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Solar Rotation Rate and Axial Orientation

Solar orientation parameters needed in space navigation can be determined using a photogrammetric technique.

Abstract: *Photographic parameters and the photogrammetric data reduction procedures are employed in the determination of the sun's rotational rate and axial orientation. The method is new and wholly photogrammetric insofar as the theory of celestial mechanics is not invoked in the reduction. The following assumptions are made in the data reduction equations of condition: (1) the sun's axis of rotation is fixed directionally in space for periods not exceeding three earth days; (2) the sun spots are significantly fixed in solar latitude for periods not exceeding three earth days during the period of minimum sun spot activity.*

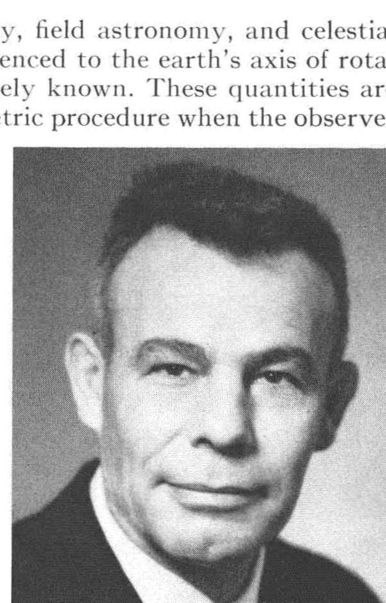
The observation consists of two photographic exposures of the sun's disk separated in time from one to three days with the shutter opening referenced to universal time. The input data consist of: (1) plate coordinates of the sun's disk; (2) plate coordinates of two or more conjugate sun spot images; (3) universal time of shutter opening and closing; (4) semi diameter and astronomic direction of sun for time and date. The output data are: (1) celestial direction of the sun's rotational axis, and (2) the rotational rate of the sun as a function of solar latitude.

Introduction

The fundamental assumptions made in geodesy, field astronomy, and celestial navigation are that the star directions are referenced to the earth's axis of rotation and that the earth's rotational rate is accurately known. These quantities are easily and accurately determined by a photogrammetric procedure when the observer occupies the body for which the determination is made. The problem is complicated with earthbound measures for reasons of weak geometry associated with small image-size and poor surface resolution for all other bodies in the solar system. The problem is further complicated for those solar system bodies not having a fixed rigid crust such as Jupiter and Saturn.

Although the sun will never be occupied or orbited closely by man, the problem of determining axial orientation and rotational rate of a solar system body with a variable surface is similar to a determination on the sun. The remaining solar system planets will eventually be orbited by man.

As preliminary to man's orbiting a solar system body, the fundamental constants of position and orientation must be determined with an unmanned orbiting sensor. A time referenced exposure camera



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offers the most efficient solution. This paper describes a general photogrammetric solution applicable to any solar system body.

The approximate rotational rate of the sun at the equator is once in twenty-five days. The rotational rate of the sun may be easily determined to the first approximation with conjugate images of sunspots referenced to time. The problem is somewhat complicated beyond the first approximation by the fact that the rotational rate varies with solar latitude. However, within this same order of accuracy the variation of the sun's rotational rate with latitude and the orientation of the sun's pole may be determined. Maximum sunspot activity has an eleven year cycle. Sunspot activity is at a minimum at the present time. The number of sunspots and surface motion of the sunspots are at a minimum during the period of minimum sunspot activity. This means an array of sunspots during this period will define nearly fixed positions relative to each other and therefore may be employed as conjugate images on the surface of the sun providing the lapsed time between exposures is not excessive.

Data Reduction (Figure 1)

Assume that two or more exposures are made of the sun at recorded times separated by one or more days. Assume also that fifty or more peripheral images and two or more conjugate sunspot images are measured on the processed negative with a coordinate comparator. Initially the center of each sun disk image is found with equations of the form:

$$a \cdot \Delta x + b \cdot \Delta y + c \cdot \Delta t + d \cdot \Delta s + e \cdot \Delta k = \Delta$$

where

$$a = \frac{(x_p - x_o')}{r'k'} \left[(1 - \sin^2 t' \cos^2 \theta')^{1/2} + \sin t' \cos \theta' \tan \sigma\right]$$

$$b = \frac{(y_p - y_o')}{r'k'} \left[(1 - \sin^2 t' \cos^2 \theta')^{1/2} + \sin t' \cos \theta' \tan \sigma\right]$$

$$c = -\frac{r'}{k'}\left[\frac{\cos t' \sin t' \cos^2 \theta'}{(1 - \sin^2 t' \cos^2 \theta')^{1/2}} - \cos t' \cos \theta' \tan \sigma\right]$$

