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Conformal Transformations and Polynomials*

Although needed for aerotrangulated strips, they are not possible for higher degrees and in three dimensions.

INTRODUCTION

IN A PAPER PUBLISHED in the March 1964[1] issue of PHOTOGRAMMETRIC ENGINEERING, this writer made the statement that conformal strip transformations by means of second- and higher-degree polynomials in the strip coordinates are not possible in three dimensions.

In two publications in subsequent issues, July 1964 and January 1965, Mikhail,[2] and Arthur,[3] discuss the possibility of such a transformation, but do not reach any firm conclusion. Mikhail, who knew of the above statement through earlier contacts with this writer, is doubtful about the possibility, whereas Arthur, who corrects what appears to be an error in name rather than in concept in Mikhail's publication, is unduly optimistic.

Because conformal transformations by means of polynomials have proved to be very useful for the horizontal adjustment of strips and blocks, the use of such transformations for simultaneous three-dimensional adjustment would deserve contemplation. It is therefore of interest to remove the uncertainty which may be caused by the two latter publications by giving the proof of the non-existence of such transformations at least for polynomials of the second degree. The proofs for higher degrees are similar, although progressively more complicated, and will be left to the reader.

THE PROOF

The proof is rather simple if vector algebra is used. The linear conformal transformation in three dimensions can be represented by the vector equation

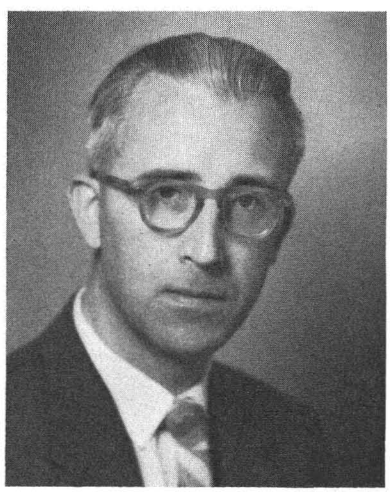
$$v = a + bX + cY + dZ. \tag{1}$$

Here, X, Y, and Z are strip coordinates, v is a vector whose three components are the three transformed coordinates, and a, b, c, and d are vectors whose components are the coefficients in the transformation formulas.

The vectors b, c, and d are the column vectors of the matrix of the transformation. This matrix serves to perform a change of scale and a rotation. Therefore, it is the product of an orthogonal matrix and a scale factor. Consequently, if λ is this scale factor,

$$b\cdot b = c\cdot c = d\cdot d = \lambda^2$$
$$b\cdot c = c\cdot d = d\cdot b = 0 \tag{2}$$

Inasmuch as the column vectors and also the row vectors of this matrix are mutually orthogonal, the matrix ought to bear the name of orthogonal matrix. This is not permitted because this name is conventionally used only for the matrix whose vectors are not only mutually orthogonal but also have unit length.



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* See FOREWORD of preceding article by Baetslé.

The most general polynomial transformation of the second degree is obtained from Equation 1 by removing the restrictions imposed on b, c, and d, and adding all possible second-degree terms:

$$v = a + bX + cY + dZ + eX^2 + fY^2 + gZ^2 + hYZ + iZX + jXY. \quad (3)$$

Differentiation of this equation with respect to X, Y, and Z gives the change dv which the vector v undergoes as the result of differential changes dX, dY, and dZ in the strip coordinates:

$$dv = (b + 2eX + jY + iZ)dX + (c + jX + 2fY + hZ)dY + (d + iX + hY + 2gZ)dZ. \quad (4)$$

ABSTRACT: *Conformal transformations by means of polynomials have been very useful for the horizontal adjustment of strips and blocks of aerial photographs, and their use for a simultaneous three-dimensional adjustment would deserve contemplation. However, conformal transformations by means of second- and higher-degree polynomials are not possible in a three-dimensional coordinate system. A proof of this statement includes the demonstration that the possibility of such transformations in two dimensions is a result of special mathematical rules. A conformal transformation of second- and higher-degree is possible in two-dimension space because the coordinates can be regarded as the parameters of a special type of matrix in which addition and multiplication produce another matrix of the same type, and multiplication is commutative. In three-dimensional space, this is not the case.*

The Transformation 3 is conformal if for an infinitely small area around any point the transformation causes only a change of scale and a rotation, but no deformation. Therefore, it is conformal if and only if for each point with coordinates X, Y, and Z Equation 4 represents a linear conformal transformation.

Consequently, if the coefficients of dX, dY and dZ are called p, q, and r, it is necessary and sufficient that for any value of X, Y, and Z:

$$p \cdot p = q \cdot q = r \cdot r$$

$$p \cdot q = q \cdot r = r \cdot p = 0. \quad (5)$$

The scalar products in these equations are second-degree polynomials in X, Y, and Z. Therefore, if in each Equation 5 all the terms are brought to the first part of the equation, each equation states that the value of a second-degree polynomial in X, Y, and Z must be equal to zero for any values of X, Y, and Z. This is possible if and only if the coefficient of each term is equal to zero.

These coefficients, insofar as they are needed for the proof, are listed in Tables 1 and 2.

