}.
\end{aligned} \tag{20}
$$

In these formulas the coefficients a_i ($i=b_z$, κ, ϕ, ω) are terms which are variable by distribution and numbers of parallax-observation-points. The coefficients a_i have been computed and the results are listed in Table 1. Since the weight number of b_y varies only by the point number it was not listed in Table 1.

In Table 1 the sign of the results for the first case are determined by A, while the coefficients for the second case are signed by M.

TABLE 1

N	a_{b_z}		a_κ		a_ϕ		a_ω		n
	A	M	A	M	A	M	A	M	
6	0.50		0.82		1.00		0.87		1
12	0.35	0.35	0.58	0.78	0.71	0.95	0.61	0.61	2
20	0.25	0.32	0.44	0.60	0.50	1.12	0.56	0.54	4
42	0.17	0.23	0.31	0.45	0.33	0.68	0.45	0.40	9
110	0.10	0.15	0.19	0.30	0.20	0.46	0.33	0.26	25
N	$\frac{0.50}{\sqrt{N}}$	$\frac{0.80}{\sqrt{N}}$	$\frac{2.00}{\sqrt{N}}$	$\frac{3.46}{\sqrt{N}}$	$\frac{1.41}{\sqrt{n}}$	$\frac{3.00}{\sqrt{n}}$	$\frac{0.50}{\sqrt{n(1-4n/N)}}$	$\frac{1.12}{\sqrt{n(1-2.2n/N)}}$	n

From Table 1 it is evident that the distribution of the points in four separate areas leads to a decrease of the mean square error of the orientation-elements b_z, κ and ϕ, but the mean square error of ω increases at a small rate. This rate may be neglected. The most significant decrease was found for ϕ, with a calculated reduction in the mean square error of about 50%. Since the orientation-element ϕ causes the largest elevation-errors in an aerotriangulation, this theoretical method for decreasing the error of ϕ should be given serious consideration.

Using the conventional technique for y-parallax observations, it is concluded by the results shown in Table 1, that a distribution of twenty points provides an optimal solution for this problem. The solution is also greatly simplified by choosing the position of the rotation center as described earlier. This is quite important if one must use a computer with limited storage capacity in performing the necessary computations.

LITERATURE

[1] Helava, "An Ultimate Solution of Relative Orientation." *Suomen Fotogramm.* Seura 1962, Nr. 3, S. 15.

[2] Jochmann, H., "Eine Möglichkeit zur Genauigkeitssteigerung des Folgebildanschlusses." *Wiss. Zeitschrift der Techn. Universität Dresden* 13 (1964) 2, S. 398.