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Resection Using Iterative Least Squares

The determination of the position and orientation parameters of the aerial camera is based on the image positions of ground control stations.

ABSTRACT: *This paper reports on the development of an iterative least squares method which can produce a theoretically "Exact" resection solution for a wide variety of aerial frame photography. The iterative method is an extension of a least squares method derived by Von Gruber and developed by Hallert. Instead of assuming vertical photography to simplify the evaluation of the observation equation coefficients, a more general solution is obtained by evaluating these coefficients in their entirety. The non-linearity of the coefficients is overcome by the use of a converging process to arrive at the space position elements. The iterative method was programmed for the computer and tested on fictitious photography. The results of the tests are shown and indicate that the iterative least squares method is both valid and versatile. It is believed that this method could provide a useful tool in both practical photogrammetry and in studies on the effects of errors.*

INTRODUCTION

THE EXISTING analytical methods for resecting a single aerial photograph can generally be separated into two basic types of mathematical solutions. These are: The "Exact"* methods which are limited to computing the six orientation elements from the photo-coordinate measurements of three control points at a time, and the "Least Squares" methods which produce the space position of best fit from the measurements of as many points as desired. Both of these resection methods have their relative advantages and disadvantages when they are used to resect aerial photography with an electronic computer. The advantage of the first method, typified by the "Church Resection Solution,"[1] is the capability of rapidly obtaining theoretically "Exact" values of the space position elements for any camera orientation other than truly vertical. Least squares resection solutions have the valuable properties of being able to weight each measurement, compute correlation numbers, and obtain the standard error estimates of the space position elements, in addition to being able to utilize many control points simultaneously. These benefits, however, are obtained at the cost of a larger amount of computer time. An important aspect of a least squares solution is that it is, of necessity, a linear process and cannot be used directly to solve exactly a non-linear relationship such as a photographic resection. This property of having to linearize non-linear functions has adverse effects upon the least squares resection methods and is the reason for the writing of this paper.

In order to produce linear observation equations from the non-linear relationship between photographic and ground coordinates, most of the least squares methods

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* In this context, "Exact" denotes the ability to arrive at a theoretically perfect answer (neglecting errors due to round off) when all measurements and constants dealt within the solution are error free.

make use of simplifying assumptions. Although these assumptions are valid and serve their purpose, they severely restrict the range of usefulness of the resection method to narrow limits of camera orientations. A variation of the photographic orientation from that assumed when the observation equation coefficients were derived will cause the results of the resection solution to differ considerably from the true values. Even variations as low as one degree can cause results to wander.

Considering the limitations that exist for this type of solution, it was felt that a method which could produce theoretically "Exact" results for a wide range of photographic orientations, yet maintain the benefits that accompany a least squares type solution, would be desirable and useful. This method could then be applied to any type of aerial photography, vertical, convergent or oblique, without the necessity of rewriting the solution for every different situation. It would also be interesting to compare the length of computer time required for solution of the "Exact" and approximate least squares methods. If times of computation were nearly the same for both methods on normal photography, it might be advantageous to employ the new approach. Finally, an "Exact" least squares method would be beneficial in research studies on the effects of errors due to various causes, since the resultant error would only be caused by these induced errors and not by the method of resection itself.

Iterative Method of Resection

Hallert Method*

In order to discuss, in specifics, the difficulties that develop in a least squares resection method and the means of overcoming them, a particular method has been selected. The least squares resection solution most familiar to photogrammetrists is that which was originally derived by Von Gruber[2] and considerably extended and modified by Hallert.[3] This solution will be referred to as the "Hallert Method," in this paper. Since the derivation of the Hallert Method can be found in his and in other texts, it will not be shown. Except for the definition of coordinate systems and rotations which were changed to those used in the U. S. Geodetic Coordinate System in order to maintain accordance with the recommendations of the Stockholm Convention of 1956, the iterative method is developed in exactly the same manner as the Hallert solution up to the formation of the condition equations. This change in system convention changes the form of the resulting equations somewhat, but does not alter any of the theoretical development.

The equations† relating the absolute ground coordinates of a control point with the photo-coordinates of that point, the focal-length and the space position elements evolve to,

$$X = \frac{X_L + h[(\cos\phi\cos\kappa)x + (\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)y + (\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)C]}{-(\sin\phi)x + (\sin\omega\cos\phi)y + (\cos\omega\cos\phi)C} \tag{1}$$

and

$$Y = \frac{Y_L + h[(-\cos\phi\sin\kappa)x + (\cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa)y + (\sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa)C]}{-(\sin\phi)x + (\sin\omega\cos\phi)y + (\cos\omega\cos\phi)C} \tag{2}$$

* See Reference 3 Pages 247–256 and 278.

