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Adjustment of a Strip Using Orthogonal Polynomials

Abstract: *In problems dealing with fitting of surfaces representable by polynomials to observations taken at points which form a uniform grid, it is often convenient to use tabulated values of orthogonal polynomials. The method offers considerable numerical advantages and simplicity in the statistical analysis of results. This article describes the method and gives an example of its application in a x-, y- and z-adjustment of a strip consisting of twelve grid plate models used as a part of a bridging test with the A-8 plotter.*

Introduction

More than a century ago, orthogonal polynomials were used in fitting of polynomials to data according to the principle of least squares by the Russian mathematician, Chebyshev. Since then this topic has been frequently discussed by others in statistical and related literature. The theory of the method is described in most texts on statistics. However, the examples usually employed to illustrate it deal with the one-dimensional case of fitting a curve. An illustration of this method in the two-dimensional case, i.e. in fitting a surface, based upon the use of tabulated values of orthogonal polynomials, is given in [2] and will be described in the following. In this case the method can be used to advantage in photogrammetry where the problem of estimating surfaces occurs very frequently.

Description of the Method

Let us suppose that we wish to fit a surface to observed values of a dependent variable z and that the z-values are observed at points which are distributed in the xy-plane in such a way that they form a rectangular grid, i.e. $x_{i+1}-x_i=d_1$ and $y_{j+1}-y_j=d_2$ for all i and j. This situation arises in photogrammetry, for example, in the studies of lens distortions, film shrinkage, model deformations, etc.. and the arrangement ought to be used whenever the experimentor has the choice of positions of the independent variables. Let us suppose that the surface to be fitted is a general polynomial in two variables given by Equation (1):

$$Z = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots \tag{1}$$

in which the a's will be estimated from the data using the principle of least squares.

If we are interested only in the nature of the surface or if we want to evaluate the surface only at the given points, a model (2) equivalent to (1) can be conveniently fitted in terms of the tabulated values of orthogonal polynomials. These polynomials are worked out for integer values 0, 1, 2 $\cdots$ of the independent variable, an arrangement which is possible to achieve with any other equally spaced values by a change of scale and origin.

In terms of these orthogonal polynomials, Equation (1) is rewritten as:

$$\begin{aligned} Z = {} & b_{00}\xi_0'(x)\xi_0'(y) + b_{10}\xi_1'(x)\xi_0'(y) \\ & + b_{01}\xi_0'(x)\xi_1'(y) + b_{20}\xi_2'(x)\xi_0'(y) \\ & + b_{11}\xi_1'(x)\xi_1'(y) + b_{02}\xi_0'(x)\xi_2'(y) + \cdots \end{aligned} \tag{2}$$

where the b's are the new coefficient and the ξ''s are the orthogonal polynomials, of appropriate dimension and degree, given in the usual notation. The polynomials may either be calculated or they are available in various sources in tabulated form (see References).

Let z_{ij} be an observed value at a point in the ith row and the jth column of a rectangular grid, $i=1, 2, \cdots m$, $j=1, 2 \cdots n$, and let there be an equal number of observations at each point. In the case of one observation (several observations are reduced to one by taking their mean), a typical observation equation, written explicitly in terms of orthogonal polynomials, will have the form:

$$
\begin{aligned}
z_{ij} = {} & b_{00}\xi_0'(i-1, m)\xi_0'(j-1, n) \\
& + b_{10}\xi_1'(i-1, m)\xi_0'(j-1, n) \\
& + b_{01}\xi_0'(i-1, m)\xi_1'(j-1, n) \\
& + b_{20}\xi_2'(i-1, m)\xi_0'(j-1, n) + \cdots
\end{aligned}
\tag{3}
$$

where e.g. $\xi'_1(j-1, m)$ is the $(j-1)$th element of the polynomial of 1st degree and dimension m.

Under the usual assumptions about the z's which underlie the use of the principle of least squares, the $m \times n$ observation equations of type (3) give rise to a set of normal equations:

$$A^T A \boldsymbol{b} = A^T \boldsymbol{z} \tag{4}$$

where A is the matrix of the coefficients of the b's, A^T its transpose and $\boldsymbol{b}$ and $\boldsymbol{z}$ are the column vectors of the b's and the z's respectively.

So far, nothing has been said about the surface which is being fitted and this question will not be settled until the end of this discussion. In the meantime, it is assumed that we are determining a surface which passes through all the points.

