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Conformal Transformations in Three Dimensions

ABSTRACT: *The properties of rigorous conformal transformations of three-dimensional euclidean space are expressed by equations that put them in a form suitable for application to adjustment of photogrammetric triangulated strips. The first step of a practical calculation is to recognize whether the deformation of the strip does show actually the characteristics of conformality in space, criteria are established to demonstrate these characteristics in a simple manner. Attention is called to the fact that such rigorous transformations cannot contain more than ten parameters. This property is likely to impose on aerotriangulation more rigidity than commonly used interpolation methods do.*

FOREWORD: *This article and the following one by Mr. Schut deal with the same topic which has been discussed fervently in this Journal for the past year or two. All of these are considered to constitute valuable, practical and interesting information. The two authors agree that a second-degree conformal transformation in three dimensions is not possible. Schut's proof of this seems easier to follow than Baetslé's; however, Baetslé shows that second-degree transformation preserves the projected angles on the three planes rather than the solid angles. In this manner he confirms the opinion of Mikhail and also explains the usefulness of this type of transformation to photogrammetry. In addition to the references listed on pages 823 and 829 is an article by Authur, Jan. 1965, page 129. The Editor (who also specializes in aerotriangulation) takes advantage of the situation by adding references of his own: Tewinkel, Jan. 1965, page 180; Keller and Tewinkel, 1964 (GPO) C&GS Tech. Bull. No. 23.*—Editor.

UTILITY OF CONFORMAL TRANSFORMATIONS

FITTING AN AEROTRIANGULATED STRIP of photographs to the available ground control is usually performed nowadays by numerical methods using transformation formulas which express the corrections to be applied to the coordinates of all points of the strip, related to some uniform rectangular cartesian system, with x-axis in the direction of the line of flight, the y-axis horizontal, and h-axis vertical:

$$\Delta x = f_1(x, y, h);\ \Delta y = f_2(x, y, h);\ \Delta h = f_3(x, y, h). \quad (1)$$

Such general formulas are, in common practice, simplified by ignoring the influence of h:

$$\Delta x = \varphi_1(x, y);\ \Delta y = \varphi_2(x, y);\ \Delta h = \varphi_3(x, y). \quad (2)$$

Another simplification is usually adopted by splitting the problem into two stages; the first for planimetry (Δx, Δy), the second for height (Δh). Although these simplifications cannot be justified mathematically, it is recognized that the assumptions on which they are based may be admitted because the results obtained are sufficient for most practical cases. Moreover a more rigorous treatment would take into account all those quantities that are actually observed throughout the entire strip, and the great number of quantities involved would necessitate a volume of calculations which would be out of proportion to the internal accuracy of aerotriangulation, which in itself is relatively weak.

Nevertheless, one can adopt another point of view, and it is the purpose of the author to bring this problem into focus.

In expressions such as Equation 2, the planimetric transformation usually adopted is a *conformal* one; there is at least one theoretical reason for adopting conformal transformation. If one admits that the relative orientations of the successive models in the strip are

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as correct as possible, then, because the so-called aerotriangulation consists of transferring orientation angles and scale from each model to the next, one may interpret the deformations along the strip as caused by the connections of the models with each other and consider the angles as "locally correct." This property differs from that of conformal transformation only by the fact that the models are of finite dimensions, while the analytical transformation is an infinitesimal one. This procedure is used of necessity. Besides the theoretical justification, a practical one may be put forward. In many cases the deformations actually show the characteristics of conformity, at least to a first approximation.

These reasons are also valid for radial triangulation, as the author pointed out more than 25 years ago[1]; but in that purely planimetric method of bridging, an actual triangulation net was measured with all the characteristics of a geodetic one. The classical adjustment was used, and the only need for applying conformal transformation was the saving of time since computers were unavailable.

But for aerotriangulation in space, due to the great number of measured quantities, continuous transformations are extremely useful, and conformal formulas are used by many authors[2]. These are expressed by polynomials derived from limited expansions in series of analytic functions of a complex variable, such as:

$$\Delta z = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{3}$$

where

$$z = x + iy$$
$$c_k = a_k + ib_k. \tag{4}$$

Because the y-dimension is small compared to the x-coordinates, higher powers of y are sometimes neglected in the polynomials.[3]

There is no theoretical reason why the polynomials should be limited to the second or third power. In other words, the plane conformal transformation expressed by Equation 3 may include as many parameters as one desires. given n control points, the transformation can fit these points exactly, provided the polynomial includes all powers up to $(n-1)$. But the coefficients of the higher powers are meaningless; therefore the expansions are limited to the second or the third power. Consequently, if there are more than three of four control points, the coefficients are to be determined by some approximation, usually by the method of least squares. In this case the sum of the squares of the residuals $(\delta x, \delta y)$ are to be minimized where

$$\delta x = x + \Delta x - X, \quad \delta y = y + \Delta y - Y \tag{5}$$

and X, Y denote the coordinates of the control points. Because the residuals are not necessarily zero, the transformed strip does not fit the control points exactly.