† Although the condition equations used in this study are written in terms of ground coordinates, the method explained herein can be easily applied to the usual analytical case where the condition equations are in terms of photographic coordinates.

where the subscripts L refer to the coordinates of the perspective center. Writing these equations in shorthand notation we have,

$$X = f(X_L, h, x, y, C, \omega, \phi, \kappa) \tag{3}$$

and

$$Y = f(Y_L, h, x, y, C, \omega, \phi, \kappa) \tag{4}$$

By using the rule that the total differential is equal to the sum of its partial differentials,* we get,

$$dX = dX_L + \frac{\partial X}{\partial Z_L}\cdot dZ_L + \frac{\partial X}{\partial \omega}\cdot d\omega + \frac{\partial X}{\partial \phi}\cdot d\phi + \frac{\partial X}{\partial \kappa}\cdot d\kappa \tag{5}$$

and

$$dY = dY_L + \frac{\partial Y}{\partial Z_L}\cdot dZ_L + \frac{\partial Y}{\partial \omega}\cdot d\omega + \frac{\partial Y}{\partial \phi}\cdot d\phi + \frac{\partial Y}{\partial \kappa}\cdot d\kappa \tag{6}$$

The term dZ_L replaces dh, since $h = Z_L - Z_p$ and Z_p is a constant for any one control point. Since x and y are observed measurements and C is considered error free for this exercise, there are no derivatives shown for these quantities.

If the true X-coordinate of a control point is known, and the X-coordinate of that point for any particular photographic orientation can be computed by use of Equation (1), the total differential dX can be found by,

$$dX = X_{\text{Control}} - X_{\text{Computed}} \tag{7}$$

By moving the known dX term to the right hand side in Equation 5, the resulting equation should equal zero, if the correct values of the unknowns are substituted. Otherwise, the sum of the terms will equal a residual V_X. The equation then appears as,

$$V_x = dX_L + \frac{\partial X}{\partial Z_L} dZ_L + \frac{\partial X}{\partial \omega} d\omega + \frac{\partial X}{\partial \phi} d\phi + \frac{\partial X}{\partial \kappa} d\kappa - dX \tag{8}$$

Similarly in the Y-direction we have

$$V_y = dY_L + \frac{\partial Y}{\partial Z_L} dZ_L + \frac{\partial Y}{\partial \omega} d\omega + \frac{\partial Y}{\partial \phi} d\phi + \frac{\partial Y}{\partial \kappa} d\kappa - dY \tag{9}$$

The six unknowns are corrections to be applied to some previously determined (estimated) space position which will produce the correct elements of orientation. But before these unknowns can be determined via the use of normal equations, the values of the partial derivatives must be evaluated.

It is in the evaluation of the partial derivative coefficients that the restrictive assumptions are made in the Hallert Method. By referring to Equations 1 and 2, it can be seen that the partial derivatives of the X and Y ground coordinates taken with respect to Z_L, ω, ϕ, and κ will result in complex non-linear functions. The Hallert Method circumvents this tedious evaluation by making several assumptions, which are quite valid, but limit the versatility of the equations. By assuming that the aerial photograph is close to vertical, the sines and cosines of the orientation angles approach zero and one respectively, and the values $d\omega$, $d\phi$, and $d\eta$ are all very small quantities. After substituting these quantities into Equations 1 and 2, neglecting

* Although, this is strictly true only when the differentials approach a zero limit, this condition is approximated with a negligible error as the iterations near the final computed space position.

products of differentials and using a series expansion (neglecting second and higher order terms), the well known observation equations of the Hallert Method are obtained. For other than vertical photography, e.g., convergent photography, the same procedure must be used to evaluate new coefficients.

HALLERT METHOD WITH ITERATIVE SOLUTION

Since the initial concept was to arrive at a least squares solution which would be both versatile and "exact," the problem to be solved consists of evaluating the partial derivative coefficients in their entirety and utilizing these non-linear coefficients such that the correct space position can be obtained. The effect of computing the orientation element corrections from the normal equations formed with these coefficients is to approximate the transcendental correction equations with linear equations. Thus, the resulting computed corrections to the initially assumed space position will be in error due to this approximation. However, the new space position elements obtained will be nearer to the correct values than the original estimates. By repeating the process of computing corrections to the orientation elements of this new space position, the linearized correction equations will approximate more closely the true transcendental functions. This effect is due to the fact that the changes in the orientation elements will be smaller than the original corrections. By iterating in this manner, the corrections to the space position elements will consistently become smaller and the coefficients of the corrections will approach linear functions. Theoretically, this procedure can be used to resect the space position elements regardless of photographic orientation to any desired degree of accuracy by performing a sufficient number of iterations.