Solution of equation (4) is immediate, since, as a consequence of orthogonality, all the off-diagonal elements in the $A^T A$ matrix vanish and the diagonal terms become products of the sum of squares of the appropriate orthogonal polynomials which also are given in the tables. A compact solution of the problem that lends itself to matrix treatment is presented in [2] and will be described in the practical example that follows.

Adjustment of an A-8 Strip

Before describing the calculations, a few words should be said about the data. These form a part of the tests carried out at the Topographical Survey, Department of Mines and Technical Surveys in Ottawa to determine the vertical bridging capability of the A-8 plotter. Grid plates were used in this particular test. The bridging procedure used with the A-8 was as follows: the first grid plate model was oriented relatively and absolutely. All the subsequent models were then individually oriented in the usual way for the A-8 by employing a cross level in the transfer of tilts. No attempt was made to control the scale accurately. x, y, z-coordinates of six symmetrically located grid points were recorded in each model. Assuming that the tilts were accurately transferred, the 'bridge' at this stage consisted of disconnected, variable scale models, each having generally a different swing, because of the absence of the b_y-motion in the A-8.

The connecting of the models to form a continuous bridge was accomplished analytically on the computer with the aid of transformation (5) which gives the relation between the coordinates of two consecutive models $(i,\ i+1)$:

$$
\begin{bmatrix} x_{i+1} \\ y_{i+1} \\ z_{i+1} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & 0 \\ -b_2 & b_1 & 0 \\ 0 & 0 & \sqrt{b_1^2 + b_2^2} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{5}
$$

The b's were computed using lateral pass-points and the c's were determined in such a way that the models matched exactly at the center pass-points. The resulting discrepancies between the z-coordinates and their true values expressed in units of 0.01 mm at model scale are displayed in Table 1. Although, because of the build-up of systematic errors, the distribution of the points in the xy-plane is not strictly uniform, the resulting effect on the z-values is negligible.

The z-values are entered in Table 1 in pairs, the top number giving the value in the previous model and the lower in the following model respectively. The three single values at the beginning and at the end of the strip were taken twice in order to maintain orthogonality.

For computation, the two values are replaced by their mean or their sum (whichever is more convenient for computation) and the

Table 1

0	1	2	9	22	33	53	78	105	128	167	198	238
0	−2	5	13	23	40	58	79	100	135	163	200	238
0	0	0	4	11	23	40	58	78	102	136	169	206
0	0	0	4	11	23	40	58	78	106	136	169	206
0	−2	−7	−5	−4	10	21	37	54	78	106	139	170
0	−6	−7	−5	3	8	25	36	57	79	110	140	170

TABLE 2

314769.55*	318348.01*	38364.31*	0.08	3.24	0.67	14.04	0.08	6.21	4.44	2.06	0.57	0.22
14858.48*	6028.88*	3.16	4.91	1.46	2.82	10.59	0.00	0.01	0.26	1.62	1.88	1.10
6.16	0.92	1.55	4.02	0.08	4.59	0.13	0.45	0.48	0.10	2.15	0.82	0.28

resulting table may be viewed as a 3×13 matrix. Let it be denoted by Z. We now compute:

$$G = P_1 Z P_2^T \tag{6}$$

where P_1 is a 3×3 matrix which has the ξ'-polynomials as rows:

$$P_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Similarly, P_2^T is a 13×13 matrix of the ξ'-polynomials, only this time the different degree polynomials are written down in columns. Product (6) yields a matrix G of g-values:

$$G = \begin{bmatrix} g_{00} & g_{01} \cdots g_{0.12} \\ g_{10} & g_{11} \cdots g_{1.12} \\ g_{20} & g_{21} \cdots g_{2.12} \end{bmatrix}$$

A typical element of G is:

$$g_{rs} = \sum_{i=1}^{3}\sum_{j=1}^{13} z_{ij}\xi_r'(i-1, 3)\xi_s'(j-1, 13),$$
$$r = 0, 1, 2,$$
$$s = 0, 1, 2 \cdots 12. \tag{7}$$

To evaluate the b's, each entry in matrix G is divided by the appropriate product of the squares of the polynomials, i.e.:

$$b_{rs} = \frac{g_{rs}}{\sum_{i=1}^{3} \xi_r'^2(i-1, 3) \sum_{j=1}^{13} \xi_s'^2(j-1, 13)} = \frac{g_{rs}}{d_{rs}} \tag{8}$$

To organize the computational work, it is convenient to arrange the d-divisors into a matrix, say D, which can be written as:

$$D = Q_1 J Q_2 \tag{9}$$

where J is a 3×13 matrix in which all the entries are ones and Q_1 and Q_2 are both diagonal matrices of order 3 and 13 whose diagonal elements are the sums of squares of the successive degree polynomials of dimension 3 and 13 respectively. Denoting by B the matrix of the b's, we can also write with the aid of (9)

$$B = Q_1^{-1} P_1 Z P_2^T Q_2^{-1} \tag{10}$$

For the purpose of computation it is best to form the product $P_1 Z P_2^T$ first and then divide each element by the corresponding element in D.

