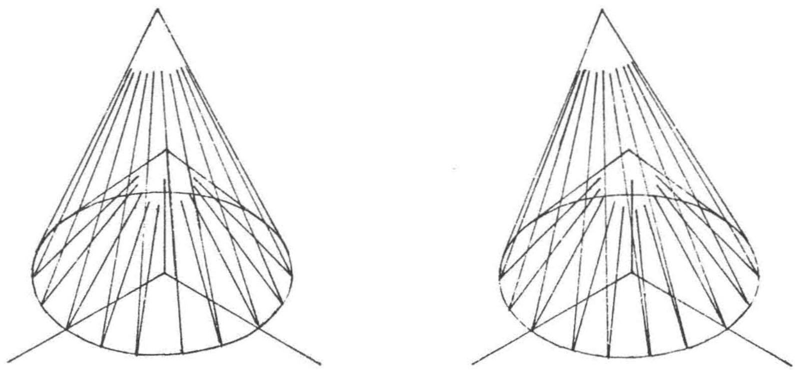

FRONTISPIECE. Model of a hypercone.

DR. LUISA BONFIGLIOLI
Israel Institute of Technology
Kiriath Bialick (Haifa), Israel

Stereoscopic Model for Four Dimensions

ABSTRACT: *A method is described for representing figures of four dimensions where the chief aim, insofar as possible, is to show them like the usual three dimensional figures to which one is accustomed. The figures can be specified by a function of four variables, or they can be defined in a geometrical way. It is also possible to represent curves or surfaces of ordinary three dimensional space where the coordinates of the points are expressed by parametric equations.*

INTRODUCTION

THE GEOMETRY of four dimensions is generally presented as an extension of the geometry of three dimensions and the study of it is based on abstract concepts[1] or on analytical calculations[2] several attempts have been made[3] in order to explain this subject, in an understandable manner, to people who have not a deep mathematical background in order to develop geometrical methods for its representation.[4]

But these approaches failed to give a graphical representation of the elements of the four dimensional geometry resembling the well-known representation of the usual space of three dimensions, 3-D. And because men can imagine only objects that are similar to to other ones to which they are accustomed, all these efforts have not had notable results.

In this article we report a simple stereoscopic method for representing bodies in four dimensional space 4-D. In these models the first three dimensions are used for drawing a usual isometric picture of the body and the fourth dimension is used as a horizontal parallax. In this manner the bodies resemble the customary ones but, because they are seen in relief, they take the appearance of the complex bodies that they actually are.

In 3-D space the perspective view of a body is generally constructed by utilizing two or more orthogonal projections of it: also in 4-D we must proceed in the same manner. Therefore, before preparing the steroscopic model, we have to draw two parallel projections of the given body,[5] and after that we have to change the fourth dimension of the points of it into horizontal parallax. Because of the impossibility of reporting here all the detail constructions of Ref. 5, we shall only explain the preparation of a few typical models in which the fourth dimension of the points is obtainable by means of calculations from a suitable equation. But if this condition is not feasible, we shall find the fourth dimension according to the suitable construction of Ref. 5, and we shall use it without attempting to explain the procedure.

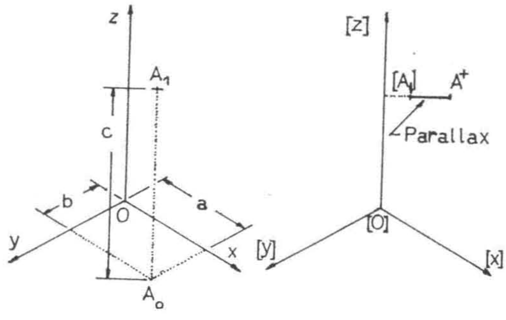

FIG. 1. Stereoscopic model of a point.

