Discussion Paper

SIMULTANEOUS THREE-DIMENSIONAL TRANSFORMATION

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THE POSSIBILITY OF a conformal representation of space upon itself aroused the interest of some writers in photogrammetry in connection with the problem of the adjustment of aerial triangulation. Judging by several recent references to it in this periodical, the topic of the three dimensional conformal transformation seems to be surrounded by a degree of uncertainty. The reason for it may be, perhaps, that its mathematical history is quite old. The following brief note is intended to add to the discussion of this subject by presenting a few results pertaining to it by some mathematicians in the past.

Conformal mapping in space interested mathematicians around the middle of the last century. As early as 1847, Joseph Liouville[1] determined all the conformal transformations in space in an analytical way. In a notable theorem bearing his name, Liouville showed that the only ways of conformal representation of space upon itself are:

1. By translation and rotation, accompanied by constant magnification,
2. By inversion with respect to the sphere.

These two methods which can be combined in any manner and repeated any number of times, generally carry a sphere into another sphere. Thus, in space, there is nothing analogous to the two-dimensional situation of mapping conformally a region of the z-plane upon a w-region by means of an arbitrary selected analytic function $w=f(z)$.

Although the Transformation 1 has a wide use in photogrammetry, interest has been shown in the possibility of a nonlinear conformal transformation in space with regard to the problem of simultaneous three dimensional adjustment of deformed strips or groups of models. As these deformations are normally very moderate, it raises the possibility of employing infinitesimal transformations with conformal properties. Around 1870 Sophus Lie introduced the concept of an infinitesimal transformation into his study of the theory of groups. He investigated the properties of the three dimensional conformal group[2] and enumerated the infinitesimal transformations associated with it. An infinitesimal transformation is a transformation whose effect differs infinitesimally from the identity transformation. In general, such a transformation is of the form

$$x_i' = x_i + t\xi_i(x_1 \cdots x_n), \qquad (i = 1, \cdots, n)$$

where t is a constant so small that its square may be neglected. In his book on the Lie's group theory[3] in 1903, John E. Campbell also derives the most general form of the infinitesimal conformal transformation in three dimensional space from the consideration of differential relationships invariant under the conformal transformation. He gives its form as

$$x' = x + t\xi, \qquad y' = y + t\eta, \qquad z' = z + t\zeta$$

where

$$\begin{aligned}\xi &= \tfrac{1}{2}a_1(x^2 - y^2 - z^2) + a_2yx + a_3xz + a_0x \\ &\quad + A_2z - A_3y + \alpha \\ \eta &= \tfrac{1}{2}a_2(y^2 - z^2 - x^2) + a_3yz + a_1xy + a_0y \\ &\quad + A_3x - A_1z + \beta \\ \zeta &= \tfrac{1}{2}a_3(z^2 - x^2 - y^2) + a_1xz + a_2yz + a_0z \\ &\quad + A_1y - A_2x + \gamma. \qquad (1)\end{aligned}$$

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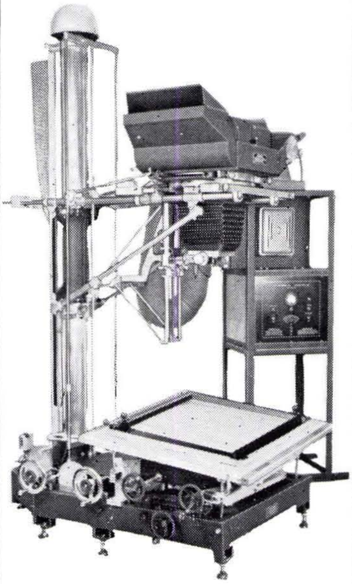

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It will be noticed that the Equations 1 are identical with those proposed by Mikhail[4] including only the terms of the second degree.