Some statistical results are now presented: If the z's form a set of observations and if the errors which we associated with these observations are uncorrelated and have a constant variance, σ^2, then the quantities

$$w_{rs} = \frac{g_{rs}}{\sqrt{d_{rs}}} = b_{rs}\sqrt{d_{rs}},$$
$$r = 0, 1, 2,$$
$$s = 0, 1, 2 \cdots 12, \tag{11}$$

are also uncorrelated, with variance σ^2. This important result is a consequence of the fact that the set of equations (11) represents an orthogonal transformation of the z's.

In matrix notation, the orthogonal transformation is:

$$W = R_1^{-1} P_1 Z P_2^T R_2^{-1}$$

where the elements in the R-matrices are the square roots of the elements in the Q-matrices Because of the orthogonal transformation, the w's possess the same statistical properties as the z's except that they have different means. Also, in view of the geometric interpretation of an orthogonal transformation as a rotation of axes, which leaves the lengths invariant, the following relation exists:

$$\sum_{r=0}^{2}\sum_{s=0}^{12} w_{rs}^2 = \sum_{i=1}^{3}\sum_{j=1}^{13} z_{ij}^2. \tag{12}$$

The w^2-values, in view of (11), are interpreted to represent the individual contributions to the total sum of squares of the corresponding b's. These values are displayed in Table 2.

With the aid of Table 2 we can now readily assess the nature of the best-fitting surface. It is quite obvious, by inspection, that the five entries marked with an asterisk in the top left corner of Table 2 represent the major contributions to the total sum of squares. By comparison, the remaining components, beside being significantly smaller in magnitude, do not appear to exhibit any definite pattern and can be said to behave randomly.

We thus conclude that all the systematic variation in the data is accounted for by the five large squares and that the remaining squares correspond to the b's whose means are zero and, therefore, can be considered to exhibit only the contributions to the error

TABLE 3. ANALYSIS OF VARIANCE

Sum of Squares (SS)		*Degrees of Freedom*	*Mean Square at Model Scale (0.01 mm units)*	*Standard Error at Diap. Scale (microns)*
Total *SS*	692609.00	78		
SS Mean	314769.55	1		
SS Linear x	318348.01	1		
SS Linear y	14858.48	1		
SS Quadratic x	38364.31	1		
SS Interaction xy	6028.88	1		
$S(z-Z)^2$	239.77	73	3.2845	$\pm 9.1\ \mu$
$S(z-\bar{z})^2$	158.50	39	4.0641	$\pm 10.1\ \mu$
$S(\bar{z}-Z)^2$	81.27	34	2.3903	$\pm 7.8\ \mu$

sum of squares. Thus the nature of the best-fitting surface is:

$$Z = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy.$$

The purpose of this study was to appraise the performance of the bridging method and to assess its precision. The precision is derived from the residual sum of squares. It is customary to arrange the results in an Analysis of Variance Table (Table 3): The column denoted "Degrees of Freedom" gives the divisors for the calculation of that particular mean square. Since there were two observations at each point, the sum of squares of the deviations from the fitted surface: $S(z-Z)^2$ was further decomposed into the sum of squares of the deviations of the observations from their means: $S(z-\bar{z})^2$ and the sum of squares of the deviations of the means from the surface: $S(\bar{z}-Z)^2$. The former part, which is calculated directly from the data, forms an independent estimate of error while the latter measures the closeness of fit of the surface to the data.

The quantity: $S(\bar{z}-Z)^2=81.27$ should be equal to the sum of the components allocated to error in Table 2.

The w^2-arrays resulting from fitting polynomials to the x- and y-discrepancies in the same strip are presented in Tables 4 and 5 respectively.