Once a conformal transformation is recognized as a useful tool in the planimetric problem, the question may be raised if a conformal transformation could be extended to three-dimensional space providing a simultaneous adjustment of all three coordinates. This idea has been studied by E. M. Mikhail[2] using polynomials exclusively. The author states that it is not possible to express a strictly conformal transformation of three-dimensional euclidean space by these means. Faced with this impossibility, *which is a geometrical fact*, (the reason will be given below) E. M. Mikhail substitutes the condition that only the *projections* of the solid angles on the three coordinate planes should be preserved for the condition of preserving the solid angles themselves. Expressing these properties he obtains three polynomials (one for each coordinate) which contain *ten* independent parameters. It will now be stated in this paper that:

a. the rigorous conformal transformation of euclidean three-dimensional space cannot introduce more than *ten* independent parameters;

b. the formulas arrived at by E. M. Mikhail may be considered as the first approximation of the rigorous transformation.

The practical use of the rigorous formulation will also be discussed.

Formulas for the General Conformal Transformation

The most general conformal transformation of three-dimensional euclidean space is a product of translation, rotation and classical inversion. A summarized proof of this fundamental property is given in Appendix I. Such transformations form a group. In the inversion, one point p is chosen, known as the "pole" (Figure 1). The transform of any point m is a point M, lying on the straight line pm; the distance $R=pM$ is determined in the functional relationship to the distance $r=pm$ by

$$R \cdot r = Q \tag{6}$$

where the constant quantity Q is called the "power" of the inversion. Although the distance R may be reckoned from p to both sides of p on the line pm, it is more convenient to

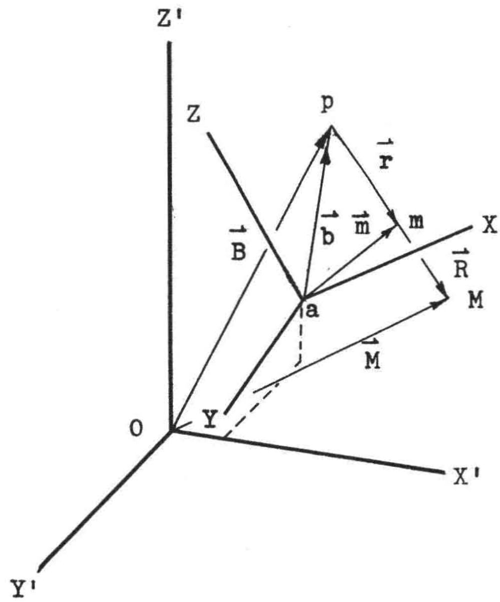

FIG. 1

consider the *vectors* $\boldsymbol{r}'$ and $\boldsymbol{R}'$, with their origin at p, and the power Q as being algebraically defined, with positive or negative sign. The choice of Q determines then the point M univocally. Denoting the scalar value of $\boldsymbol{r}'$ by r, the vectorial form of Equation 6 is

$$\boldsymbol{R} = Qr^{-2}\boldsymbol{r}. \tag{7}$$

An inversion is defined by *four* parameters—the three coordinates of the chosen pole and the value of Q. Obviously, translation and rotation with six parameters altogether for the most general euclidean displacement will preserve all solid angles. Consequently, the total number of parameters of the most general conformal transformation is precisely *ten*, as stated before. One might think that an additional parameter could be introduced by a modification of scale, which also preserves all the angles, but this factor is combined with Q, so that a modification of scale is already included in the choice of that quantity. The sign of Q has its effect on the orientation of solid angles, which, in the most general transformation, may be preserved or inverted.

In the inversion, the pole itself is transformed to a point at infinity, which must be considered as unique; with this convention any point is transformed to a unique point without exception. A sphere is transformed to a sphere. To visualize the generality of this assertion, one must consider a plane as being a sphere going through the point at infinity; its radius is infinite. As a sphere remains a sphere by translation and rotation, the general conformal transformation *preserves the spheres.* The intersection of two different spheres being a circle, the general transformation preserves the circles, which are to be generalized as "circles and straight lines" for the same reason that spheres are to be generalized as "spheres and planes." A straight line going through the pole remains a straight line; an actual circle going through the pole is transformed to a straight line; conversely, a straight line which does not contain the pole is transformed to an actual circle; the transform of an actual circle not passing through the pole is an actual circle.

The analytical expression for the most general conformal transformation can be established in vectorial form by applying to the right side of Equation 7 a translation (which is expressed by adding a constant vector $\boldsymbol{B}$) and a rotation which is obtained from premultiplying the vector $\boldsymbol{r}$ by a rotation matrix $\mathbf{S}$. This matrix contains *nine* elements (3×3); all are functions of *three* independent parameters. Then

$$\boldsymbol{B} + Qr^{-2}\mathbf{S}\boldsymbol{r}'. \tag{8}$$

Let the point a be the origin of a rectangular cartesian coordinate system for the space before transformation, and let us denote by $\boldsymbol{b}$ the vector $\boldsymbol{ap}$ (the components of this vector are the coordinates of the pole p). The coordinates of some point m in this space are the components of the vector

$$\boldsymbol{m} = \boldsymbol{b} + \boldsymbol{r} \tag{9}$$

The coordinates of the transformed point M of m are, according to Equations 8 and 9;

$$\boldsymbol{M} = \boldsymbol{B} + Qr^{-2}\mathbf{S}(\boldsymbol{m} - \boldsymbol{b}). \tag{10}$$

Note that all vectors symbolized with lower case letters are referenced to the original coordinate system while those symbolized by capital letters are referenced to the transformed coordinate system.