STEREOSCOPIC MODEL OF A POINT

Let A be a point in 4-D whose coordinates are a, b, c, d. Draw (Figure 1) three concurrent lines x, y, z, starting from a point O, so that z is *always* vertical and x, y are inclined towards z at angles of 120°. Locate a point A_0 whose coordinates are a and b in the xy-plane, and along the parallel to z through A_0, plot a segment A_0A_1 whose length is c. It is clear that A_1 is the usual isometric picture of a point A' in 3-D (the associate point of A) whose coordinates are a, b, and c.

Now draw concurrent lines [x], [y], [z], correspondingly parallel to the previous lines x, y, z, whose starting point [O] is connected to the former point O by means of a *horizontal* segment. This segment O [O] is assumed to be the *base* and its length is about 63 mm. It follows that the horizontal parallax of [O] is zero.

Now on a line parallel to O[O], starting from A_1 locate a point A^* whose horizontal parallax with respect of [O] is proportional to the value of the fourth coordinate d of A. The ratio of proportionality is arbitrary but must be suited to the limits of the stereoscopic vision. By looking at the drawing stereoscopically, one can merge or fuse the two identical pencils of lines into a unique pencil seen lying on the sheet of the drawing while the two points A_1 and A^* fuse into a unique point floating below or above the sheet of the drawing according to the sign of the parallax of A.

This construction is fundamental and we shall always use it for preparing further models. Its letters should have been rewritten in each model but, because this is not the case, we must remember that their location in respect of the axes is always as shown in Figure 2. In the further explanations we refer to the figure constructed by means of the x, y, z axes as the L-part (left), and to the figure constructed by means of the [x], [y], [z] axes as the R-part (right) of the drawing. You will note that all the models of this article are prepared for observing with the naked eye, but they can be used also with a stereoscope after a suitable translation of the R-part withrespect to the L-part.

STEREOSCOPIC MODEL OF A HYPERPLANE

Let

$$ax + by + cz + dt = e$$

be the equation of the given hyperplane W. (Figure 3). W cuts the x-axis at a point A whose distance from O is equal to e/a; therefore, the coordinates of A are (e/a, 0, 0, 0). In an analogous manner the coordinates of the points of intersection of W with the y- and z-axes are B (0, e/b, 0, 0) and C (0, 0, e/c, 0).

The value of the parallax t at the point D of W whose three first coordinates are 0, 0, 0 is e/d and the value of the parallax of the points A, B, C is zero. Because of its coordinates the point D_1 coincides with O in the L-part. It follows that A_1, B_1, C_1 and A^*, B^*, C^* have an identical location with respect to x, y, z and [x], [y], [z], and D^* is translated with respect to D_1. The lines connecting these points two by two are the lines of intersection between W and the coordinate planes. The particular equation of the model is

$$4x - 3y + 2z + 3t = 12.$$

STEREOSCOPIC MODEL OF THE HYPERSPHERE

Let

$$x^2 + y^2 + z^2 + t^2 = r^2$$

be a function of the four variables x, y, z, t where r is a constant (Figure 4). Give to t a certain fixed value t; thus the equation can be rewritten as

$$x^2 + y^2 + z^2 = r^2 - t^2.$$

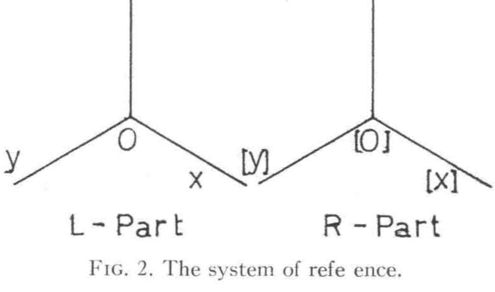

FIG. 2. The system of refe ence.

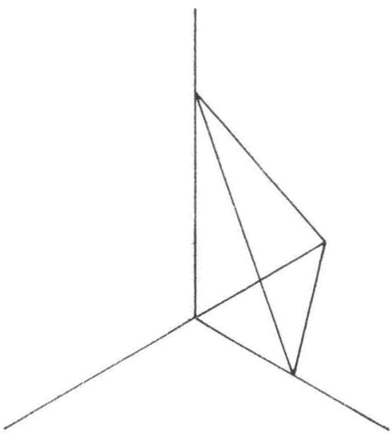

FIG. 3. Model of a hyperplane.