The partial derivatives appearing in Equations 8 and 9 have been evaluated in their entirety. This evaluation consists of differentiating Equations 1 and 2 with respect to each of the orientation elements and is shown in Appendix A. The results of this effort are as follows:

A. Observation Equation Coefficients for X-Direction

$$1.\quad \frac{\partial X}{\partial Z_L} = \frac{1}{F}\Big[(\cos\phi\cos\kappa)x + (\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)y + (\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)C\Big] \tag{10}$$

$$2.\quad \frac{\partial X}{\partial \omega} = \frac{h}{F^2}\Big[xy(\sin\omega\sin\phi\sin\kappa - \cos\omega\cos\kappa) + xC(\sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa) - y^2(\cos\phi\sin\kappa) - C^2(\cos\phi\sin\kappa)\Big] \tag{11}$$

$$3.\quad \frac{\partial X}{\partial \phi} = \frac{k}{F^2}\Big[x^2(\cos\kappa) + xy(\cos\omega\cos\phi\sin\kappa) - xC(\sin\omega\cos\phi\sin\kappa) + y^2(\sin^2\omega\cos\kappa + \sin\omega\cos\omega\sin\phi\sin\kappa) + yC(2\sin\omega\cos\omega\cos\kappa + [\cos^2\omega - \sin^2\omega]\cdot\sin\phi\sin\kappa) + C^2(\cos^2\omega\cos\kappa - \sin\omega\cos\omega\sin\phi\sin\kappa)\Big] \tag{12}$$

$$4.\quad \frac{\partial X}{\partial \kappa} = \frac{h}{F}\,[-(\cos\phi\sin\kappa)x + (\cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa)y$$
$$- (\cos\omega\sin\phi\sin\kappa + \sin\omega\cos\kappa)C] \qquad (13)$$

$$5.\quad -dX = \frac{h}{F}\,[(\cos\phi\cos\kappa)x + (\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)y$$
$$+ (\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)C] + X_L - X_C \qquad (14)$$

B. Correction Equation Coefficients for Y-Direction

$$1.\quad \frac{\partial Y}{\partial Z_L} = \frac{1}{F}\,[-(\cos\phi\sin\kappa)x + (\cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa)y$$
$$- (\sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa)C] \qquad (15)$$

$$2.\quad \frac{\partial Y}{\partial \omega} = \frac{h}{F^2}\,[xy(\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)$$
$$+ xC(\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa) \qquad (16)$$
$$- y^2(\cos\phi\cos\kappa)$$
$$- C^2(\cos\phi\cos\kappa)]$$

$$3.\quad \frac{\partial Y}{\partial \phi} = \frac{h}{F^2}\,[-x^2(\sin\kappa)$$
$$+ xy(\cos\omega\cos\phi\cos\kappa)$$
$$- xC(\sin\omega\cos\phi\cos\kappa) \qquad (17)$$
$$+ y^2(\sin\omega\cos\omega\sin\phi\cos\kappa - \sin^2\omega\sin\kappa)$$
$$+ yC(-2\sin\omega\cos\omega\sin\kappa + [\cos^2\omega - \sin^2\omega]\sin\phi\cos\kappa)$$
$$- C^2(\sin\omega\cos\omega\sin\phi\cos\kappa + \cos^2\omega\sin\kappa)]$$

$$4.\quad \frac{\partial Y}{\partial \kappa} = \frac{h}{F}\,[-(\cos\phi\cos\kappa)x - (\cos\omega\sin\kappa + \sin\omega\sin\phi\cos\kappa)y$$
$$+ (\sin\omega\sin\kappa - \cos\omega\sin\phi\cos\kappa)C] \qquad (18)$$

$$5.\quad -dY = \frac{h}{F}\,[-(\cos\phi\sin\kappa)x + (\cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa)y$$
$$- (\sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa)C] + Y_L - Y_C \qquad (19)$$
$$F = -(\sin\phi)x + (\sin\omega\cos\phi)y + (\cos\omega\cos\phi)C$$

where

By substituting the equivalent values for the partial derivatives as shown in Equations 10 through 14 into Equation 8, and those shown in Equations 15 through 19 into Equation 9, we arrive at the observation equations with the partial derivatives completely evaluated.