Now Q is a constant scalar factor and S is a constant matrix. We may write with only one symbol

$$\mathbf{K} = \mathbf{Q}\mathbf{S}, \tag{11}$$

$\mathbf{K}$ being also a square matrix (3×3) with the following property:

$$\mathbf{K}\mathbf{K}^{\tau} = \mathbf{K}^{\tau}\mathbf{K} = \mathbf{Q}^2\mathbf{I} \tag{12}$$

(the superscript τ denoting transposition). It is well known that $\mathbf{S}\mathbf{S}^{\tau} = \mathbf{S}^{\tau}\mathbf{S} = \mathbf{I}$ (unit matrix). These relations together with Equation 11 lead obviously to Equation 12. The transformation Equation 10 may now be written as follows:

$$M = B + r^{-2}\mathbf{K}(m - b). \tag{13}$$

To calculate the value of the scalar r from coordinates, use Equation 9; the scalar multiplication of $\boldsymbol{r} = \boldsymbol{m} - \boldsymbol{b}$ by itself gives

$$r^2 = b^2 + m^2 - 2(b \cdot m) \tag{14}$$

where $(\boldsymbol{b} \cdot \boldsymbol{m})$ denotes the scalar product of the vectors $\boldsymbol{b}$ and $\boldsymbol{m}$.

Although the following notations will not be necessary for the further development, the expansion is given for the sake of clarity. Let us write the equations equivalent to Equation 13 in an explicit form denoting by u_1, u_2, u_3 the components of any vector such as $\boldsymbol{u}$, and by K_{ij} the elements of K:

$$\begin{aligned} M_1 &= B_1 + [b_1^2 + b_2^2 + b_3^2 + m_1^2 + m_2^2 + m_3^2 - 2(b_1m_1 + b_2m_2 + b_3m_3)]^{-1} \\ &\quad \cdot \{K_{11}(m_1 - b_1) + K_{12}(m_2 - b_2) + K_{13}(m_3 - b_3)\} \\ M_2 &= B_2 + [\cdots]^{-1}\{K_{21}(m_1 - b_1) + K_{22}(m_2 - b_2) + K_{23}(m_3 - b_3)\} \\ M_3 &= B_3 + [\cdots]^{-1}\{K_{31}(m_1 - b_1) + K_{32}(m_2 - b_2) + K_{33}(m_3 - b_3)\}. \end{aligned} \tag{15}$$

The terms m_1, m_2, m_3 are the coordinates of the point to be transformed; b_1, b_2, b_3 are constants, being the coordinates of the pole; B_1, B_2, B_3 are the coordinates of a point P which is the transform of the point at infinity, and consequently the pole of the converse transformation. The symbol [. . .] stands for the expression between square brackets in the fully written value of M_1.

Practical Rigorous Formulas

Equations 13, 14, and 15 are sufficient to express the most general conformal transformation. We already see why the rigorous transformation cannot be expressed by polynomials. This impossibility is shown by the presence of the factor r^{-2}, which means the existence of a pole in the analytical sense of the word. Consequently the formulas must be put in a more practical form to be applied to our photogrammetric problem. In that problem we know that the deformations are relatively small; for instance, the x-axis of the strip, which becomes a circle if the deformation is conformal, will have a slight curvature. That means that the pole will lie at a very great distance. Coordinates such as b_i and B_i will therefore be very large numbers, a circumstance which is unfavorable for practical calculations. We must, therefore, try to introduce small quantities with the hope that their higher powers may be neglected if necessary.

When the deformations are small, the matrix $\mathbf{K}$ must have special properties. To investigate these properties, we first consider inversion alone, as expressed by Equation 7. What is the transform of an infinitesimal vector $\boldsymbol{dr}$, introduced as an increment of $\boldsymbol{r}$? Differentiating Equation 7 will provide the answer:

$$\boldsymbol{dR} = Q(-2r^{-3}dr\boldsymbol{r} + r^{-2}\boldsymbol{dr}). \tag{16}$$

But from

$$r^2 = (\boldsymbol{r} \cdot \boldsymbol{r}) \tag{17}$$

we get

$$\begin{aligned} rdr &= (\boldsymbol{r} \cdot \boldsymbol{dr}) \\ dr &= r^{-1}(\boldsymbol{r} \cdot \boldsymbol{dr}) \end{aligned}$$

and substituting in Equation 16 yields

$$\boldsymbol{dR} = \{Qr^{-2}\boldsymbol{dr} - 2r^{-2}(\boldsymbol{r} \cdot \boldsymbol{dr})\boldsymbol{r}\}. \tag{18}$$