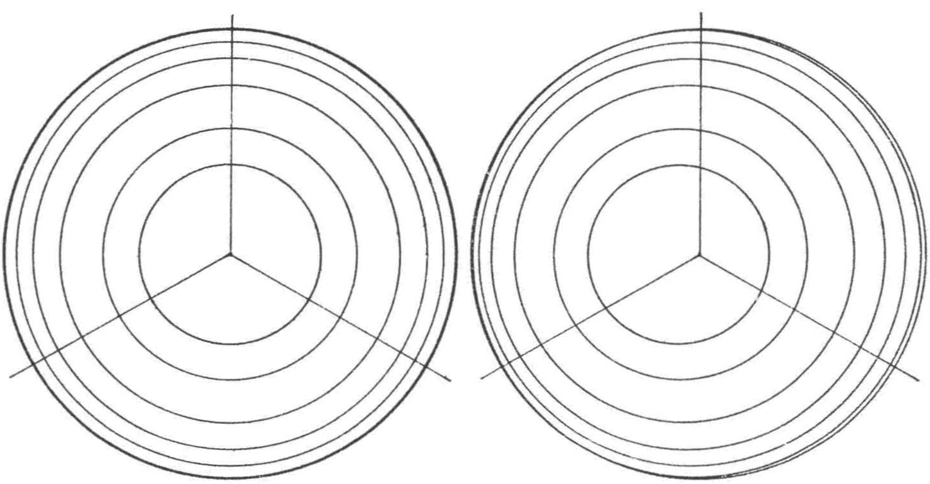

FIG. 4. Model of a hypersphere.

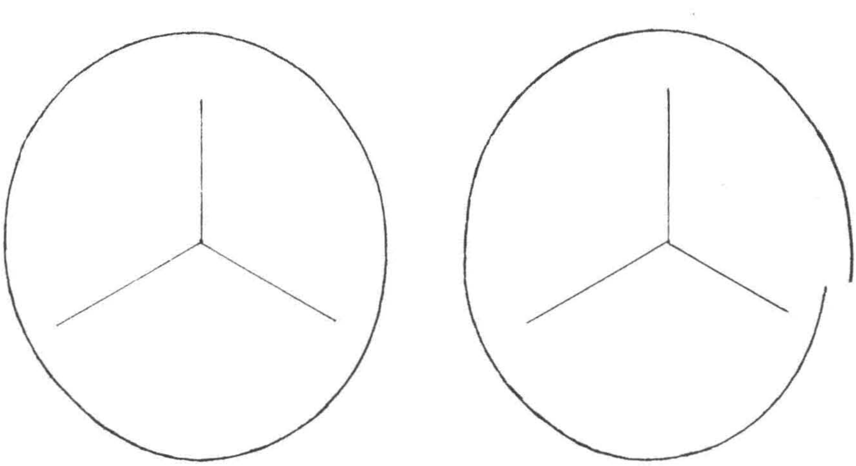

FIG. 5. Model of a parametric curve.