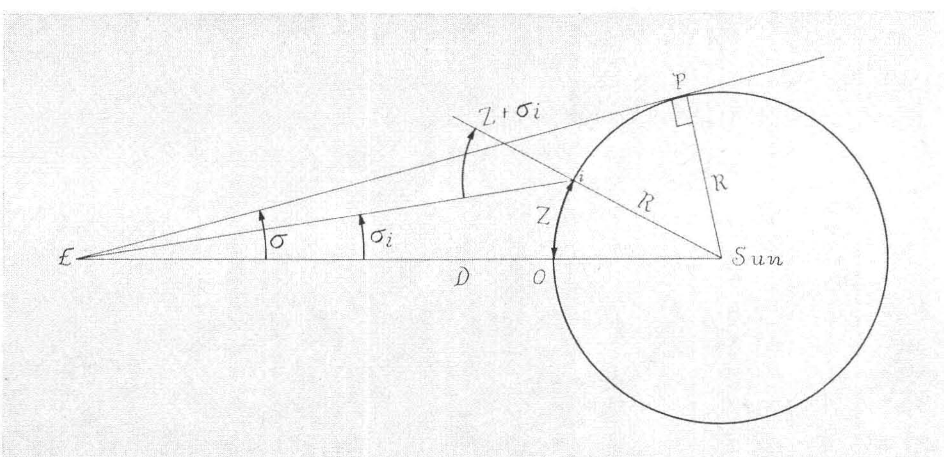

FIG. 1. Relation of σ, D, and R to $\Delta\sigma$ and Z.

$$d = +\frac{r'}{k'}\left[\frac{\sin^2 t' \sin\theta' \cos\theta'}{(1 - \sin^2 t' \cos^2\theta')^{1/2}} - \sin t' \sin\theta' \tan\sigma\right]$$

$$e = -\frac{r'}{k'^2}\left[(1 - \sin^2 t' \cos^2\theta')^{1/2} + \sin t' \cos\theta' \tan\sigma\right]$$

$$\Delta = 1 - \frac{r'}{k'}\left[(1 - \sin^2 t' \cos^2\theta')^{1/2} + \sin t' \cos\theta' \tan\sigma\right]$$

$$r' = \left[(x_p - x_o')^2 + (y_p - y_o')^2\right]^{1/2}$$

$$k' = \frac{r_1' + r_2' + \cdots + r_n'}{n}$$

$$\sin\theta' = \frac{(x_p - x_o')\cos s' - (y_p - y_o')\sin s'}{r'}$$

$$\cos\theta' = \frac{(x_p - x_o')\sin s' + (y_p - y_o')\cos s'}{r'}$$

σ is the sun's semidiameter and is found in the American Ephemeris for the date.

Initial estimates of the coordinates of the sun's disk x_0', y_0' are made with which values of t' and s' may be approximated:

$$\tan t' = \frac{(x_o'^2 + y_o'^2)^{1/2}}{f}$$

$$\tan s' = \frac{x_o'}{y_o'}$$

on the nth iteration

$$x_o = x_o' + \Sigma\Delta x$$
$$y_o = y_o' + \Sigma\Delta y$$
$$s = s' + \Sigma\Delta s$$
$$t = t' + \Sigma\Delta t$$
$$k = k' + \Sigma\Delta k$$

whence for any θ value

$$\tan\theta = \frac{(x_p - x_o)\cos s - (y_p - y_o)\sin s}{(x_p - x_o)\sin s + (y_p - y_o)\cos s}$$

Since k is the constant radius of the rectified sun image

$$\tan\sigma = \frac{k}{L_o}$$

$$L_o = (x_o^2 + y_o^2 + f^2)^{1/2}$$

Any conjugate image i referred to the apparent center is

$$\tan\sigma_i = \frac{\left[(x_i - x_o)^2 + (y_i - y_o)^2\right]^{1/2}\left[(1 - \sin^2 t \cos^2\theta_i)^{1/2} + \sin t \cos\theta_i \tan\sigma_i\right]}{L_o}$$

where

x_i and y_i denote coordinates of any image of a sunspot.