In the last term we find the product of a scalar product $(\boldsymbol{r} \cdot \boldsymbol{dr})$ by a vector $\boldsymbol{r}$. This can be written in another form by introducing the "dyadic" product of vector $\boldsymbol{r}$ by itself. The matrix

$$\langle \boldsymbol{rr} \rangle = \begin{bmatrix} r_1^2 & r_1r_2 & r_1r_3 \\ r_2r_1 & r_2^2 & r_2r_3 \\ r_3r_1 & r_3r_2 & r_3^2 \end{bmatrix} \tag{19}$$

and Equation 18 becomes (see Appendix I)

$$\begin{aligned} \boldsymbol{dR} &= Qr^{-2}\{\boldsymbol{dr} - 2r^{-2}\langle \boldsymbol{rr} \rangle \boldsymbol{dr}\} \\ &= Qr^{-2}\{I - 2r^{-2}\langle \boldsymbol{rr} \rangle\}\boldsymbol{dr} \end{aligned}$$

or, if we put

$$\mathrm{L_r} = 2r^{-2}\langle \boldsymbol{rr} \rangle - \tag{20}$$

$$\boldsymbol{dR} = -Qr^{-2}\mathrm{L_r}\boldsymbol{dr}. \tag{21}$$

Now the matrix $\mathbf{L}_r$ transforms any vector into its image with respect to the straight line on which the vector $\boldsymbol{r}$ lies (see Appendix II). The unit vector on the line ar is $r^{-1}\boldsymbol{r}$, whereupon Equation 20 is identical to (II–7). Consequently, Equation 21 shows that the direction of the vector $\boldsymbol{dR}$ is symmetrical to that of $\boldsymbol{dr}$. The angle $(\boldsymbol{r}, \boldsymbol{dR})$ is thus equal to the angle $(\boldsymbol{r}, \boldsymbol{dr})$, and this is an analytical proof of the inversion being a conformal transformation.

If we apply the same calculation to the general conformal transformation expressed by Equation 8, because $\boldsymbol{B}$ and $\mathbf{S}$ are constant elements we get, instead of Equation 21,

$$d\boldsymbol{R} = -Qr^{-2}SL_r d\boldsymbol{r} \tag{22}$$

or using matrix $\mathbf{K}$ (Equation 11),

$$d\boldsymbol{R} = -r^{-2}KL_r d\boldsymbol{r}. \tag{23}$$

In particular, according to Equation 9 at the origin a of the coordinate system, $\boldsymbol{m}=0$; $\boldsymbol{r}=-\boldsymbol{b}$; then

$$d\boldsymbol{R}_a = -b^{-2}KL_b d\boldsymbol{r}_a. \tag{24}$$

If the deformations are small, $d\boldsymbol{R}_a$ must be approximately equal to $d\boldsymbol{r}_a$, which means that the matrix

$$-b^{-2}\mathbf{K}\mathbf{L}_b$$

must be nearly equal to the unit matrix. Therefore, since the inverse of $\mathbf{L}_b$ is $\mathbf{L}_b$ itself (see Appendix II),

$$\mathbf{K} \sim -b^2\mathbf{L}_b \tag{25}$$

or, according to Equation 20

$$K \sim b^2\{I - 2b^{-2}\langle \boldsymbol{bb}\rangle\}. \tag{26}$$

We are now able to introduce the small quantities we need for practical calculations.

Although the vector $\boldsymbol{ap}=\boldsymbol{b}$ is likely to be very large, we take the transform c of the pole p in an inversion with pole a and the power $Q=+1$. The vector $\boldsymbol{c}=\boldsymbol{ac}$ is given by Equation 7,

$$\boldsymbol{c} = b^{-2}\boldsymbol{b} \tag{27}$$

and, conversely, if c denotes the scalar value of $\boldsymbol{c}$,

$$\boldsymbol{b} = c^{-2}\boldsymbol{c}. \tag{28}$$

The scalar values of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are related by

$$bc = 1; \tag{29}$$

thus, if $\boldsymbol{b}$ is large, $\boldsymbol{c}$ is small.

The approximate value of $\mathbf{K}$ given by Equation 26 can now be written as follows

$$K \sim c^{-2}\{\mathbf{I} - 2c^{-2}\langle \boldsymbol{cc}\rangle\}, \tag{30}$$

and the exact value of $\mathbf{K}$ as the product of the approximate value pre-multiplied by a matrix $\mathbf{N}$. The $\mathbf{N}$ matrix is the product of a rotation matrix $\mathbf{W}$ and a scalar quantity n, which differs only slightly from unity. Then

$$\mathbf{N} = n\mathbf{W} \tag{31}$$

$$\mathbf{K} = c^{-2}\mathbf{N}\{\mathbf{I} - 2c^{-2}\langle \boldsymbol{cc}\rangle\}. \tag{32}$$

The rotation expressed by $\mathbf{W}$ may also be assumed as having a small amplitude.