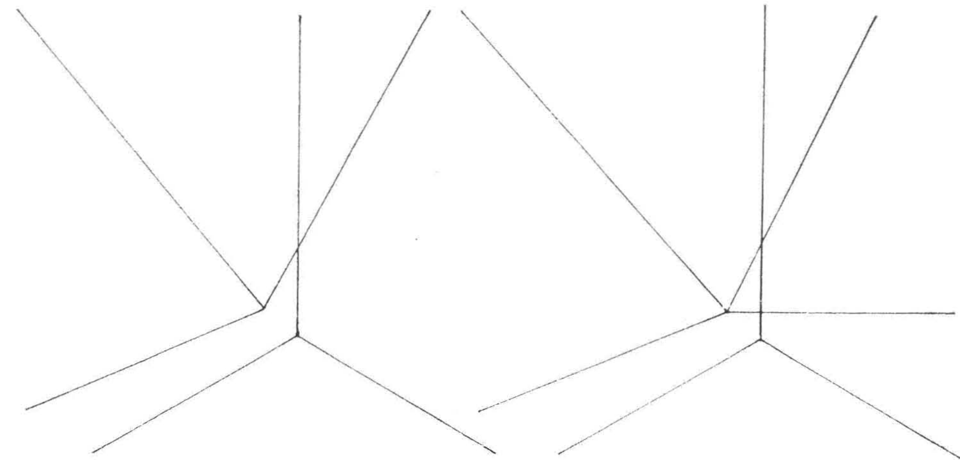

FIG. 6. Model of four concurrent perpendicular lines.

In the 3-D space having x, y, z axes, this equation determines a sphere whose radius is equal to $(r^2-t^2)^{\frac{1}{2}}$ and whose center has coordinates (0, 0, 0). The isometric picture of *this* sphere is a circle whose value of t, and in the R-part draw the corresponding points whose parallaxes are proportional to the same t. Connect the points of the L-part by means of a continuous line and do likewise for the points of the R-part. The particular equations of the model are

$$x = 20 \sin (t + 90°)$$
$$y = 10 \sin (t - 60°)$$
$$z = 25 \sin t.$$

Stereoscopic Model of the Hypercube

As a matter of definition, a hypercube is generated by the translatory motion of a cube in a direction perpendicular to all its edges through a distance equal to the length of its edge. Instead of calculating the coordinates of the vertices of the hypercube, it is preferable to draw the model by using the constructions of Ref. 5. Therefore, first of all we draw the parallel projections of *any set* of four concurrent lines perpendicular two by two and according to this we draw the model of the set (Figure 6). It is worthwhile pointing out that the L-part the fourth perpendicular to the first three lines *always* dwindles at their point of concurrence and in the R-part its direction is *always* horizontal. We find graphically the true length of these lines and along them we plot segments of the same length (in the model this length center is the origin O and whose radius is

$$(3/2)^{1/2}(t^2 - t^2)^{1/2}.$$

Because the fourth coordinate of all the points of this sphere is always $\bar{t}$, the R-part of

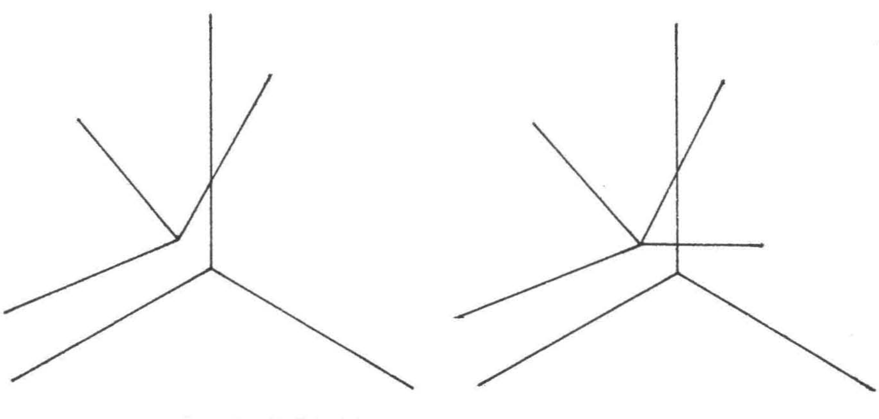

FIG. 7. Model of four equal perpendicular segments.