Once the coefficients of the observation equations have been determined, the residuals are set equal to zero and the normal equations are formed in the usual manner. The resulting six normal equations are solved by use of the Gauss-Jordan Matrix Reduction Method using pivotal elements for a minimum loss of significant digits. After each solution of the normal equations, the orientation element corrections obtained are algebraically added to the previous (assumed or computed) space position elements and the entire cycle is repeated. Since this is a converging process, the iterations should continue until the change between successive space position values is less than an arbitrary amount. Because both position and rotation elements are involved, there should be one check value for each type on successive iterations. Once sufficiently accurate values of the space position have been obtained, the solution for weight numbers and standard errors is carried out in a manner identical to the Hallert Method and thus will not be discussed.

WRITING AND TESTING THE COMPUTER PROGRAM

In order to evaluate the iterative modification to the Hallert Method and to test its versatility and exactness, a computer program was written for the IBM 650 Digital Computer. Although the program was written with the intent of making it general enough to resect for as wide a range of photographic orientations as practical, no attempt was made to time optimize the program.

The two values for checking successive iteration results are read into the program as data. This allows the accuracy of the solution to be as high or low as desired and thus automatically determines the number of iterations to be performed. The initial values of the space position orientations are not assumed to be zero as in the Hallert solution, but form part of the input data so that good estimates of these values can be utilized by the program.

The computer program was tested on hypothetical photography. Nine evenly distributed photo-points and their corresponding ground elevations were assigned arbitrarily on a fictitious photograph having a known space position. With these data and the camera focal length as input, a Church intersection computer program which had been previously written then computed the horizontal ground coordinates of these points. Using the same photo-coordinates, elevations, and computed ground coordinates, the resection program, if correct, should then resect to exactly the same space position originally assigned. Since the photograph orientation angles in the Church Intersection are described in terms of tilt, swing, and azimuth while the resection program utilizes ω, ϕ, and κ, a method for converting between the two systems had to be developed in order to have a complete check on the resection program. A computer program for the orientation angle system conversion was written and tacked on to the intersection program. By using the intersection program to produce fictitious data for any assigned space position, a thorough check could then be made on the iterative resection program being tested.

A fairly extensive number of resection tests were run on the IBM 650 Computer in order to obtain sufficient information on certain properties of the program. The two qualities of principle interest were the range of orientations that could be resected and the number of iterations required for each solution. Also of interest was the effect of the size of the checking values and of the initial assumed space position elements on the number of iterations required. Since fictitious photography was used, the accuracy of the solution was of interest only to the extent of the possible loss of significant digits in the computations.

RESULTS OF PROGRAM TESTS

As was stated earlier, the program tests were run in order to gain information about its properties under varying conditions. The tests are described and some of the results tabulated in the order that they were run.

Varying Checking Values

The values for checking successive values of position and rotation space position elements were varied to find the most suitable values to be used for all other tests. The camera position elements used for this series and for all tests run were $X_L = 50{,}000$ ft., $Y_L = 30{,}000$ ft., $Z_L = 20{,}000$ ft. The orientation angles used for this test

TABLE 1

Varying ϵ_1, $\epsilon_2=30'$				Varying ϵ_2, $\epsilon_1=.05$ Ft.			
Test #	ϵ_1 (Ft)	No. of Iterations	IBM 650 Time	Test #	ϵ_2	No. of Iterations	IBM 650 Time
1	5.0	3	7 min–30s	6	10′	3	7 min-30s
2	1.0	3	7–30	7	1′	3	7-30
3	0.1	3	7–30	8	30″	3	7-30
4	0.05	3	7–30	9	10″	3	7-30
5	0.01	4	8–55	10	1″	3	7-30

Test #11, $\epsilon_1=.01$ ft., $\epsilon_2=.01''$, No. Iter. = 4, Time 8 min-55 sec.

were tilt $t=1°$; swing $s=180°$; azimuth $\alpha_{vo}=0°$. The checking values selected for use on all other following tests were $\epsilon_1=.01$ ft. for position elements and $\epsilon_2=1''$ for rotation elements. (Table 1.)

Comparison with Non-Iterative Hallert Solution

A photograph which could be considered in the range of normal aerial photography was resected with both an existing Hallert Method program and the iterative resection method. This photograph had a tilt of one degree, a swing of 180 degrees and a zero azimuth angle ($t=1°$, $s=0°$, $\alpha_{vo}=0°$). The relative errors in the resection solutions are given in Table 2.

Varying Tilt, Swing, and Azimuth Singularly

This series of tests was run to determine the effect on the number of iterations required when each church orientation angle parameter was varied separately. While each angle was varied, the two remaining parameters were kept constant except when they caused κ to be greater than 90 degrees. The orientation angles used and the results are shown in Table 3.