To introduce Equation 32 in general Equation 13, we first calculate the vector

$$\begin{aligned}\boldsymbol{v} &= \{\mathbf{I} - 2c^{-2}\langle \boldsymbol{cc}\rangle\}(\boldsymbol{m} - \boldsymbol{b})\\ &= \{\mathbf{I} - 2c^{-2}\langle \boldsymbol{cc}\rangle\}(\boldsymbol{m} - c^{-2}\boldsymbol{c})\\ &= \boldsymbol{m} - c^{-2}\boldsymbol{c} - 2c^{-2}\langle \boldsymbol{cc}\rangle \boldsymbol{m} + 2c^{-4}\langle \boldsymbol{cc}\rangle \boldsymbol{c}.\end{aligned}$$

The last two terms can be transformed using Appendix I becoming

$$-2c^{-2}(\boldsymbol{c}\cdot\boldsymbol{m})\boldsymbol{c} + 2c^{-2}\boldsymbol{c}.$$

Therefore,

$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{m} + c^{-2}\boldsymbol{c} - 2c^{-2}(\boldsymbol{c}\cdot\boldsymbol{m})\boldsymbol{c}\\ &= \boldsymbol{m} + c^{-2}\boldsymbol{c}\{1 - 2(\boldsymbol{c}\cdot\boldsymbol{m})\}.\end{aligned} \tag{33}$$

Equation 13 can now be written as

$$\boldsymbol{M} = \boldsymbol{B} + c^{-2}r^{-2}\mathbf{N}\boldsymbol{v} \tag{34}$$

where r^2, given by Equation 14, is now to be expressed by means of $\boldsymbol{c}$:

$$\begin{aligned}r^2 &= c^{-2} + m^2 - 2c^{-2}(\boldsymbol{c}\cdot\boldsymbol{m})\\ &= c^{-2}[1 - 2(\boldsymbol{c}\cdot\boldsymbol{m}) + c^2m^2].\end{aligned} \tag{35}$$

Therefore,

$$c^2r^2 = 1 - 2(\boldsymbol{c}\cdot\boldsymbol{m}) + c^2m^2. \tag{36}$$

Using this in Equation 33 we get

$$\boldsymbol{v} = \boldsymbol{m} + (r^2 - m^2)\boldsymbol{c}$$

and Equation 34 becomes

$$\mathbf{M} = \boldsymbol{B} + c^{-2}r^{-2}\mathbf{N}[\boldsymbol{m} + (r^2 - m^2)\boldsymbol{c}]. \tag{37}$$

This gives for the transform of the origin to $\boldsymbol{m}=0$, $r=b$

$$\boldsymbol{B} + c^{-2}\mathbf{N}\boldsymbol{c} \tag{38}$$

and this again must be a small vector $\boldsymbol{d}$. Therefore, we put

$$\boldsymbol{B} = \boldsymbol{d} - c^{-2}\mathbf{N}\boldsymbol{c}; \tag{39}$$

Equation 37 gives now

$$\boldsymbol{M} = \boldsymbol{d} + c^{-2}r^{-2}\mathbf{N}(\boldsymbol{m} - m^2\boldsymbol{c}) \tag{40}$$

or, taking Equation 36 into account,

$$\boldsymbol{M} = \boldsymbol{d} + [1 - 2(\boldsymbol{c}\cdot\boldsymbol{m}) + c^2m^2]^{-1}\mathbf{N}(\boldsymbol{m} - m^2\boldsymbol{c}) \tag{41}$$

which is the desired equation; i.e., $\boldsymbol{c}$ and $\boldsymbol{d}$ are small vectors, $\mathbf{N}$ is the product of a rotation matrix with small amplitude multiplied by a scalar factor near unity. We have always our ten parameters; namely, three for $\boldsymbol{d}$, three for $\boldsymbol{c}$, and four in $\mathbf{N}$.

First Approximation

Considering $\boldsymbol{c}$ as being small enough, the linear approximation of the fractional factor gives

$$[1 - 2(\boldsymbol{c}\cdot\boldsymbol{m}) + c^2m^2]^{-1} \sim 1 + 2(\boldsymbol{c}\cdot\boldsymbol{m}). \tag{42}$$

Using an approximation of the same order in the expression of **N**,

$$\mathbf{N} \sim \begin{bmatrix} 1+n & -s_3 & s_2 \\ s_3 & 1+n & -s_1 \\ -s_2 & s_1 & 1+n \end{bmatrix}, \qquad (43)$$

we get explicitly, neglecting all products such as nc, cs, etc. . . .

$$\begin{aligned} M_1 &\sim d_1 + (1+n)m_1 - s_3m_2 + s_2m_3 + c_1m_1^2 - c_1m_2^2 - c_1m_3^2 + 2c_3m_1m_3 + 2c_2m_1m_2 \\ M_2 &\sim d_2 + s_3m_1 + (1+n)m_2 - s_1m_3 - c_2m_1^2 + c_2m_2^2 - c_2m_3^2 + 2c_3m_2m_3 + 2c_1m_1m_2 \\ M_3 &\sim d_3 - s_2m_1 + s_1m_2 + (1+n)m_3 - c_3m_1^2 - c_3m_2^2 + c_3m_3^2 + 2c_2m_2m_3 + 2c_1m_1m_3 \end{aligned} \qquad (44)$$

or, turning to the usual notations (X, Y, Z) for (M_1, M_2, M_3) and (x, y, z) for (m_1, m_2, m_3)

$$\begin{aligned} X &\sim d_1 + (1+n)x - s_3y + s_2z + c_1x^2 - c_1y^2 - c_1z^2 + 2c_3xz + 2c_2xy \\ Y &\sim d_2 + s_3x + (1+n)y - s_1z - c_2x^2 + c_2y^2 - c_2z^2 + 2c_3yz + 2c_1xy \\ Z &\sim d_3 - s_2x + s_1y + (1+n)z - c_3x^2 - c_3y^2 + c_3z^2 + 2c_2yz + 2c_1xy. \end{aligned} \qquad (45)$$

