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Projective Methods

Resection is solved in a short, direct manner using four or more control points where one is near the center.

(*Abstract on next page*)

FOREWORD. *A numerical example of this method applied to the determination of the tilt of an aerial photograph is included in the Third Edition of the* Manual of Photogrammetry, *page 43, 1966, in which it was noted that the original work had not been published. The procedure was also described by Mr. D. T. Giles, Jr., "Space Resection and Orientation by the Morse Method," a memorandum, Cornell Aeronautical Laboratory, Inc., Arlington, Virginia, 1961. The system is relatively easy to compute, the mathematics is rigorous, and the accuracy of the results depends only on the amount, accuracy and character of the input. In those applications where it is best suited, the method probably constitutes the easiest precise solution in existence. Even though publication is a quarter of a century late, we are pleased to present Professor Morse's original paper.*—Editor

1. THE CAMERA AND PROBLEM

THE ESSENTIALS OF THE CAMERA are a lens which operates as a point O through which rays of light from ground points pass and strike a plate. The distance from O to the plate is called the *principal distance* and designated by f. The foot of a perpendicular from O on the plate is called the *principal point* M. It is determined as the intersection of two perpendicular lines on the plate. These lines are themselves determined by marks on the plate made by the photographing apparatus. The length f is most important as it enters essentially as a factor of the height h of the camera above the ground in the formula giving h.

The constant f should be known with a relative error less than that desired in h. By the *relative error* of h we mean $\Delta h/h$ where Δh is the error in h; similarly with f. The constants of the camera may be determined at the National Bureau of Standards at Washington.

Points on the ground are referred to a rectangular system of coordinates u, v, w known as *ground coordinates*, with $w=0$, a convenient horizontal plane. Points on the plate are referred to an arbitrary rectangular system x, y, z of which $z=0$ gives the plate. Viewed from above, we suppose that the u-axis rotates into the v-axis counterclockwise. Similarly with the x- and y-axes. The origin on the plate is sometimes taken at the principal point M and sometimes at a point p_0 whose image on the ground is known.

The ground coordinates of the lens point will be denoted by U, V, h. Theoretically they are determined, possibly multiply, when three ground points and their images on the plate are known together with f, and the principal point M. If, however, four or more *ground* points and their images on the plate are known, the problem of finding U, V, h can be given a relatively simple explicit solution, as we shall see, as distinguished from the complicated successive approximations required to find U, V, h when only three points are known.

We shall start with the assumption that the ground points are at the same elevation. This can be brought about in fairly level country by raising or lowering discs representing the points. Or if this is impractical, a preliminary correction for the difference in elevation can be made as will be indicated in §7. This preliminary correction is easily made if the plate is tilted from the horizontal by at most 2°. In practice the average tilt comes out less than 1°. The preliminary correction is made by first calculating U, V, h as if the plate were level, and then proceeding as follows. If p' is the true image on the plate of a known ground

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point $q'=(u, v, w)$, p' is replaced mathematically by the image p on the plate of $q=(u, v, O)$. One then proceeds with p and q as if q were given in the plane $w=0$. This correction of p' to obtain p is absolutely correct if the plate is horizontal, and is accurate in practice beyond the accuracy of reading the plate if the tilt is at most 1° and if $w<20$ feet.

In the following, we first show how to calculate the tilt angle t. The angle t is the angle which the normal to the plate makes with the vertical.

2. Determination of the Tilt Angle t

In determining t we also determine the *principal direction* on the plate. This is the direction on the plate of steepest ascent relative to the horizontal. It is orthogonal to the horizontal lines on the plate. We measure all angles on the plate in counterclockwise sense as seen from above. Let φ be the angle from the positive x-axis to the principal direction. We shall give $\cos\varphi$ and $\sin\varphi$.

We start with four known ground points q_0, q_1, q_2, q_3 with no three points collinear. A good distribution would be for q_1, q_2, q_3 to form a very rough equilateral triangle and for q_0 to be very roughly near the center of the triangle. We shall see that five or more points are better than four points, so that we are now proceeding in a preliminary way. Let p_i be the image of q_i on the plate. We shall take p_0 as the origin on the plate and q_0 as the origin on the ground. The directions of the u-axis and x-axis are arbitrary and independent.