Division of the second by the first gives

$$\tan \sigma_i = \frac{r_i}{k} \left[(1 - \sin^2 t \cos^2 \theta_i)^{1/2} + \sin t \cos \theta_i \tan \sigma_i\right] \tan \sigma$$

or

$$\tan \sigma_i = \frac{(1 - \sin^2 t \cos^2 \theta_i)^{1/2} \tan \sigma r_i}{k - \sin t \cos \theta_i \tan \sigma r_i}$$

The particular approach outlined above is intended to bypass the weak geometry associated with small cone angles. It may be seen from Figure 1 that

$$\sin \sigma = \frac{R}{D}$$

where

R = the radius of the sun
D = the earth-sun distance.

With the law of sines,

$$\sin (Z_i + \sigma_i) = \frac{D}{R} \sin \sigma_i = \frac{\sin \sigma_i}{\sin \sigma},$$

whence

$$Z_i = (Z + \sigma_i) - \sigma_i$$

where Z_i is the angle at the center of the sun between the apparent center of the sun's image o and the sunspot image i. The angle θ_i is in the plane of the film. To be compatible with Z_i it must be in a plane tangent to the sun at o

$$\tan \theta_i'' = \frac{\tan \theta_i}{\cos t}$$

The computed spherical angles Z_i and θ_i'' are illustrated in Figure 2. Z and θ'' correspond to zenith and arbitrary azimuth angles on the sun's surface. The above reduction is performed on at least two exposures having at least two sun spot images. The relation between the two apparent centers is established explicitly with a pair of Z and θ'' angles illustrated in Figure 3.

$$\theta_{1ij}'' = \theta_{i1}'' - \theta_{j1}''$$

$$\theta_{2ij}'' = \theta_{i2}'' - \theta_{j2}''$$

$$\begin{aligned} \cos ij &= \cos Z_{i1} \cos Z_{j1} + \sin Z_{i1} \sin Z_{j1} \cos \theta_{1ij}'' \\ &= \cos Z_{i2} \cos Z_{j2} + \sin Z_{i2} \sin Z_{j2} \cos \theta_{2ij}'' \end{aligned}$$

$$\sin o_{1ij} = \frac{\sin Z_{1j} \sin \theta_{1ij}'}{\sin ij}$$

$$\sin o_{1ji} = \frac{\sin Z_{1j} \sin \theta_{1ij}''}{\sin ij}$$

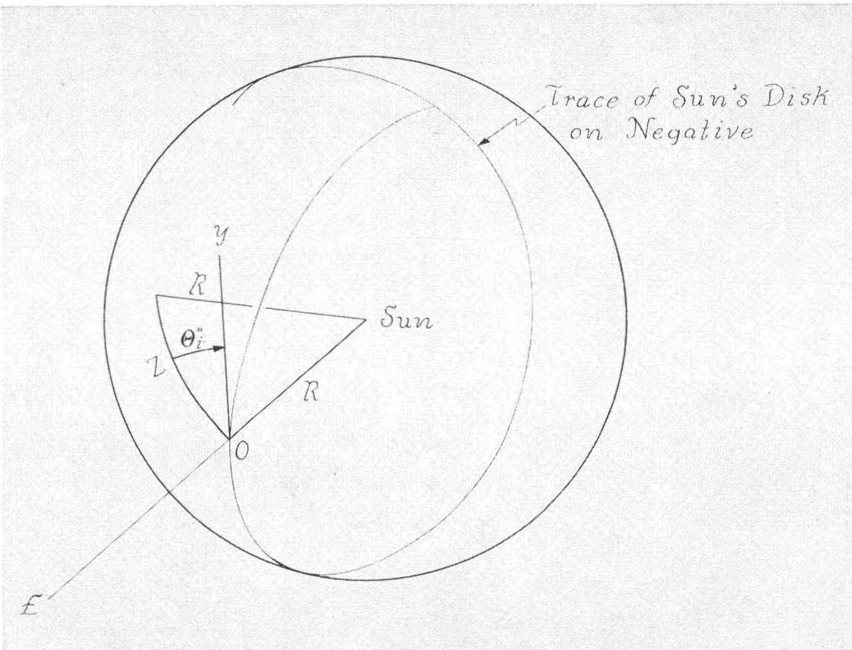

FIG. 2. Z and θ'' Angles.