As far as the use of conformal transformations in photogrammetric adjustments is concerned, it would be presumptions to take the view that these transformations are the only ones that faithfully describe the nature of the discrepancies between the photogrammetric model and its control. However, in practical adjustments, where we often carry little hope of accomplishing more than smoothing of these discrepancies, conformal transformations do offer some advantages. By imposing restrictions on the coefficients of the interpolating polynomials they provide some insurance against undue deformations in small regions as already pointed out by Schut.[5] Also, by consuming fewer degrees of freedom than the general polynomials, they would seem to be safer to use with sparse control.

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IN A NOTE PUBLISHED recently (D. W. G. Arthur, PHOTOGRAMMETRIC ENGINEERING, January 1965) a few critical remarks of my earlier paper: "Simultaneous Three-Dimensional Transformation of Higher Degrees" (PHOTOGRAMMETRIC ENGINEERING, July 1964) were presented. It is the purpose of this note (1) to analyze and discuss these remarks and perhaps cast more light on the principal aspects of the original paper, and (2) to present results of tests performed subsequent to the publication of the original paper.

Arthur's criticisms can be summarized as claiming that:

a. The condition for conformality was "mis-stated."
b. My "fears about non-existence of second and third degree conformal transformation in three dimensions may be quite without foundation."[1]
c. Arthur's equations developed in Reference 3 and summarized in Reference 1 are "considerably more general for the strip[1] than are mine.

In reply to these criticisms, each point will be dealt with independently.

a. The entire text of my original article leaves no doubt in one's mind that the equations under consideration were of the polynomial type. As an example, the starting point of the development gave the general form of the three-dimensional transformation equations as:

$$\begin{aligned} X &= A_0 + A_1x + A_2y + A_3z + A_4x^2 + A_5y^2 + A_6z^2 \\ &\quad + A_7xy + A_8yz + A_9zx + \cdots \\ Y &= B_0 + B_1x + B_2y + B_3z + B_4x^2 + B_5y^2 + B_6z^2 \\ &\quad + B_7xy + B_8yz + B_9zx + \cdots \\ Z &= C_0 + C_1x + C_2y + C_3z + C_4x^2 + C_5y^2 + C_6z^2 \\ &\quad + C_7xy + C_8yz + C_9zx + \cdots \end{aligned} \tag{1}$$

where x,y,z are model coordinates and X, Y, Z are ground coordinates.

In the original paper, the condition for Equations 1 to represent conformal transformation was given as the orthogonality of the matrix:

$$\begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{bmatrix} \tag{2}$$

whereas in Arthur's paper this condition was stated as the orthogonality of the matrix:

$$\begin{bmatrix} \frac{1}{f}\frac{\partial X}{\partial x} & \frac{1}{f}\frac{\partial X}{\partial y} & \frac{1}{f}\frac{\partial X}{\partial z} \\ \frac{1}{f}\frac{\partial Y}{\partial x} & \frac{1}{f}\frac{\partial Y}{\partial y} & \frac{1}{f}\frac{\partial Y}{\partial z} \\ \frac{1}{f}\frac{\partial Z}{\partial x} & \frac{1}{f}\frac{\partial Z}{\partial y} & \frac{1}{f}\frac{\partial Z}{\partial z} \end{bmatrix}. \tag{3}$$

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It is evident that the only difference between the two Matrices 2 and 3 is simply f which represents a scale factor. *There should be no doubt in one's mind that the orthogonality of Matrix 3 is the proper condition.* However, it is very easily seen that the use of Matrix 2 instead was only for simplification in which the scale factor f was not considered. This should have been quite obvious from the fact that, even after the simplification, it was clearly stated in the paper that "since this derivative is quite complicated, it has not been attempted."[2] Furthermore, the same point was considered by another prominent author in regard to the same problem when Schut wrote, "Consequently, disregarding a scale factor, the matrix of the nine differential quotients (i.e. Matrix 2) must be *orthogonal* for any values of x, y and z."[7] Therefore the claim that there was a "mistake" or a "misstatement" in the original paper is clearly a misunderstanding.