The lens point O may be regarded as a point of *perspective* by virtue of which a point u, v, O on the ground and its image x, y on the plate suffer a non-singular collineation. (See Bocher, Introduction to Higher Algebra pp. 68–74, MacMillan, 1907). The form is

$$u = (a_1x + b_1y)/(ax + by + 1)$$

$$v = (a_2x + b_2y)/(ax + by + 1) \qquad (2.1)$$

the constant terms in the numerators being zero because origin corresponds to origin. We need to determine a and b. The fundamental theorem here is as follows.

THEOREM 2.1. *If* $(\bar{x},\bar{y})$ *are the coordinates of the principal point, then*

$$\cos\varphi = a/(a^2+b^2)^{1/2} \qquad \sin\varphi = b/(a^2+b^2)^{1/2} \qquad (2.2)$$

$$\tan t = [f(a^2+b^2)^{1/2}]/(1 + a\bar{x} + b\bar{y}) \qquad (2.3)$$

In practice $f=200$ mm. approximately, $t<1°$ the plane is flying at a height of one to five miles $\bar{x}$ and $\bar{y}$ are small relative to f and a and b are less than 10^{-3}. Thus the term $ax+by$ is small.

Before proving Theorem 2.1 we shall give the theorem evaluating a and b.

Let u_i, v_i be the coordinates of q_i and x_i, y_i the coordinates of p_i, $(i=1, 2, 3)$. Set

$$d_{0jk} = (x_jy_k - x_ky_j), \qquad (j \neq k) \qquad (2.4)$$

where j and k range over the values 1, 2, 3. Equation 2.4 expresses twice the area of the triangle $p_0p_jp_k$, positive if these points run in counterclockwise sense, negative otherwise. Similarly set

$$D_{0jk} = (u_jv_k - u_kv_j), \qquad (j \neq k) \qquad (2.5)$$

$$D = D_{012} + D_{023} + D_{031}. \qquad (2.6)$$

The number D is twice the area of the triangle $p_1p_2p_3$, positive if these points run in counterclockwise sense, negative otherwise. The numbers from Equations 2.5 and 2.6 should be permanently known and tabulated. The three numbers from Equation 2.4 will be calculated when needed. This is all that is needed to find a and b,

ABSTRACT. *A relatively short direct method is used to determine the tilt data for an aerial photograph based on four or more ground control stations of nearly equal elevation where one of them is near the center. The derivation, based on principles of projective geometry, utilizes the ratios of areas on the photograph relative to the ground of central-point triangles. The solution determines the tilt angle first, then the flight height, nadir point, and camera coordinates. A means is provided if the elevations of the control points differ. The system is particularly well suited for testing instruments repeatedly over the same area.*

THEOREM 2.2. *The constants* a *and* b *are given by the formulas*

$$a = (1/D)[(y_1 - y_2)(D_{012}/d_{012}) + (y_2 - y_3)(D_{023}/d_{023}) + (y_3 - y_1)(D_{031}/d_{031})]$$
$$b = (-1/D)[(x_1 - x_2)(D_{012}/d_{012}) + (x_2 - x_3)(D_{023}/d_{023}) + (x_3 - x_1)(D_{031}/d_{031})]. \tag{2.7}$$

Note that the subscripts advance cyclically. The ratios D_{012}/d_{012} etc. are all positive regardless of the order of the points, but D may be positive or negative. For the applications, the details of the following proof are not necessary.

PROOF OF THEOREM 2.2

To determine a and b note that

$$(ax_i + by_i + 1)u_1 = a_1x_i + b_1y_i \qquad (i = 1, 2, 3) \tag{2.8}$$

We regard these equations as linear and homogeneous in a_1, b_1, and 1, the last constant multiplying the left members. Hence the determinant of the coefficients of a_1, b_1, 1 must vanish. Thus

$$\begin{vmatrix} ax_1u_1 + by_1u_1 + u_1 & x_1 & y_1 \\ ax_2u_2 + by_2u_2 + u_2 & x_2 & y_2 \\ ax_3u_3 + by_3u_3 + u_3 & x_3 & y_3 \end{vmatrix} = 0. \tag{2.9}$$