$$\sin o_{2ij} = \frac{\sin Z_{2j} \sin \theta_{2ij}''}{\sin ij}$$

$$\sin o_{2ij} = \frac{\sin Z_{2i} \sin \theta_{2ij}''}{\sin ij}$$

$$\begin{aligned}\cos o_1 o_2 &= \cos Z_{1i} \cos Z_{2i} + \sin Z_{1i} \sin Z_{2i} \cos (o_{1ij} + o_{2ij}) \\ &= \cos Z_{1j} \cos Z_{2j} + \sin Z_{1j} \sin Z_{2j} \cos (o_{1ji} + o_{2ji})\end{aligned}$$

These preliminary determinations are employed in the fundamental reduction which is the astronomic direction of the sun's rotational axis. The explicit equations given below embrace the geometric basis of the determination

$$\begin{aligned}\cos Z_{1i} \sin \phi_{o1} + \sin Z_{1i} \cos \phi_{o1} \cos (\theta_{1i}'' - Az_{o1i}) &= \sin \phi_i \\ \cos Z_{2i} \sin \phi_{o2} + \sin Z_{2i} \cos \phi_{o2} \cos (\theta_{2i}'' - Az_{o2i}) &= \sin \phi_i \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cos Z_{ni} \sin \phi_{on} + \sin Z_{ni} \cos \phi_{on} \cos (\theta_{ni}'' - Az_{oni}) &= \sin \phi_i\end{aligned} \qquad (1)$$

where ϕ_0 is the solar latitude of the apparent center of the image of the sun's disk.

$$\cos \theta_{1i}'' = \frac{\cos Z_{2i} - \cos Z_{1i} \cos o_1 o_2}{\sin Z_{1i} \sin o_1 o_2}$$

$$\cos Az_{o1} = \frac{\sin \phi_{o2} - \sin \phi_{o1} \cos o_1 o_2}{\cos \phi_{o1} \sin o_1 o_2}$$

These equations are illustrated in Figure 4. The equations assume that for short periods the sun's axis of rotation is fixed in space and the sunspot latitudes are fixed with