Arthur also wrote further, "That Mikhail's condition (meaning that for the Matrix 2) is too restrictive is easily seen when we consider that it rules out even the general transformation of rectangular coordinates."[1] In light of the discussion given above, any further comment on this statement would be superfluous, as one agrees that when the condition given by Matrix 2 is applied to the linear terms of Equations 1, one gets the orthogonal rotation matrix quite familiar to the readers. This matrix premultiplied by a scale factor would obviously represent the "general transformation of rectangular coordinates."

b. The second critical remark was stated as, "Mikhail's fears about the non-existence of second and third degree conformal transformation in three dimensions may be quite without foundation."[1] This statement was perhaps on regard to my remark "this treatment casts a shadow of doubt as to whether three-dimensional conformal transformation of second and higher degrees even exist. This fact is left for future investigation to prove its definitive validity."[2] Where I merely raised the doubt and left the door open for more investigation, I should have thought that Arthur would rather have considered the following more definite statement published by Schut: "Unfortunately, it turns out that the conditions (for conformality) can be satisfied only if the coefficients of all terms that are higher than the first degree with respect to x, y, and z are equal to zero. This proves that a conformal transformation in three dimensions which includes strip deformation *is not possible* by means of polynomials."[7] At any rate, as there was a disagreement with these views, the equations which were believed to exist were not presented.

Concurrent with the foregoing discussion, the following example representing "the well-known inversion with respect to a sphere" was cited:

$$X = k^2x/(x^2 + y^2 + z^2)$$
$$Y = k^2y/(x^2 + y^2 + z^2)$$
$$Z = k^2z/(x^2 + y^2 + z^2). \qquad (4)$$

Where, as mentioned in *a* above, we are only concerned with polynomials, the above example cannot support any argument and seems to be out of place as Equations 4 can hardly be considered as polynomials.

c. The last remark concerns the claim that the equations given in Reference 1 are "considerably more general" than those given in Reference 2. First, the principal application of the former is for the strip because they were developed for that purpose, while the latter, or this writer's equations, "were developed for an almost ideal situation of adjusting square sub-blocks"[2], and it was suggested that they "may not be limited to sub-blocks and can be used for strip adjustment, either directly or after minor modifications."[2] Secondly, it may be indeed appropriate to present at this point some of the results obtained using the writer's equations *as applied to strips* and *even without any modifications*. These results are obtained from tests performed subsequent to the publication of the original paper and reported at the last Congress of the International Society of Photogrammetry in Lisbon (Reference 4).

Table 1* shows the results obtained from a strip formed by using grid plates in the Wild Autograph A7. The strip is 17 models long at an equivalent scale of 1:40,000 and all ground coordinates are in meters. Three bands of three control points each were used in the beginning, middle and end of the strip. Each control point was fixed by all three coordinates.

For the sake of comparison, a test run on the same data was performed using the programs developed at the National Research Council of Canada (NRC) and currently in use by the Topographical Division of the Department of Mines and Technical Surveys.

* Photogrammetric Engineering, v. 31, n. 5, p. 917, September 1965.

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The results of this test are also included in Table I.

A second test was performed on data for the Savona Test Range taken from References 5 and 6. This Test Range is a strip some 8 miles in length along the south side of Kamloops Lake in Southern British Columbia (B.C.), Canada, and includes relief amounting to 10 per cent of the flying height. Table II* shows the results obtained on check points using the author's equations in comparison with the results published in the references cited.

The least one can say about the results displayed in the foregoing two tables is that they are certainly adequate. Therefore the questioning of the generality of this author's equations appears to be without foundation as it was shown that they apply not only to subblocks but also to strips.

IT IS REGRETTABLE that the criticisms have been based on scattered portions of the original article taken out of context rather than on the article *in toto*. It is therefore hoped that the foregoing discussion together with the original paper will help clarify this subject to the interested readers.

* Photogrammetric Engineering, v. 31, n. 5, p. 917, September 1965.

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ERRATA

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Because the January issue was already in press at the time of the final decisions, the subscription rates stated at the bottom of page 3 are incorrect, but they are correctly stated herein on page 163.

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