Expanding,

$$0 = a\begin{vmatrix} x_1u_1 & x_1 & y_1 \\ x_2u_2 & x_2 & y_2 \\ x_3u_3 & x_3 & y_3 \end{vmatrix} + b\begin{vmatrix} y_1u_1 & x_1 & y_1 \\ y_2u_2 & x_2 & y_2 \\ y_3u_3 & x_3 & y_3 \end{vmatrix} + \begin{vmatrix} u_1 & x_1 & y_1 \\ u_2 & x_2 & y_2 \\ u_3 & x_3 & y_3 \end{vmatrix}. \tag{2.10}$$

Equation 2.10 is *one* condition on a and b. We write it in the form

$$A_1a + B_1b + C_1 = 0 \tag{2.11}$$

where A_1, B_1, C_1 are the respective determinants in (2.10). Equations 2.9 and 2.10 hold if we replace u_i by v_i, $i = 1, 2, 3$. Equation 2.11 then takes the form

$$A_2a + B_2b + C_2 = 0, \tag{2.12}$$

where A_2 is obtained from A_1 by replacing u_i by v_i, and B_2 from B_1, C_2 from C_1 similarly. Solving Equations 2.11 and 2.12 for a and b we find that

$$a = \frac{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}} \qquad b = \frac{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{2.13}$$

Explicitly,

$$\begin{aligned}
\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} &= \begin{vmatrix} x_1u_1d_{023} + x_2u_2d_{031} + x_3u_3d_{012} & y_1u_1d_{023} + y_2u_2d_{031} + y_3u_3d_{012} \\ x_1v_1d_{023} + x_2v_2d_{031} + x_3v_3d_{012} & y_1v_1d_{023} + y_2v_2d_{031} + y_3v_3d_{012} \end{vmatrix} \\
&= d_{023}d_{031}D_{012}(x_1y_2 - y_1x_2) + d_{031}d_{012}D_{023}(x_2y_3 - y_2x_3) + d_{012}d_{023}D_{031}(x_3y_1 - y_3x_1) \\
&= d_{012}d_{023}d_{031}(D_{012} + D_{023} + D_{031}) \\
&= d_{012}d_{023}d_{031}D \\
\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} &= \begin{vmatrix} y_1u_1d_{023} + y_2u_2d_{031} + y_3u_3d_{012} & u_1d_{023} + u_2d_{031} + u_3d_{012} \\ y_1v_1d_{023} + y_2v_2d_{031} + y_3v_3d_{012} & v_1d_{023} + v_2d_{031} + v_3d_{012} \end{vmatrix} \\
&= d_{023}d_{031}D_{012}(y_1 - y_2) + d_{031}d_{012}D_{023}(y_2 - y_3) + d_{012}d_{023}D_{031}(y_3 - y_1).
\end{aligned} \tag{2.14}$$

Equations 2.14 and 2.15 yield the formula for a in the theorem. Quite similarly we obtain the formula for b.

In these formulas for a and b the ratios such as D_{023}/d_{023} will tend to h^2/f^2 as t tends to 0. Here h is the height of the lens point above the ground. Moreover the coefficients

$$y_1 - y_2, \qquad y_2 - y_3, \qquad y_3 - y_1$$

sum to 0, and similarly with the x_i's. Hence a and b tend to 0 as t tends to 0.

PROOF OF THEOREM 2.1. The line $(ax+by+1=0)$ on the plate $z=0$ is the intersection I of the plate with a horizontal plane passing through the lens point O. For points (x, y) on I correspond under our perspective to

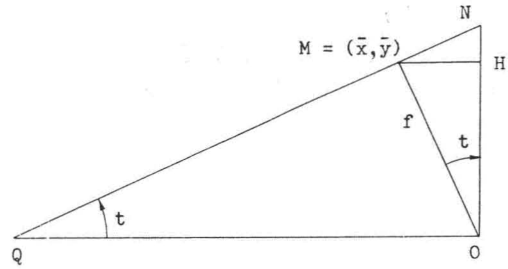

FIG. 1

points (u, v) at infinity. Points for which $(ax+by+1)$ is null clearly have this property since they appear in the denominator of the expressions for u and v. One needs to know that there is no point at which

$$ax + by + 1 = 0$$
$$a_1x + b_1y = 0$$
$$a_2x + b_2y = 0.$$

This is true in accord with the general theory of non-singular collineations.