TABLE 1

Equation	$p \cdot p - q \cdot q = 0$	$q \cdot q - r \cdot r = 0$
Coefficient of X^2	$4e \cdot e - j \cdot j = 0$	$j \cdot j - i \cdot i = 0$
of Y^2	$j \cdot j - 4f \cdot f = 0$	$4f \cdot f - h \cdot h = 0$
of Z^2		$h \cdot h - 4g \cdot g = 0$

The coefficients in Table 1 show that the vectors $2e$, $2f$, $2g$, h, i, and j must have the same length. None can be equal to zero, since otherwise all would be equal to zero and the transformation would have no second-degree terms.

The coefficients in the first three rows of Table 2 show that since e, i, and j cannot be equal to zero, they must be mutually orthogonal. Also, h is orthogonal to i and to j, and therefore e and h must be parallel. Similarly, it follows that f and i must be parallel and that g and j must be parallel.

Each of the coefficients in rows 4 and 5 of Table 2 contains one term which according to the equations in the first three rows is equal to zero, and therefore the second term must also be equal to zero. In each case, this second term is a scalar product of two parallel vectors. It can be equal to zero only if at least one

TABLE 2

Equation	$p \cdot q = 0$	$q \cdot r = 0$	$r \cdot p = 0$
Coefficient of X^2	$e \cdot j = 0$	$j \cdot i = 0$	$i \cdot e = 0$
of Y^2	$j \cdot f = 0$	$f \cdot h = 0$	$h \cdot j = 0$
of Z^2	$i \cdot h = 0$	$h \cdot g = 0$	$g \cdot i = 0$
of XY		$h \cdot j + 2f \cdot i = 0$	$j \cdot i + 2e \cdot h = 0$
of YZ			$i \cdot h + 2g \cdot j = 0$

of the vectors is equal to zero. Therefore, at least three of the vectors e, f, g, h, i, and j must be equal to zero.

According to the conclusions derived from Table 1, this means that all six vectors must be equal to zero. Therefore, the transformation can have no second-degree terms!

An Explanation

It may appear strange that a conformal transformation by means of second- and higher-degree polynomials in the strip coordinates is possible in two-dimensional space but not in three-dimensional space. The explanation is that this is a consequence of rules which are valid for a special type of matrices of the second order, but not for such matrices of the third order.

The conformal transformation in two-dimensional space can be written with complex numbers as

$$X_t = a + bX + cX^2 + \cdots \quad (6)$$

where X represents the complex number $X+iY$, the subscript t indicates transformed coordinates, and the coefficients a, b, c, $\cdots$ are also complex numbers.

Now, to any complex number a_1+ia_2 corresponds a unique matrix

$$\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}.$$

The rules for addition and multiplication of complex numbers have been chosen in such a way that the result of the operation corresponds to the result of the same operation performed on the corresponding real matrices. Therefore, the symbols in the transformation equation can represent such matrices as well as complex numbers.

The transformation is conformal if and only if for each X the differentiated equation

$$dX_t = b(dX) + c(dX)X + cX(dX) + \cdots \quad (7)$$

represents a conformal transformation. Since multiplication of these matrices (and of complex numbers) is commutative,

$$dX_t = (b + 2cX + \cdots)dX. \quad (8)$$

This shows that the differentiated equation is of the type $X_t=a+bX$ and, therefore, does represent a conformal transformation, as required.

In four-dimensional space, each symbol in the transformation Equation 6 can represent either a quaternion or a matrix, for instance

$$(T, X, Y, Z) \text{ and } \begin{bmatrix} T & -X & -Y & -Z \\ X & T & Z & -Y \\ Y & -Z & T & X \\ Z & Y & -X & T \end{bmatrix}.$$

As in two-dimensional space, this matrix is the product of an orthogonal matrix and a scale factor. Therefore, the linear transformation $X_t=a+bX$ can be said to be conformal.

Here also, the sum and the product of two such matrices is again a matrix of the same type, and the rules for addition and multiplication of quaternions have been chosen in accordance with the rules for these matrices.

However, the multiplication of two such matrices, and of two quaternions, is not commutative. Therefore, the differentiated equation can be written only in the first of the above two forms. It cannot in any way be reduced to the second form, in which the coefficient of dX is the product of an orthogonal matrix and a scale factor. As a result, the second- and higher-degree polynomial transformations are not conformal here.

In three-dimensional space, the coefficient b in Equation 6 can represent the transformation matrix, and as before the coordinates X, Y, and Z can be written as the first column of a matrix which is the product of an orthogonal matrix and a scale factor. However, now the elements in the second and third column cannot be rational functions of X, Y, and Z. As a result, the transformation equation of second- or higher-degree is not a polynomial in X, Y, and Z. Also, addition of two such matrices does not produce a matrix with the same properties. Further, the multiplication of two such matrices is not commutative. As a result, the differentiated equation can be written only in the first of the above two forms, the transformation is not conformal, and it is not even a polynomial transformation.

Summarizing, a conformal transformation of second- and higher-degree in the strip coordinates is possible in two-dimensional space because the coordinates can be regarded as the parameters of a special type of matrix. These matrices have the properties that addition and multiplication produce another matrix of the same type, and the multiplication is commutative. In three-dimensional space, this is not the case.

Consequently, there is no reason to expect that such a transformation should be possible here.

References

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