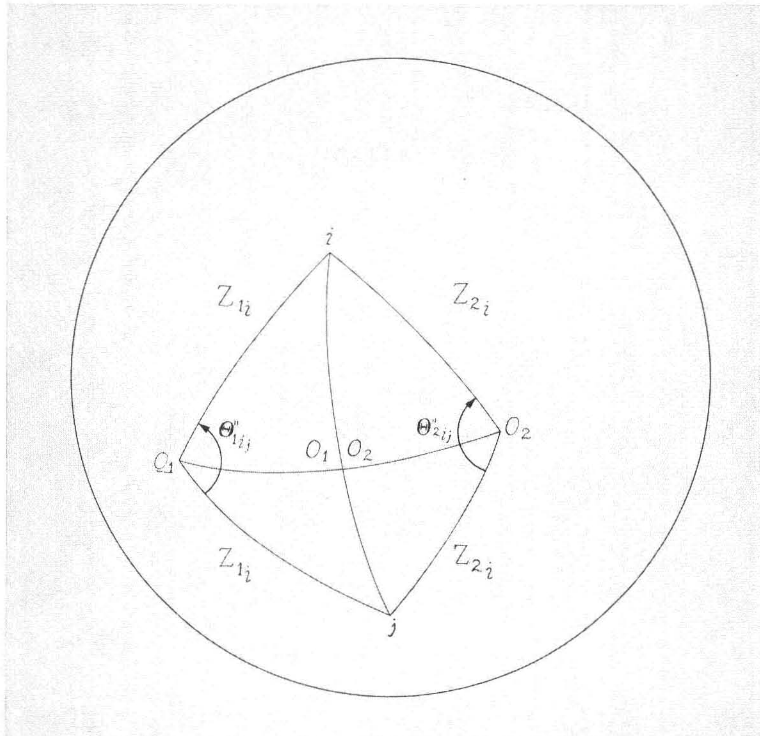

FIG. 3. Relative Orientation of Apparent Centers

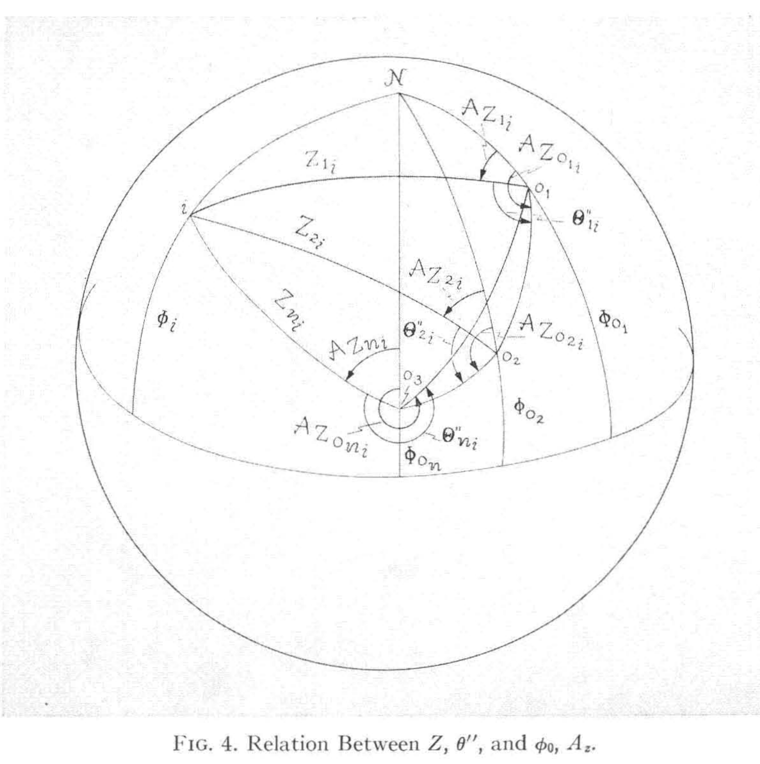

FIG. 4. Relation Between Z, θ'', and ϕ_0, A_z.

respect to the sun's rotational axis. It may be noted that the unknowns are directly and indirectly in terms of the latitude of the apparent center of the sun's image. While these angles change from exposure to exposure they are expressible in terms of the astronomic direction of the sun's axis and the right ascension and declination of the sun. The latter quantities may be interpolated from the ephemeris employing the date and GCT of the exposure as the argument:

$$\sin \delta_0 \sin \delta_z + \cos \delta_0 \cos \delta_z \cos (RA_0 - RA_z) = \sin \phi_o$$

The subscript z denotes the sun's rotational axis.

Taking the first order differentials of ϕ_i with respect to ϕ_0 and Az_0 in equation (1) gives

$$[\cos Z \cos \phi_o - \sin Z \sin \phi_o \cos (\theta'' - Az_o)]\Delta\phi_o + \sin Z \cos \phi_o \sin (\theta'' - Az_o)\Delta Az_o = \cos \phi_i \Delta\phi.$$

But

$$\Delta\phi_o = \frac{[\sin \delta_o \cos \delta_z - \cos \delta_o \sin \delta_z \cos (RA_o - RA_z)]}{\cos \phi_o} \Delta\delta_z + \frac{\cos \delta_o \cos \delta_z \sin (RA_o - RA_z)}{\cos \phi_o} \Delta RA_z$$

$$\Delta Az_{o1} = \frac{(\cos o_1o_2 \cos \phi_{o1} - \cos Az_{o1} \sin o_1o_2 \sin \phi_{o1})}{\sin Az_{o1} \cos \phi_{o1} \sin o_1o_2} \Delta\phi_{o1} - \frac{\cos \phi_{o2} \cdot \Delta\phi_{o2}}{\sin Az_{o1} \cos \phi_{o1} \sin o_1o_2}.$$