In figure 1, O is the lens point, M the principal point, OQ is horizontal, QN is a line in the plate of maximum inclination, ON is vertical. The point Q is on the line $ax+by+1=0$. The line $ax+by+1=0$ is orthogonal to the plane of the figure. The distance QM is the distance of M from the line $ax+by+1=0$, that is,

$$|QM| = |(1 + a\bar{x} + b\bar{y})/(a^2 + b^2)^{1/2}|$$

The absolute value sign on the right may be dropped since $1+a\bar{x}+b\bar{y}>0$. This is clear if the plate is horizontal in which case $a=b=0$, and is seen to be true in general by continuous variation from this case, the plate never becoming vertical. From Figure 1, we thus see that

$$\tan t = f/|QM|$$

so that

$$\tan t = f(a^2 + b^2)^{1/2}/(1 + a\bar{x} + b\bar{y})$$

as stated.

We come to the formulas for $\cos\varphi$, $\sin\varphi$. The directed line QN is orthogonal to $ax+by+1=0$ and so has direction numbers in the plane $z=0$ which are $\pm a$, $\pm b$. To eliminate the minus sign we understand that the principal direction is sensed so that the height above the ground increases along it in its positive sense. The function $ax+by+1$ is null at Q and positive at M. This is possible only if $ax+by+1$ increases in the positive sense of QN. This in turn is possible only if in the relations

$$\cos\varphi = e \quad a/(a^2 + b^2)^{1/2}$$
$$\sin\varphi = e \quad b/(a^2 + b^2)^{1/2}$$

the constant $e=1$.

The proof of Theorem 2.1 is complete.

3. THE HEIGHT h OF THE LENS POINT

Let E be a horizontal plane through the principal point M. Let $p=(x, y)$ be a point on the plate. Let $P=(x^*, y^*, z^*)$ be the intersection of the line Op with the plane E. The coordinates (x^*, y^*, z^*) are in the system (x, y, z) belonging to the plate, and O is the lens point. Let E^0 be a horizontal plane through O. In Figure 2, the point K is taken in the plate, as the foot of the perpendicular from p on the line $ax+by+1=0$. Then SP is in E, and KO is in E^0. The distance $|SK|$ equals the distance from the principal point $\bar{x}$, $\bar{y}$, O to the line $ax+by+1=0$, and $|pK|$ equals the distance of x, y from the line $ax+by+1=0$. Hence

$$\frac{|SK|}{|pK|} = \frac{a\bar{x} + b\bar{y} + 1}{ax + by + 1}. \tag{3.1}$$

From Figure 2 we see that

$$\frac{|PO|}{|pO|} = \frac{|SK|}{|pK|} = \frac{a\bar{x} + b\bar{y} + 1}{ax + by + 1} = T(x, y) \tag{3.2}$$

introducing T.

The (x, y, z) coordinates of p are (x, y, O); those of P, (x^*, y^*, z^*); those of O, $(\bar{x}, \bar{y}, -f)$. From Equation 3.2 we see that

$$\frac{x^* - \bar{x}}{x - \bar{x}} = \frac{y^* - \bar{y}}{y - \bar{y}} = \frac{z^* + f}{0 + f} = T(x, y)$$

or

$$x^* = (x - \bar{x})T + \bar{x}$$
$$y^* = (y - \bar{y})T + \bar{y}$$
$$z^* = f(T - 1)$$
$$T = \frac{a\bar{x} + b\bar{y} + 1}{ax + by + 1} \tag{3.3}$$

Let $P_i = x_i^*, y_i^*, z_i^*$ be obtained from $p_i = x_i, y_i$ in accordance with Equation 3.3. In Figure 3, P_1 and P_2 define a horizontal line. They are in perspective through O with the known ground points q_1 and q_2. The distance