Varying Tilt, Swing and Azimuth Simultaneously

For this series of tests, the values assigned to the orientation angles were chosen semi-indiscriminantly in order to test the versatility of the program as far as the possibility of "getting lost" is concerned and to gain further knowledge about times required for solution. The results are shown in Table 4.

Varying Initial Approximations.

To observe the influence of the initial approximations on the number of iterations necessary for solution, a photograph which required a large number of iterations was selected for further investigation. The photograph chosen (Test #30, $t=20°$, $s=30°$, $\alpha_{vo}=150°$) required 13 iterations when the initial approximations were 1000 feet in error for position coordinates and started with all rotation angles equal to zero. The

TABLE 2

Solution	Errors						No. of Iterations	IBM 650 Time Min-Sec
	X_L(Ft)	Y_L(Ft)	Z_L(Ft)	ω	φ	κ		
Hallert	.43	.26	6.72	1′-33″	0′-4″	0′-10″	—	4-30
Test #12	.53	.26	6.65	1′-03″	0′-5″	<1″	1	4-40
Test #12	<.01	<.01	0.02	<1″	<1″	<1″	2	6-05
Test #12	<.01	<.01	<.01	<1″	<1″	<1″	3	7-30
Test #12	0	0	0	0	0	0	4	8-55

TABLE 3

Test	*Tilt*	*Swing*	*Azimuth*	*Approximate* ω	*Approximate* ϕ	*Approximate* κ	*No. of Iterations*	*IBM 650 Time (M-S)*
				Varying Tilt				
13	1°	180°	0°	1°	0°	0°	4	8-55
14	5°	180°	0°	5°	0°	0°	4	8-55
15	10°	180°	0°	10°	0°	0°	5	10-20
16	20°	180°	0°	20°	0°	0°	6	11-45
17	30°	180°	0°	30°	0°	0°	9	16-00
				Varying Swing				
18	5°	0°	180°	5°	0°	0°	4	8-55
19	5°	30°	180°	$4\frac{1}{2}°$	$2\frac{1}{2}°$	30°	6	11-45
20	5°	150°	0°	$4\frac{1}{2}°$	$2\frac{1}{2}°$	30°	6	11-45
21	5°	210°	0°	$4\frac{1}{2}°$	$2\frac{1}{2}°$	30°	6	11-45
22	5°	300°	180°	$2\frac{1}{2}°$	$4\frac{1}{2}°$	60°	9	16-00
				Varying Azimuth				
23	5°	180°	10°	5°	0°	10°	4	8-55
24	5°	180°	30°	5°	0°	30°	6	11-45
25	5°	180°	60°	5°	0°	60°	8	14-35

tests were run to see how much the number of iterations would be reduced by using more correct initial approximations. Due to limited computer availability, the number of tests run were limited. The results of these tests are shown in Table 5.

CONCLUSIONS

Based on the Results of the Computer Tests, the following conclusions were reached:

1. There is no theoretical limit to the degree of precision which can be obtained for the space position elements by this iterative method of resection, other than computer limitations.
2. For "Normal" vertical aerial photography, (ω, ϕ, and κ fairly small), nominal accuracy for space position elements can be achieved with two or three iterations, and high accuracy (less than .01 ft. in position and 1 second in rotation) with four iterations.

TABLE 4

Test No.	*Tilt*	*Swing*	*Azimuth*	*Approximate* ω	*Approximate* ϕ	*Approximate* κ	*No. of Iterations*	*Time (M-S)*
26	1°	45°	150°	43′	43′	75°	8	14-35
27	2°	45°	180°	$1\frac{1}{2}°$	$1\frac{1}{2}°$	45°	7	13-10
28	5°	30°	210°	$4\frac{1}{2}°$	$2\frac{1}{2}°$	0°	4	8-55
29	10°	30°	270°	9°	5°	60°	12	20-15
30	20°	30°	150°	$17\frac{1}{2}°$	10°	59°	13	21-40
31	1°	90°	210°	0°	1°	60°	7	13-10
32	2°	150°	210°	2°	1°	60°	7	13-10
33	5°	200°	45°	$4\frac{3}{4}°$	$1\frac{1}{2}°$	25°	6	11-45
34	10°	270°	150°	0°	10°	45°	10	17-25
35	20°	300°	150°	$10\frac{1}{2}°$	$17\frac{1}{2}°$	29°	9	16-00

TABLE 5

Test No.	Error in Initial Approximations						No. Iter.	Time (Min-Sec)
	XL (Ft)	*YL (Ft)*	*ZL (Ft)*	ω	ϕ	κ		
30	1000	1000	1000	$17\frac{1}{2}°$	10°	$58\frac{1}{2}°$	13	21-40
36	1000	1000	1000	2°	2°	2°	10	17-25
37	1000	1000	1000	1°	1°	1°	10	17-25
38	100	100	100	2°	2°	2°	10	17-25
39	0	0	0	1°	1°	1°	10	17-25
40	0	0	0	0°	0°	0°	1	4-40

3. Comparison of the iterative and non-iterative Hallert Methods for resection of a particular photograph showed that after only two iterations, the maximum errors in position and rotation were reduced from comparable values to .02 feet and 1 second respectively. Since only one additional iteration (about $1\frac{1}{2}$ minutes) is required to reduce the errors to nominal values, the increase in time required by the iterative method is not large enough to make it non-competitive.