These equations are nothing else than those arrived at by E. M. Mikhail.[2] The correspondence of notation is as follows

Mikhail:	A_0	B_0	C_0	A	B	C	D	E	F	G
here:	d_1	d_2	d_3	$1+n$	$-s_3$	$-s_2$	$-s_1$	c_1	c_2	c_3

We see now that, as stated before, conformity in the three orthogonal projections on the coordinate planes as considered by E. M. Mikhail is equivalent to the linear approximation of rigorous conformality in space.

Let us repeat that rigorous conformality contains no more than *ten* parameters, which we have held in the linear approximation. A higher approximation would contain the same parameters. Introducing more coefficients would destroy conformality. Because the simple and practical form of Equation 41 of the rigorous conformal transformation, no need exists for determining polynomials of higher approximation. The linear form is able to give a first approximation of the parameters, and is at once suitable for treatment by least squares. A better approximation, if required, would be given by the rigorous formulas.

Criteria for Conformal Transformation

But before calculating parameters, one should investigate whether the deformation of the strip does actually show the characteristics of a conformal one with a sufficient likelihood. It is now our aim to establish some criteria to facilitate this investigation.

The fundamental Equation 41 can be written as

$$\boldsymbol{M} = \boldsymbol{d} + \mathbf{N}\boldsymbol{M}^* \qquad (46)$$

whereby we have put

$$\boldsymbol{M}^* = [1 - 2(\boldsymbol{c}\cdot\boldsymbol{m}) + c^2m^{2-1}](\boldsymbol{m} - m^2\boldsymbol{c}). \qquad (47)$$

Equation 46 expresses the linear part of the transformation; it includes translation $\boldsymbol{d}$, rotation and scale modification **N**. Equation 47 contains the non-linear part and can be interpreted as "inversion plus symmetry." There are three parameters in it, the components of the vector $\boldsymbol{c}$.

The transformation expressed by Equation 46 is the linear conformal one and is well known. If $\boldsymbol{c}=0$, the general transformation reduces to the linear one. It contains seven parameters and possesses many invariants which can be used as criteria to recognize it, such as angles between straight lines, ratio of two distances, etc. It will be only when these criteria are not fulfilled that one should investigate if the deformation is more generally conformal. The simple criteria will be those which are independent of $\boldsymbol{d}$ and **N**, in other words, those which are invariant by a linear conformal transformation. They should determine, at least in a first approximation, the values of c_1, c_2, c_3. These represent the change of orientation along the strip, the change of scale along the strip, and the curvature of some deformed straight line.

Change of Orientation

Equation 23 gives, in the general conformal transformation, the transform $\boldsymbol{dR}$ of an infinitesimal vector $\boldsymbol{dr}$. If we replace the matrix **K** by its expression Equation 32, we get

$$\boldsymbol{dR} = -c^{-2}r^{-2}\mathbf{N}\{\mathbf{I} - 2c^{-2}\langle\boldsymbol{cc}\rangle\}\mathbf{L}_r\boldsymbol{dr} \qquad (48)$$

or, using the symmetry matrix $\mathbf{L}_c = \mathbf{L}_b$ according to Appendix III,

$$\boldsymbol{dR} = c^{-2}r^{-2}\mathbf{N}\mathbf{L}_b\mathbf{L}_r\boldsymbol{dr}. \qquad (49)$$

The non-linear part of $\boldsymbol{dR}$, according to Equations 46 and 47, is

$$\boldsymbol{dR} = k\mathbf{L}_b\mathbf{L}_r\boldsymbol{dr} \qquad (50)$$

where

$$k = c^{-2}r^{-2} = [1 - 2(\boldsymbol{c}\cdot\boldsymbol{m}) + c^2m^2]^{-1} \quad (51)$$

is a scalar quantity which is of no interest in the direction of our vectors.