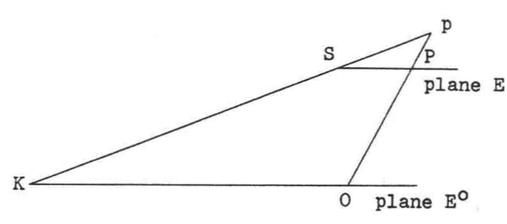

FIG. 2

of O from the horizontal plane E through the principal point is $f \cos t$, and this horizontal plane contains P_1 and P_2. The distance of O from the plane $w=O$ is h. From Figure 3 then

$$\frac{h}{f \cos t} = \frac{|q_1q_2|}{|P_1P_2|}$$

thereby determining h in terms of numerically known quantities.

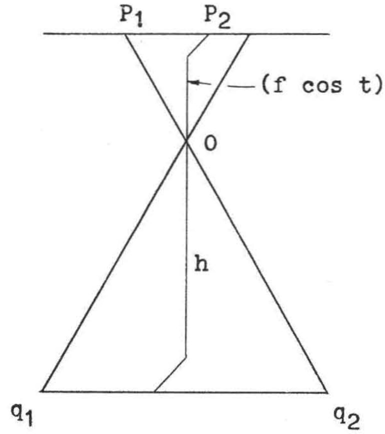

FIG. 3

4. The Over-Determination of the Tilt Angle t

It may be desirable to get greater accuracy in the determination of the tilt angle t by the use of more points, and for that purpose $n+1$ ground points $q_0, \cdots, q_n$ and their images $p_0, \cdots, p_n$ may be used as follows. Let D_{ijk} represent twice the area of the triangle $q_iq_jq_k$, with positive sign if $q_iq_jq_k$ run counterclockwise, otherwise negative. These numbers D_{ijk} may be supposed computed and tabulated for repeated use. A good basis for most purposes would be a roughly rectangular grid of points with distances of a mile between points. Let d_{ijk} denote the corresponding double-signed area of the triangle $p_ip_jp_k$ on the plate.

We have the following theorem:

THEOREM 4.1. *The collineation constants* a *and* b, *determining* t *and* φ *as in* §2 *are given by the formulas,*

$$aD = (D_{012}/d_{012})(y_1 - y_2) + (D_{023}/d_{023})(y_2 - y_3 + \cdots + (D_{0,n-1,n}/d_{0,n-1,n})(y_{n-1} - y_n) + (D_{0n1}/d_{0n1})(y_n - y_1)$$

$$-bD = (D_{012}/d_{012})(x_1 - x_2) + (D_{023}/d_{023})(x_2 - x_3) + \cdots + (D_{0,n-1,n}/d_{0,n-1,n})(x_{n-1} - x_n) + (D_{0n1}/d_{0n1})(x_n - x_1)$$

where

$$D = D_{012} + D_{023} + \cdots + D_{0n1}.$$

The points $q_1, \cdots, q_n$ will ordinarily form a polygon without self-intersection, and q_0 will be taken as an interior point. The constant D is the area of this polygon, positive if the points $q_1, \cdots, q_n$ run counterclockwise, negative otherwise.

From Theorem 2.2 we have the expression shown in Figure 4.

But

$$D_{123} + D_{134} + \cdots + D_{1,n-1,n} = D$$

and the theorem follows.

5. The Choice of Points

The ground distances can be measured with a relative accuracy considerably better than the corresponding accuracy on the plate. We suppose this done. We suppose that a polygon of *2m* points has been chosen with rough regularity and a point q_0 roughly near the center. For example, a rough square or a square with the midpoints of its sides added would do very well. *In estimating the maximum error in* h, *errors in the coordinates cancel more, the more nearly the polygon is symmetric with respect to* q_0. The distance from p_0 to the nearest point of the polygon on the plate should be as large as convenient, perhaps $f/3$ or greater. All this refers to the finding of a and b.

In obtaining h *a pair* p_1p_2 *is preferred. This pair should be chosen so that its midpoint is as near the principal point* M *as possible.*

It should be possible to choose p_1p_2 so that the distance of its mid-point from M is less than $|p_1p_2|/4$. Naturally greater accuracy will be obtained if several pairs such as p_1p_2 are used to obtain h and the results averaged in some way.