4. The iterative resection method was completely versatile in its ability to resect for all photographic orientations tested. It is felt that this ability will hold for all orientations within the limitations incorporated (ω, ϕ, and κ all less than 90 degrees).

5. The number of iterations required for solution appears to be related to the angular orientation of the photograph. The relationship between church orientation angles and iterations required does not seem to fit a definite pattern; except for tilt, where the number of iterations appear to increase exponentially for a tilt greater than ten degrees. In terms of ω, ϕ, κ rotations, κ has the greatest affect on the number of iterations required. In either case, the number of iterations required for solution increase as the photograph deviates further from the vertical.

6. For photographs with high values of ω, ϕ, and κ, rough initial approximations for these rotation elements reduced the required number of iterations somewhat, but further refinement had little effect. Position element approximations had no effect at all on iterations. It is felt that additional study on this aspect, e.g., continued refinement of approximations, or the use of a church resection to get approximations of high quality, would be beneficial.

RECOMMENDATIONS

Based on the effort expended on the development and testing of the iterative adaptation of the Hallert Method for resection, the following recommendations are made:

1. Considering the increased computer time necessary for solution, the gain in accuracy obtained from the iterative least squares method warrants its use in place of the existing Hallert Method for near vertical photography.
2. Because of its inherent properties of versatility and error free least square solution, the iterative resection adaptation of the Hallert Method is ideally suited for use as a tool in studies made on the effects of errors and on error propagation.
3. Additional investigations should be performed on this method of resection in order to gain more complete knowledge of its properties; particularly with regard to the affect that initial approximations have on the number of iterations required for resection solution.

Appendix A

Evaluating Partial Derivatives

Although the partial derivatives in each observation equation have been evaluated separately, only the derivation for the X-Equation will be shown. The Y-Equation is handled in an identical manner.

From the scale factor relationship of coordinates we have

$$X = x'h/(-z') \tag{A-1}$$

where x' and z' are the rectified photo-coordinates and h is the difference in elevation between the aerial photograph and a control point. By treating this equation as a fraction with $x'h$ replaced by E and $-z'$ by F, we get for the partial derivative of X with respect to a general variable g,

$$\frac{\partial X}{\partial g} = \frac{F\dfrac{\partial E}{\partial g} - E\dfrac{\partial F}{\partial g}}{F^2} \tag{A-2}$$

where

$$E = x'h = \left[(\cos\phi\cos\kappa)x + (\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)y + (\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)C\right]$$

and

$$F = -z' = -(\sin\phi)x + (\sin\omega\cos\phi)y + (\cos\omega\cos\phi)C$$

Evaluating each coefficient separately by substituting the particular variable for the general variable g, we have the following:

1. EVALUATING $\partial X/\partial Z_L$.

Here

$$\frac{\partial X}{\partial Z_L} = \frac{F\dfrac{\partial E}{\partial Z_L} - E\dfrac{\partial F}{\partial Z_L}}{F^2}$$

where

$$\frac{\partial E}{\partial Z_L} = x' \quad \text{and} \quad \frac{\partial F}{\partial Z_L} = 0$$

Thus

$$\frac{\partial X}{\partial Z_L} = \frac{Fx'}{F^2} = \frac{x'}{F}$$

or

$$\frac{\partial X}{\partial Z_L} = \frac{1}{F}\left[(\cos\phi\cos\kappa)x + (\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)y + (\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)C\right] \tag{A-3}$$

2. EVALUATING $\partial X/\partial\omega$

Now

$$\frac{\partial X}{\partial \omega} = \frac{F\dfrac{\partial E}{\partial \omega} - E\dfrac{\partial F}{\partial \omega}}{F^2}$$

where

$$\frac{\partial E}{\partial \omega} = h[(\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)y + (-\sin\omega\sin\phi\cos\kappa - \cos\omega\sin\kappa)C]$$

and

$$\frac{\partial F}{\partial \omega} = (\cos\omega\cos\phi)y - (\sin\omega\cos\phi)C$$