We can interpret Equation 50 as follows. If $\boldsymbol{dR}$ and $\boldsymbol{dr}$ are translated to a common origin, $\mathbf{L}_r\boldsymbol{dr}$ is symmetrical to $\boldsymbol{dr}$ with respect to the direction of $\boldsymbol{r}$, and $\mathbf{L}_b\mathbf{L}_r\boldsymbol{dr}$ is again symmetrical to $\mathbf{L}_r\boldsymbol{dr}$ with respect to the direction of $\boldsymbol{b}$. Now, the product of two symmetries is a rotation with amplitude equal to twice the angle between the axes of symmetry. The axis of rotation is perpendicular to both axes of symmetry, and, in particular, to $\boldsymbol{b}$, the direction of which is our unknown. The datum of our problem consists of the direction of vectors like $\boldsymbol{dr}$ and $\boldsymbol{dR}$ as given by their direction cosines. One pair of such vectors is not sufficient to determine the direction of the axis of the rotation which brings one vector on the other. If we had two pairs of vectors, the problem can be solved, but we must be sure that the axis of rotation has the same direction for the second pair also. This is obtained when $\boldsymbol{b}$, $\boldsymbol{r}$ (for the first pair) and $\boldsymbol{r}'$ (for the second pair) lie in the same plane. Remember that $\boldsymbol{b}$ is the vector $\boldsymbol{ap}$, where a denotes the origin, and p the pole, and that $\boldsymbol{r}$ is the vector $\boldsymbol{pm}$, where m is the point under consideration on the nondeformed model; if the second point m' where a direction is known lies on the straight line am, then $\boldsymbol{b}$, $\boldsymbol{r}$ and $\boldsymbol{r}'$ will be coplanar wherever p lies. Our conclusion is then that we must know the alteration of directions in three points a, m, m' of the strip lying on the same straight line. In practical cases, the alteration will be zero at the origin, and the straight line will be the x-axis.

In this particular case, the direction of rotation can be found as follows: if $\boldsymbol{u}$ and $\boldsymbol{U}$ are corresponding unit vectors, the first in the space of control points m, the latter in the deformed space M, the axis of the rotation sought must lie in the bisecting plane of both vectors. (There are two such planes, but because only the small angle is to be considered, the correct one can easily be identified.) Similarly, vectors $\boldsymbol{u}'$ and $\boldsymbol{U}'$ corresponding to the points m', M' will give a second plane. The axis of rotation is then obtained as the intersection of the two planes. Lastly, because the vector $\boldsymbol{c}$ is perpendicular to the axis of rotation, one gets, between the components of $\boldsymbol{c}$, a linear relation, such as

$$C_1c_1 + C_2c_2 + C_3c_3 = 0. \quad (52)$$

The most general case can always be reduced to the particular one by an adequate rotation.

CHANGE OF SCALE

In the pure inversion, the scale factor e is immediately given by differentiating Equation 6

$$e = \left|\frac{dR}{dr}\right| = |Q|\, r^{-2}. \quad (53)$$

As stated before, Q may be regarded as containing also the scale factor in the general transformation, and consequently Equation 53 remains valid in all cases. From Equation 12 we see that Q is the absolute value of the determinant of $\mathbf{K}$, and Equations 31 and 32 show that this value is

$$|Q| = nc^{-2}. \quad (54)$$

Therefore,

$$e = \left|\frac{dR}{dr}\right| = nc^{-2}r^{-2}$$

and from Equation 36

$$f = \frac{1}{e} = n[1 - 2(\boldsymbol{c}\cdot\boldsymbol{m}) + c^2m^2]. \quad (55)$$

This means that along a straight line in the nondeformed space the inverse of the scale factor is a quadratic function, or, conversely, that along a straight line in the deformed space the scale factor itself is a quadratic function.

In practical cases, e will be unity at the origin ($\boldsymbol{m}=0$), so that $n=1$, and the straight line will be the x-axis. In any point thereon $(m_1, 0, 0)$, Equation 55 becomes

$$f = 1 - 2c_1m_1 + c^2m_1^2. \quad (56)$$

This function shows a minimum for $m_1 = c_1c^{-2} = b_1$ (see Equation 28). Such peculiarity has never been reported, but b_1 is nothing but the abscissa of the pole, which, as we know, must be very far away, and is therefore likely to lie outside the x-range of the strip. This indicates at the same time that Equation 56 will furnish a good value for c_1, but a poorer one for c.

CURVATURE OF A DEFORMED STRAIGHT LINE

We know from the properties of inversion that any straight line not passing through the pole will be a circle. If l is the distance from the pole to some straight line, the radius of the transform is given by

$$\rho = Q/2l. \quad (57)$$

This gives for the curvature, according to Equation 54

$$\gamma = 1/\rho = 2n^{-1}lc^2. \quad (58)$$

If the straight line is the x-axis,

$$l = (b_2^2 + b_3^2)^{1/2}$$

or, from Equation 28,

$$l = c^{-2}(c_2^2 + c_3^2)^{1/2}. \tag{59}$$

therefore

$$\gamma = 2n^{-1}(c_2^2 + c_3^2)^{1/2}. \tag{60}$$

For $n=1$,

$$\gamma = 2(c_2^2 + c_3^2)^{1/2}. \tag{61}$$

CALCULATION OF THE VECTOR $\boldsymbol{c}$

We have collected so far three sorts of information about the vector $\boldsymbol{c}$. The easiest to express are those given by the scale factor (Equation 56) and the curvature of the x-axis (Equation 61). We may hope that Equation 56 will give a good value of c_1, and Equation 61 a good value for $(c_2^2+c_3^2)^{1/2}$; these are respectively the longitudinal and the transverse components of $\boldsymbol{c}$. To separate c_2 and c_3 it is then sufficient to introduce Equation 52, and to solve the system (Equations 52, 61).