In finding (U, V) *a point* p_0 *is preferred. Choosing* p_0 *near the principal point reduces the error in* (U, V) *due to errors in* a *and* b.

The pair p_1, p_2 used in finding (U, V) will presumably not be the pair p_1, p_2 used in finding h. The triangle $p_0p_1p_2$ used in finding (U, V) should be large and regular. See §6.

If the above precautions are taken an error of at most .01 mm. in measurements on the plate will lead to a *maximum estimate of error* in h of about 2 in 10,000. Naturally the probable error is much less. We presuppose a

$$aD_{123} = \frac{D_{012}}{d_{012}}(y_1 - y_2) + \frac{D_{023}}{d_{023}}(y_2 - y_3) + \frac{D_{031}}{d_{031}}(y_3 - y_1)$$

$$aD_{134} = \frac{D_{013}}{d_{013}}(y_1 - y_3) + \frac{D_{034}}{d_{034}}(y_3 - y_4) + \frac{D_{041}}{d_{041}}(y_4 - y_1)$$

$$aD_{145} = \frac{D_{014}}{d_{014}}(y_1 - y_4) + \frac{D_{045}}{d_{045}}(y_4 - y_5) + \frac{D_{051}}{d_{051}}(y_5 - y_1)$$

$$\cdot\ \cdot\ = \cdot\ \cdot$$

$$aD_{1,n-1,n} = \frac{D_{0,1,n-1}}{d_{0,1,n-1}}(y_1 - y_{n-1}) + \frac{D_{0,n-1,n}}{d_{0,n-1,n}}(y_{n-1} - y_n) + \frac{D_{0,n,1}}{d_{0,n,1}}(y_n - y_1)$$

FIG. 4

polygon of, say, eight sides. With a polygon of four sides we estimate an error in 3 in 10,000 in h, and a probable error which is much less.

For accuracy of this sort a correction for the contraction of the focal length due to the cold may be necessary, and perhaps a small correction for refraction of light. The latter can be made as a small preliminary correction in the form of a contraction of the coordinates on the plate towards the principal point. This is approximate but adequate.

If the ground points are not at the same elevation preliminary corrections can be made as indicated in §7.

6. THE COORDINATES U, V OF THE LENS POINT BY THE COLLINEATION METHOD

To determine the image u, v on the ground of a point x, y on the plate, we choose an origin p_0 as near as possible to x, y. We suppose p_0 the image of $q_0=(0, 0)$ on the ground. We also suppose the constants a and b of the collineation

$$\begin{aligned} u &= (a_1x + b_1y)/(ax + by + 1) \\ v &= (a_2x + b_2y)/(ax + by + 1) \end{aligned} \qquad (6.1)$$

have been found as previously indicated. If a and b are known for one choice of origin p_0, they can be determined for any other parallel choice of axes x', y' by making the translation of axes in Equation 6.1 and bringing the new denominator to the form

$$a'x' + b'y' + 1$$

by dividing by the constant term. Then a', b' are the constants replacing a and b, in the new system with coordinates x', y'.

Returning to Equation 6.1, suppose that p_1 and p_2 are two known points, forming with p_0 a large and fairly regular triangle $p_0p_1p_2$. Let q_1 and q_2 be the known ground images of p_1 and p_2. Then

$$\begin{aligned} (ax_1 + by_1 + 1)u_1 &= a_1x_1 + b_1y_1 \\ (ax_2 + by_2 + 1)u_2 &= a_1x_2 + b_1y_2 \end{aligned}$$

and for the general point x, y

$$(ax + by + 1)u = a_1x + b_1y.$$

Eliminating a_1 and b_1

$$\begin{vmatrix} u_1(ax_1 + by_1 + 1) & x_1 & y_1 \\ u_2(ax_2 + by_2 + 1) & x_2 & y_2 \\ u(ax + by + 1) & x & y \end{vmatrix} = 0.$$

We set

$$r_i = ax_i + by_i + 1, \quad r = ax + by + 1 \quad (i = 1, 2)$$

and find that

$$u = [(xy_2 - yx_2)r_1u_1 + (x_1y - y_1x)r_2u_2]/(x_1y_2 - x_2y_1)r \qquad (6.2)$$

Similarly

$$v = [(xy_2 - yx_2)r_1v_1 + (x_1y - y_1x)r_2v_2]/(x_1y_2 - x_2y_1)r \qquad (6.3)$$

Formulas 6.2 and 6.3 give the collineation explicitly. To apply these formulas to find U, V we have merely to find the coordinates x^N, y^N of the point N on the plate directly over the lens and substitute x^N, y^N for x, y in the right of Formulas 6.3 and 6.2.