Then

$$\begin{aligned}\frac{\partial X}{\partial \omega} = \frac{h}{F^2}\{&[-(\sin\phi)x + (\sin\omega\cos\phi)y + (\cos\omega\cos\phi)C]\cdot[\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)y \\ &-(\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)C] \\ &+[(\cos\phi\cos\kappa)x + (\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)y \\ &+(\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)C]\cdot[-(\cos\omega\cos\phi)y + (\sin\omega\cos\phi)C]\}.\end{aligned}$$

After multiplying we get

$$\begin{aligned}\frac{\partial X}{\partial \omega} = \frac{h}{F^2}[&xy(-\cos\omega\sin^2\phi\cos\kappa + \sin\omega\sin\phi\sin\kappa) \\ &+ xC(\sin\omega\sin^2\phi\cos\kappa + \cos\omega\sin\phi\sin\kappa) \\ &+ y^2(\sin\omega\cos\omega\sin\phi\cos\phi\cos\kappa - \sin^2\omega\cos\phi\sin\kappa) \\ &+ yC(-\sin^2\omega\sin\phi\cos\phi\cos\kappa - \sin\omega\cos\omega\cos\phi\sin\kappa) \\ &+ yC(\cos^2\omega\sin\phi\cos\phi\cos\kappa - \sin\omega\cos\omega\cos\phi\sin\kappa) \\ &+ C^2(-\sin\omega\cos\omega\sin\phi\cos\phi\cos\kappa - \cos^2\omega\cos\phi\sin\kappa) \\ &+ xy(-\cos\omega\cos^2\phi\cos\kappa) \\ &+ xC(\sin\omega\cos^2\phi\cos\kappa) \\ &+ y^2(-\sin\omega\cos\omega\sin\phi\cos\phi\cos\kappa - \cos^2\omega\cos\phi\sin\kappa) \\ &+ yC(\sin^2\omega\sin\phi\cos\phi\cos\kappa + \sin\omega\cos\omega\cos\phi\sin\kappa) \\ &+ yC(-\cos^2\omega\sin\phi\cos\phi\cos\kappa + \sin\omega\cos\omega\cos\phi\sin\kappa) \\ &+ C^2(\sin\omega\cos\omega\sin\phi\cos\phi\cos\kappa - \sin^2\omega\cos\phi\sin\kappa)\end{aligned}$$

By regrouping common terms, cancelling and using trigonometric identities, the partial derivative appears as

$$\frac{\partial X}{\partial \omega} = \frac{h}{F^2}\left[xy(\sin\omega\sin\phi\sin\kappa - \cos\omega\cos\kappa)\right.$$
$$+ xC(\sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa) \tag{A-4}$$
$$- y^2(\cos\phi\sin\kappa)$$
$$\left.- C^2(\cos\phi\sin\kappa)\right]$$

3. EVALUATING $\partial X/\partial\phi$.

Now

$$\frac{\partial X}{\partial \phi} = \frac{F\dfrac{\partial E}{\partial\phi} - E\dfrac{\partial F}{\partial\phi}}{F^2}$$

where

$$\frac{\partial E}{\partial \phi} = h[-(\sin\phi\cos\kappa)x + (\sin\omega\cos\phi\cos\kappa)y + (\cos\omega\cos\phi\cos\kappa)C]$$

and

$$\frac{\partial F}{\partial \phi} = -(\cos\phi)x - (\sin\omega\sin\phi)y - (\cos\omega\sin\phi)C$$

Then

$$\frac{\partial X}{\partial \phi} = \frac{h}{F^2}\{[-(\sin\phi)x + (\sin\omega\cos\phi)y + (\cos\omega\cos\phi)C]$$
$$\cdot[-(\sin\phi\cos\kappa)x + (\sin\omega\cos\phi\cos\kappa)y + (\cos\omega\cos\phi\cos\kappa)C]$$
$$+ [(\cos\phi\cos\kappa)x + (\sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa)y$$
$$+ (\cos\omega\sin\phi\cos\kappa - \sin\omega\sin\kappa)C]$$
$$\cdot[(\cos\phi)x + (\sin\omega\sin\phi)y + (\cos\omega\sin\phi)C]\}$$