Linearization of ϕ_{o2}, ϕ_{on}, Az_{oi} and Az_{on} yields cyclically similar equations. Substituting,

$$\left\{[\cos Z_1 \cos \phi_{o1} - \sin Z_1 \sin \phi_{o1} \cos (\theta_1'' - Az_{o1})] + \sin Z_1 \cos \phi_{o1} \sin (\theta_1'' - Az_{o1}) \frac{[\cos o_1o_2 \cos \phi_{o1} - \sin o_1o_2 \sin \phi_{o1} \cos Az_{o1}]}{\sin o_1o_2 \cos \phi_{o1} \sin Az_{o1}}\right\} \Delta\phi_{o1} - \frac{\sin Z_1 \sin (\theta_1'' - Az_{o1}) \cos \phi_{o2}}{\sin Az_{o1} \sin o_1o_2} \Delta\phi_{o2} = \cos \phi \Delta\phi$$

$$\left\{[\cos Z_2 \cos \phi_{o2} - \sin Z_2 \sin \phi_{o2} \cos (\theta_2'' - Az_{o2})] + \sin Z_2 \cos \phi_{o2} \sin (\theta_2'' - Az_{o2}) \frac{[\cos o_2o_n \cos \phi_{o2} - \sin o_2o_n \sin \phi_{o2} \cos Az_{o2}]}{\sin o_2o_n \cos \phi_{o2} \sin Az_{o2}}\right\} \Delta\phi_{o2} - \frac{\sin Z_2 \sin (\theta_2'' - Az_{o2}) \cos \phi_{on}}{\sin Az_{o2} \sin o_2o_n} \Delta\phi_{on} = \cos \phi \Delta\phi$$

$$\left\{[\cos Z_n \cos \phi_{on} - \sin Z_n \sin \phi_{on} \cos (\theta_n'' - Az_{on})]\right.$$

$$\left. + \sin Z_n \cos \phi_{on} \sin (\theta_n'' - Az_{on})] \frac{[\cos o_n o_1 \cos \phi_{on} - \sin o_n o_1 \sin \phi_{on} \cos Az_{on}]}{\sin o_n o_1 \cos \phi_{on} \sin Az_{on}}\right\} \Delta\phi_{on}$$

$$- \frac{\sin Z_n \sin (\theta_n'' - Az_{on}) \cos \phi_{o1}}{\sin Az_{on} \sin o_n o_2} \Delta\phi_{o1} = \cos \phi \Delta\phi.$$

Let the coefficients be expressed with simplified notation:

$$a_1 \Delta\phi_{o1} + b_1 \Delta\phi_{o2} = \Delta$$

$$a_2 \Delta\phi_{o2} + b_2 \Delta\phi_{on} = \Delta$$

$$\cdots\cdots\cdots$$

$$a_n \Delta\phi_{on} + b_n \Delta\phi_{o1} = \Delta$$

also let

$$\Delta\phi_{o1} = m_1 \Delta\delta_z + n_1 \Delta RA_z$$

$$\Delta\phi_{o2} = m_2 \Delta\delta_z + n_2 \Delta RA_z$$

$$\Delta\phi_{on} = m_n \Delta\delta_z + n_n \Delta RA_z$$

where

$$m_1 = \frac{\sin \delta_{o1} \cos \delta_z' - \cos \delta_{o1} \sin \delta_z' \cos (RA_{o1} - RA_z')}{\cos \phi_{o1}'}$$

$$m_2 = \frac{\sin \delta_{o2} \cos \delta_z' - \cos \delta_{o2} \sin \delta_z' \cos (RA_{o1} - RA_z')}{\cos \phi_{o2}'}$$

$$\cdots\cdots\cdots$$

$$m_n = \frac{\sin \delta_{on} \cos \delta_z' - \cos \delta_{on} \sin \delta_z' \cos (RA_{on} - RA_z')}{\cos \phi_{on}'}$$

$$n_1 = \frac{\cos \delta_{o1} \cos \delta_z' \sin (RA_{o1} - RA_z')}{\cos \phi_{o1}}$$

$$n_2 = \frac{\cos \delta_{o2} \cos \delta_z' \sin (RA_{o2} - RA_z')}{\cos \phi_{o2}}$$

$$\cdots\cdots\cdots$$

$$n_n = \frac{\cos \delta_{on} \cos \delta_z' \sin (RA_{o2} - RA_z')}{\cos \phi_{on}}.$$