The distance of the principal point from N is $f\ tan\ t$ so that

$$\begin{aligned} x^N &= \bar{x} + f \tan t \cos \varphi = \bar{x} + f^2a/\bar{r} \\ y^N &= \bar{y} + f \tan t \sin \varphi = \bar{y} + f^2b/\bar{r} \end{aligned}$$

where $\bar{x}$, $\bar{y}$ is the principal point and

$$\bar{r} = a\bar{x} + b\bar{y} + 1.$$

Thus one computes

$$x^N = \bar{x} + \frac{f^2 a}{\bar{r}}$$

$$y^N = \bar{y} + \frac{f^2 b}{\bar{r}}$$

and substitutes for x, y in Formulas 6.3 and 6.2 to find U, V.

Note that this method does not require h, φ, nor t, merely a, b and the three points p_0, p_1, p_2 and their images. The principal error in the result will be caused by errors in a and b. We emphasize the need of determining a and b as well as possible.

In computing a and b, certain double areas were computed. A simple check on these areas is that given by equating the cross ratio of four lines through p_0 with the cross ratio of their images on the ground. (See Introduction to Higher Geometry, Graustein, Exercise 3, page 77.) We have the true relation

$$d_{031}d_{042}/d_{032}d_{041} = D_{031}D_{042}/D_{032}D_{041}$$

As these triangle areas have to be computed anyhow, this forms a convenient check on the trueness of the collineation as given by the measured points. Signs of the areas can be neglected by taking absolute values of both sides. This check is independent of the correctness of f which is not used.

7. Corrections if the Ground Points Differ in Elevation

If the ground points differ in elevation, it is advisable to select a horizontal plane of reference, $w=0$, (Figure 5) as a plane whose elevation is the average of the elevations of the ground points to be used. Then replace each real* ground point u, v, w by the ideal point u, v, $O=q$, and replace the real image p' of u, v, w on the plate, by the image p of q. If the maximum difference of elevation of points used is not more than 20 feet, and if the tilt angle $t<1°$, this correction can be made approximately but with sufficient accurracy under the assumption that the plate is horizontal. This assumption is used to make an easy approximate computation of U, V, h.

*We apply the term *real* to a point physically identifiable.

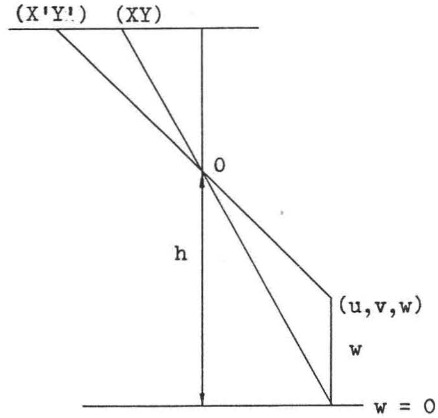

FIG. 5

Let the coordinates on the plate be referred to the principal point of origin. With these coordinates, let X', Y' be the image on the plate of the real point u, v, w on the ground, and let $p=(X, Y)$ be the image on the plate of the ideal ground point $(u, v, O)=q$.

If the plate is horizontal and the ground points are at the same level

$$h = f\frac{|q_1q_2|}{|p_1p_2|},$$

as we have seen. We suppose h so determined in approximate fashion. If the plate is horizontal

$$(X' - X)\ X' = w/h$$

as one sees from similar triangles. This gives

$$X = X' - tX'$$
$$Y = Y' - tY'$$

where

$$t = |p_1p_2/q_1q_2|\ (w/f)$$

This correction should be applied to each point $p_i' = (X_i' Y_i')$ imaging a real ground point $u_iv_iw_i$ for which $w_i \neq 0$.