While, in the previous example, the division of the w^2-components into those representing systematic variation in the data and those reflecting error posed no difficulties, these examples are less clear-cut in this respect. Here the transition between the two groups of components is less abrupt and no definite boundary is readily apparent. Even in situations like these there is no great danger of misinterpretation of the results if the decision is made just on the basis of visual inspection of the magnitude and the spread of the w^2-components. For instance, the component $w_{13}^2=9.97$ in Table 4 might be interpreted to represent systematic variation rather than error. While this may well be so, the decision to the contrary is going to make very little difference in the results.

In some situations, assistance in reaching a decision can be obtained from statistical tests which deal with testing the homogeneity of a group of variances. Some of these tests are listed in [2]. This requires, however, that we are prepared to make an additional assumption about the errors—the assumption being that the errors are distributed normally.

As in the case of the z-adjustment, the residual sums of squares were obtained for the x- and y-adjustments yielding the following standard errors of the residuals at the scale of the diapositives:

TABLE 4

74,277.55*	78,625.23*	7,083.10*	326.77*	33.61*	0.81	3.16	5.03	6.26	2.95	1.03	0.12	2.87
11,550.48*	2,730.91*	26.42*	9.97	1.02	1.20	0.67	0.13	0.11	1.90	1.05	0.18	3.76
0.01	4.40	0.07	0.00	0.23	1.22	0.77	3.77	0.43	0.93	2.20	0.14	0.09

TABLE 5

362851.28*	235018.68*	10,383.30*	34.08*	0.22	53.55	14.45	8.84	0.92	11.52	7.16	2.66	0.00
2464.69*	997.12*	226.91*	10.10	0.14	2.25	0.01	7.82	0.43	1.68	2.07	0.03	1.75
43.10*	4.22	6.40	7.57	0.38	0.00	0.37	0.96	0.13	1.99	0.92	1.92	0.36

$$S_x = \pm 8.1\mu$$
$$S_y = \pm 14.8\mu$$

NOTES

In conclusion it should be remarked that, if the original polynomial form is desired, i.e., if the a-coefficients, which refer to the original values of x and y, are needed, then they can be obtained from the b's by replacing the ξ'-polynomials by their equivalent polynomial functions (see e.g. [1]).

Another point, worth mentioning, is the fact that, while the existing tables of orthogonal polynomials apply to uniformly spaced values of the independent variables, the method is general and can be advantageously applied in some instances to non-uniform data. In the latter case, the orthogonal polynomials would have to be calculated but, once found, they can be used with different sets of observations. Such a situation exists, for example, in test areas where the same set of control points is repeatedly used for different tests.

Lastly, the method can readily be extended to the case of three and more independent variables.

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GRAVITY MAP OF UNITED STATES

A gravity map of the United States (exclusive of Alaska and Hawaii), the first of its kind, and described as a significant contribution to the growth of basic scientific knowledge about the earth, has just been published by the Department of the Interior's Geological Survey.

The comprehensive, 2-sheet, wall-sized map, two decades in the making, is at a scale of 1:2,500,000 (about 40 miles to the inch). On the map varying measurements of the force of gravity are shown by contour lines.

It will be of particular interest to scientists studying the structure of the earth's crust and upper Mantle, and to those studying regional geologic problems.

WHITE PLAINS FIRM SIGNS CONTRACT WITH PANAMA

International Resources and Geotechnics, Inc. of White Plains, New York, announced today that it had signed a contract with the Government of Panama to undertake a 4.2 million dollar survey of Panamanian natural resources, property ownership and land valuation.

The survey will cover some 40,000 square kilometers, or approximately 63 percent of Panama.

Financing was provided with the help of a 2.4 million dollar loan from the United States Agency for International Development. To this, the Panamanian Government will add 1.8 million dollars from its internal budget to cover the costs of supplies, materials, transport, equipment, and Panamanian personnel.

The contract, signed in Panama before United States and Panamanian officials, is an important element of newly elected President Marco A. Robles' program for Panama's economic development. Most of the country's lands are occupied but unowned, unregistered and untaxed. The survey will identify all lands being used and determine whether they are owned or merely occupied. Land capabilities, based upon natural resources characteristics, will be mapped. Tenure and ownership will be mapped. Valuation of lands will be based upon land capabilities. Taxes will be levied and collected upon the basis of valuations. It is expected that the resultant increase in taxation revenue will increase Panama's internal revenue by several times. Mr. D. R. Lueder, President of IRG, said today that this is one of the most diverse and ambitious programs of this nature ever attempted.