Substituting

$$a_1 m_1 \Delta\delta_z + a_1 n_1 \Delta RA_z + b_1 m_2 \Delta\delta_z + b_1 n_2 \Delta RA_z = \Delta$$

$$a_2 m_2 \Delta\delta_z + a_2 n_2 \Delta RA_z + b_2 m_n \Delta\delta_z + b_2 n_n \Delta RA_z = \Delta$$

$$\cdots\cdots\cdots$$

$$a_n m_n \Delta\delta_z + a_n n_n \Delta RA_z + b_n m_1 \Delta\delta_z + b_n n_1 \Delta RA_z = \Delta$$

or

$$(a_1m_1 + b_1m_2)\Delta\delta_z + (a_1n_1 + b_1n_2)RA_z = \Delta$$
$$(a_2m_2 + b_2m_n)\Delta\delta_z + (a_2n_2 + b_2n_n)RA_z = \Delta$$
$$(a_nm_n + b_nm_1)\Delta\delta_z + (a_nn_n + b_nn_1)RA_z = \Delta.$$

Simplifying further,

$$A_1\Delta\delta_z + B_1\Delta RA_z = \Delta$$
$$A_2\Delta\delta_z + B_2\Delta RA_z = \Delta$$
$$A_n\Delta\delta_z + B_n\Delta RA_z = \Delta$$

or

$$\eta_1\Delta\delta_z + \xi_1\Delta RA_z = \Delta\phi_1 = \phi_{i1}' - \phi_{i2}'$$
$$\eta_2\Delta\delta_z + \xi_2\Delta RA_z = \Delta\phi_2 = \phi_{i2}' - \phi_{in}'$$

where

$$\eta_1 = \frac{A_1 - A_2}{\cos\phi'} \qquad \xi_1 = \frac{B_1 - B_2}{\cos\phi'}$$
$$\eta_2 = \frac{A_2 - A_n}{\cos\phi'} \qquad \xi_2 = \frac{B_2 - B_n}{\cos\phi'}\,.$$

When the right hand terms vanish,

$$\delta_z = \delta_z' + \Sigma\Delta\delta_z$$
$$RA_z = RA_z' + \Sigma\Delta RA_z\,.$$

Then

$$\sin\delta_{o1}\sin\delta_z + \cos\delta_{o1}\cos\delta_z\cos(RA_{o1} - RA_z) = \sin\phi_{o1}$$
$$\sin\delta_{o2}\sin\delta_z + \cos\delta_{o2}\cos\delta_z\cos(RA_{o2} - RA_z) = \sin\phi_{o2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sin\delta_{on}\sin\delta_z + \cos\delta_{on}\cos\delta_z\cos(RA_{on} - RA_z) = \sin\phi_{on}$$

$$\cos Az_{o1} = \frac{\sin\phi_{o2} - \sin\phi_{o1}\cos o_1o_2}{\cos\phi_{o1}\sin o_1o_2}$$

$$\cos Az_{o2} = \frac{\sin\phi_{on} - \sin\phi_{o2}\cos o_2o_n}{\cos\phi_{o2}\sin o_2o_n}$$

$$\cos Az_{on} = \frac{\sin\phi_{o1} - \sin\phi_{on}\cos o_no_1}{\sin\phi_{on}\sin o_no_1},$$

and then the solar latitude for any image i is

$$\cos Z_{1i}\sin\phi_{o1} + \sin Z_{1i}\cos\phi_{o1}\cos(\theta_{1i}'' - Az_{o1}) = \sin\phi_i$$
$$\cos Z_{2i}\sin\phi_{o2} + \sin Z_{2i}\cos\phi_{o2}\cos(\theta_{2i}'' - Az_{o2}) = \sin\phi_i$$

which is the original equation before linearization. A similar reduction is performed for image j. The astronomic direction of i and j for times T_1 and T_n are determined next. These incremental rotations are illustrated in Figure 5. Consider image i:

$$\sin \delta_z(\sin \delta_i) + \cos \delta_z \cos RA_z(\cos \delta_i \cos RA_i) + \cos \delta_z \sin RA_z(\cos \delta_i \sin RA_i) = \sin \phi_i$$
$$\sin \delta_o(\sin \delta_i) + \cos \delta_o \cos RA_o(\cos \delta_i \cos RA_i) + \cos \delta_o \sin RA_0(\cos \delta_i \sin RA_i) = \cos Z_i.$$