After multiplying,

$$\frac{\partial X}{\partial \phi} = \frac{h}{F^2}[x^2(\sin^2\phi\cos\kappa)$$
$$- xy(\sin\omega\sin\phi\cos\phi\cos\kappa)$$
$$- xC(\cos\omega\sin\phi\cos\phi\cos\kappa)$$
$$- xy(\sin\omega\sin\phi\cos\phi\cos\kappa)$$
$$+ y^2(\sin^2\omega\cos^2\phi\cos\kappa)$$
$$+ yC(\sin\omega\cos\omega\cos^2\phi\cos\kappa)$$
$$- xC(\cos\omega\sin\phi\cos\phi\cos\kappa)$$
$$+ yC(\sin\omega\cos\omega\cos^2\phi\cos\kappa)$$
$$+ C^2(\cos^2\omega\cos^2\phi\cos\kappa)$$

$$
\begin{aligned}
&+ x^2(\cos^2 \phi \cos \kappa)\\
&+ xy(\sin \omega \sin \phi \cos \phi \cos \kappa + \cos \omega \cos \phi \sin \kappa)\\
&+ xC(\cos \omega \sin \phi \cos \phi \cos \kappa - \sin \omega \cos \phi \sin \kappa)\\
&+ xy(\sin \omega \sin \phi \cos \phi \cos \kappa)\\
&+ y^2(\sin^2 \omega \sin^2 \phi \cos \kappa + \sin \omega \cos \omega \sin \phi \sin \kappa)\\
&+ yC(\sin \omega \cos \omega \sin^2 \phi \cos \kappa - \sin^2 \omega \sin \phi \sin \kappa)\\
&+ xC(\cos \omega \sin \phi \cos \phi \cos \kappa)\\
&+ yC(\sin \omega \cos \omega \sin^2 \phi \cos \kappa + \cos^2 \omega \sin \phi \sin \kappa)\\
&+ C^2(\cos^2 \omega \sin^2 \phi \cos \kappa - \sin \omega \cos \omega \sin \phi \sin \kappa)]
\end{aligned}
$$

This reduces to

$$
\begin{aligned}
\frac{\partial X}{\partial \phi} = \frac{h}{F^2}\, [x^2(\cos \kappa)&\\
&+ xy(\cos \omega \cos \phi \sin \kappa)\\
&- xC(\sin \omega \cos \phi \sin \kappa)\\
&+ y^2(\sin^2 \omega \cos \kappa + \sin \omega \cos \omega \sin \phi \sin \kappa)\\
&+ yC(2 \sin \omega \cos \omega \cos \kappa + [\cos^2 \omega - \sin^2 \omega] \sin \phi \sin \kappa)\\
&+ C^2(\cos^2 \omega \cos \kappa - \sin \omega \cos \omega \sin \phi \sin \kappa)]
\end{aligned}
\tag{A-5}
$$

4. EVALUATING $\partial X/\partial \kappa$.

Now

$$
\frac{\partial X}{\partial \kappa} = \frac{F \dfrac{\partial E}{\partial \kappa} - E \dfrac{\partial F}{\partial \kappa}}{F^2}
$$

where

$$
\begin{aligned}
\frac{\partial E}{\partial \kappa} = h[- (\cos \phi \sin \kappa)x &+ (-\sin \omega \sin \phi \sin \kappa + \cos \omega \cos \kappa)y\\
&- (\cos \omega \sin \phi \sin \kappa + \sin \omega \cos \kappa)C]
\end{aligned}
$$

and

$$
\frac{\partial F}{\partial \kappa} = 0
$$

Then

$$
\frac{\partial X}{\partial \kappa} = \frac{F \dfrac{\partial E}{\partial \kappa}}{F^2} = \frac{\dfrac{\partial E}{\partial \kappa}}{F}
$$

or

$$\frac{\partial X}{\partial \kappa} = \frac{h}{F}\left[-(\cos\phi \sin\kappa)x + (\cos\omega \cos\kappa - \sin\omega \sin\phi \sin\kappa)y - (\cos\omega \sin\phi \sin\kappa + \sin\omega \cos\kappa)C\right] \tag{A-6}$$

5. EVALUATING $-dX$.

Previously it was shown that dX was equal to the difference between the true and computed x-coordinate of a point, both quantities being measured from the nadir point, or

$$dX = (X_C - X_L) - \frac{x'h}{-z'}$$

where X_C is the coordinate of the control point. Then

$$-dX = \frac{x'h}{-z'} + X_L - X_C$$

Substituting for x' (remembering that $F = -z'$) we get

$$-dX = \frac{h}{F}\left[(\cos\phi \cos\kappa)x + (\sin\omega \sin\phi \cos\kappa + \cos\omega \sin\kappa)y + (\cos\omega \sin\phi \cos\kappa - \sin\omega \sin\kappa)C\right] + X_L - X_C \tag{A-7}$$

By substituting the equivalent values of the partial derivatives into the correction equation, we now have the coefficients entirely evaluated without any assumptions or approximations.

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