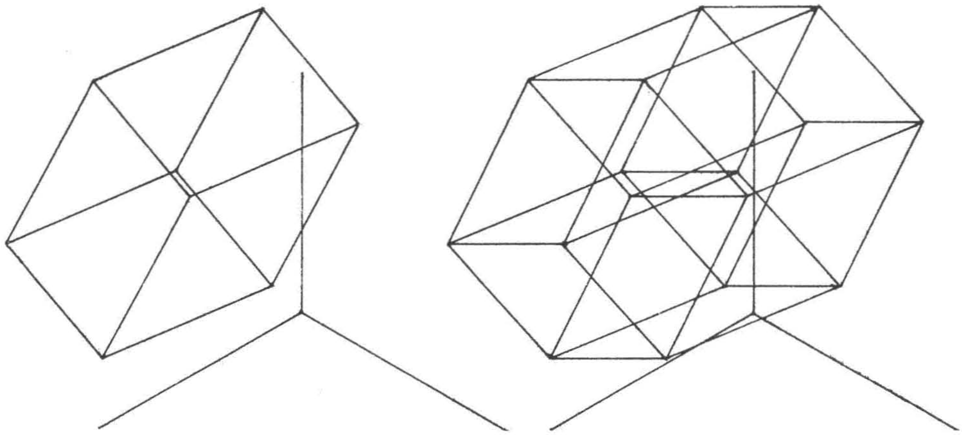

FIG. 8. Model of a hypercube.

its drawing consists of a circle identical to the previous one whose center has a horizontal parallax proportional to $\bar{t}$, with respect to the center of the previous circle.

Now give to t all the values from zero to r and for each value repeat the previous construction. For the sake of clearness only half a hypersphere has been drawn. In this example, the equation is

$$x^2 + y^2 + z^2 + t^2 = 6$$

and the unit is 1 cm.

Stereoscopic Model of a Curve

Let a curve be defined by

$$x = A_1 \sin (t + \alpha_1°)$$
$$y = A_2 \sin (t + \alpha_2°)$$
$$z = A_3 \sin (t + \alpha_3°).$$

Give to t various values $\bar{t}$ between zero and 2π and calculate the corresponding values $\bar{x}$, $\bar{y}$, $\bar{z}$ of the x, y, z (Figure 5). In the L-part draw the isometric view of all the points whose coordinates are values of x, y, z corresponding to the same is 32 mm. (the horizontal segment is not at the same scale) (Figure 7). Using the first three segments, we complete the representation of a cube in both the L-part and R-part (Figure 8). After that, through each vertix of the cube of the R-part we draw a segment parallel and equal to the fourth segment of Figure 7, and we complete the drawing of the hypercube. Viewing the model, the four pairs of identical cubes can be distinguished; they constitute the eight cells of the hypercube.

Stereoscopic Model of a Hypercone

The definition is as follows: a hypercone is generated connecting the points on a *base* of a cone with any point not belonging to the base. In this case the elements of the hypercone are not linked by any special geometrical relations; therefore, the drawing is made without the help of the parallel projections (Frontispiece).

Conclusion

The method of graphic construction is very simple yet it leads to intuitive drawings that give the observer the feeling that the bodies of four dimensions resemble the usual bodies of three dimensions. The method is applicable to bodies determined by equations of four variables or by suitable definitions of four-dimensional geometry, as well as to the curves and surfaces of the 3-D space whose points have coordinates depending on the same parameter (parametric equations).

References

1. Henry Parker Manning, Geometry of Four Dimensions, Dover Publications.
2. Frederick S. Woods, Higher Geometry, Dover Publications.
3. M. Boucher, Introduction a la géométrie a quatre dimensions d'aprè's les méthodes de la géométrie élémentaire, *Librairie Scientifique A. Herman.*
   Henry P. Manning, The Fourth Dimension Simply Explained, Dover Publications.
4. J. Maurin/ Géométrie Descriptive a 4 Dimensions, Gauthier-Villars.
   Ernesto S. Lindgren, Descriptive Geometry of Four Dimensions, Technical Seminar Series of the Dept. of Graphics, Princeton University.
5. Luisa Bonfiglioli, Parallel Projection for Euclidean Geometry of Four Dimensions, in printing.