Linearizing,

$$a_z \Delta\delta_i + b_z \Delta RA_i = \Delta\phi_i$$
$$a_o \Delta\delta_i + b_o \Delta RA_i = \Delta Z_i$$

where

$$a_z = \frac{\sin \delta_z \cos \delta_i - \cos \delta_z \sin \delta_i \cos (RA_z - RA_i)}{\cos \phi_i}$$

$$b_z = \frac{\cos \delta_z \cos \delta_i \sin (RA_z - RA_i)}{\cos \phi_i}$$

$$\Delta\phi_i = \phi_i - \phi_i'$$

$$a_o = \frac{\sin \delta_o \cos \delta_i - \cos \delta_o \sin \delta_i \cos (RA_o - RA_i)}{\sin Z_i}$$

$$b_o = \frac{\cos \delta_o \cos \delta_i \sin (RA_o - RA_i)}{\sin Z_i}$$

$$\Delta Z_i = Z_i - Z_i'.$$

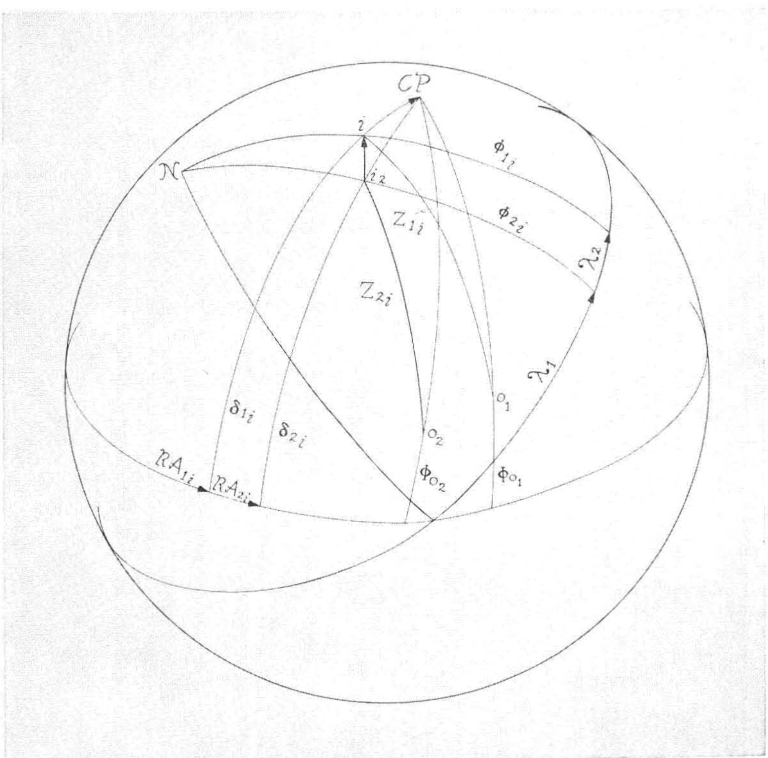

FIG. 5. Rotational Rate at ϕ_i.

The primed quantities are obtained by substitution of the approximate values in the basic equations. The first reduction is for time T_1 and the second for time T_n. With two values of the astronomic direction of i

$$m = \frac{\left[(\delta_{1i} - \delta_{ni})^2 + (RA_{1i} - RA_{ni})^2 \cos^2 \delta_i\right]^{1/2}}{(Tn - T_1) \cos \phi_i}$$

which is the rotational rate at latitude i for the time interval $T_n - T_1$. Presumably j, not being identical with i, would produce a different rotational rate. According to Duncan the inclination of the sun is 83° to the plane of the ecliptic and the rotational rate varies with latitude as follows:

25 days	$\phi = 0$
27.2 days	$\phi = \pm 45°$
33 days	$\phi = \pm 80°$.

Sunspots are found in definite zones chiefly where ϕ is between 10° and 30°. They are occasionally found where ϕ is 45° but never at the poles. Photographic determination by Adams at Mount Wilson gives

24.6 days	$\phi = 0$
33.3 days	$\phi = 80°$.

The rotational rate according to Carrington and Faye has the value

$$m = 14.°37 - 3.°10 \sin^2 \phi,$$

and Adams

$$m = 14.°61 - 3.°99 \sin^2 \phi.$$

Conclusion and Recommendation

It is concluded that the problem of weak geometry and variable surface can be solved with tangential equations outlined applied to short interval exposures.

It is recommended that this basic problem, not heretofore considered, be evaluated by the professional photogrammetrist, since it is here that he can make his greatest initial contribution to the technology of solar system mechanics.

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