Other information might be called upon to provide redundant equations, but let us remember that we are only seeking the criteria for conformality and not calculating the transformation without knowing that it can be conformal.

Conclusion

If the deformation of a triangulated strip is likely to be conformal in space, closed formulas with only ten parameters can be written to express the transformation (Equation 41). Before calculating these parameters, it is advisable to test the conformality by some criteria which are given in the foregoing section, leading to values of a first approximation of the three parameters c_1, c_2, c_3. Applying then the transformation expressed by Equation 47, if the deformation is nearly conformal, the residuals will show that there is some linear transformation such as Equation 46 which will fit the strip to the control points with reasonable accuracy. This leads to the determination of the seven parameters contained in that transformation, which is the same problem as absolute orientation of a single model. Improved values of all ten parameters can then be obtained by least squares using Equation 41 and the first approximation of the parameters, leading to a linear form in the usual way.

The author would be grateful to colleagues who try to apply the proposed method will provide him information concerning actual triangulated strips which show that the deformation can be regarded as conformal in space.

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Appendix I

PROOF OF THE IDENTITY: $\boldsymbol{u}(\boldsymbol{v}\cdot \boldsymbol{w}) = \boldsymbol{uvw}$ $(\boldsymbol{v}\cdot \boldsymbol{w})$ denotes the scalar product:

$$(v\cdot w) = v_1w_1 + v_2w_2 + v_3w_3; \tag{I-1}$$

$\langle \boldsymbol{uv}\rangle$ denotes the dyadic product:

$$\langle \boldsymbol{uv}\rangle = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix} \tag{I-2}$$

Both sides of the asserted equality are vectors the components of which are obviously:

$$u_1(v_1w_1 + v_2w_2 + v_3w); \quad u_2(v_1w_1 + v_2w_2 + v_3w_3);$$
$$u_3(v_1w_1 + v_2w_2 + v_3w_3) \text{ q.e.d.} \tag{I-3}$$

Using pure matrix-notation, denoting u, v, w as column vectors

$$(\boldsymbol{v}\cdot \boldsymbol{w}) = \mathbf{u}^T\mathbf{w} \tag{I-4}$$

$$\langle \boldsymbol{uv}\rangle = \mathbf{uv}^T. \tag{I-5}$$

If both sides ($\mathbf{u}$ and $\mathbf{w}$) from the original identity are written as column vectors;

$$\mathbf{uv}^T\mathbf{w} \tag{I-6}$$

and the equality results immediately from the associative law applied to matrix-multiplication.

Appendix II

THE SYMMETRY-MATRICES $\mathbf{L}$

Given in a three-dimensional euclidean space, a straight line s and a point u, determine the point v symmetrical to u with respect to s.

To solve this problem with vectors, select some point g on s, and consider the vector $\boldsymbol{gu}=\boldsymbol{u}$. Let $\boldsymbol{w}$ be the vector which is the orthogonal projection of $\boldsymbol{u}$ on s. Then, the vector $(\boldsymbol{u}-\boldsymbol{w})$ must be perpendicular to s or, if $\boldsymbol{s}$ is a unit-vector on s, the scalar product $\boldsymbol{s}\cdot(\boldsymbol{u}-\boldsymbol{w})$ must be zero. But $\boldsymbol{w}$, lying also on s, is proportional to $\boldsymbol{s}$:

$$w = ks. \qquad \text{(II–1)}$$

Whence

$$s\cdot(u - ks) = 0, \qquad \text{(II–2)}$$

and because $\boldsymbol{s}\cdot\boldsymbol{s}=1$,

$$k = (s\cdot u) \qquad \text{(II–3)}$$

$$w = s(s\cdot u). \qquad \text{(II–4)}$$

If $\boldsymbol{v}$ is the vector $\boldsymbol{gv}$,

$$v - u = 2(w - u)$$

$$v = 2w - u = 2s(s\cdot u) - u \qquad \text{(II–5)}$$

or, according to Appendix I,

$$v = 2\langle ss\rangle u - u = \mathbf{L}_s u. \qquad \text{(II–6)}$$

If we use **I** denoting the unit-matrix,

$$\mathbf{L}_s = 2\langle ss\rangle - \mathbf{I}. \qquad \text{(II–7)}$$

Interpreting Equation II–6, we see that $\mathbf{L}_s$ is the matrix by which one must premultiply a vector $\boldsymbol{u}$ to find the vector $\boldsymbol{v}$. The latter is symmetrical to $\boldsymbol{u}$ with respect to the straight line and goes through the origin of $\boldsymbol{u}$; it also contains the unit vector $\boldsymbol{s}$. In pure matrix-notation:

$$\mathbf{L}_s = 2\mathbf{s}\mathbf{s}^T - \mathbf{I}. \qquad \text{(II–8)}$$

Obviously, the square of $\mathbf{L}_s$ is the unit-matrix and the inverse of $\mathbf{L}_s$ is L_s itself, since the symmetrical of $\boldsymbol{v}$ is $\boldsymbol{u}$. These properties can easily be verified analytically using Equation II–8.
