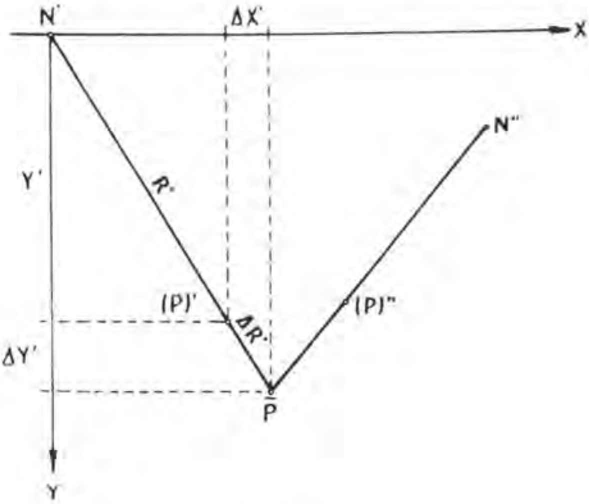

FIG. 1.1

B. SHMUTTER
L. BONFIGLIOLI
Technion—Israel Institute of Technology
Haifa, Israel

Orientation Problems in Two-Medium Photogrammetry

A two-medium photo is of the projection of the object onto the interface rather than of the object itself.

(*Abstract on page 1423*)

1. Basic Geometric Relations

In two-medium photogrammetry, where the object to be mapped is located in one medium and the camera in another, the bundle of rays issuing from the photographed object is refracted at the interface of the two media. The bundle converges towards the projection center of the camera and is deformed compared to its unrefracted counterpart. As the photograph obtained on the image plane of the camera is not of the object itself, but of its projection on the interface, and the image points correspond to those formed by the incident rays on the interface, the entire image is distorted by the refractive effect.

This distortion can be analyzed by comparing two projections of the photographed object on the refraction surface: the real projection formed by the incident and refracted rays, and an imaginary orthographic projection. Such a comparison would readily show how the coordinates of the point of incidence on the refracting surface (or on the photograph) should be corrected to coincide with the orthographic projection. Assuming the refracting surface to be a horizontal plane, the orthographic projection of the object on it is, in fact, identical with the desired map.

As the incident and refracted rays are coplanar with the normal to the refracting plane at the point of incidence, it is seen at once that, by projecting both rays onto the plane, the photographed points are displaced towards the nadir.

In the following, expressions are formulated for solving the exterior orientation of a stereopair. The required equations are derived from the general projective relations between points on the refraction plane and their images on the photographs, allowing for the refractive effect and utilizing the fact that the coordinates of a given point in the common region of the pair have to be found from both photographs. This last condition corresponds to the intersection of restituted rays in single-medium photogrammetry.

Figure 1.1 represents a view of the refracting plane π. $\overline{P}$ is the orthographic projection of the object point P on π, and $(P)'$, $(P)''$ are the points at which the rays issuing from P impinge on π. The distance (depth) of P from the refraction plane is given by $P\overline{P}$ and denoted by t. The coordinates of $\overline{P}$, identical with those of P, are obtainable by means of both the left-hand and right-hand photographs. This is reflected in the formula:

$$X' + \Delta X' = X'' + \Delta X'' = X$$
$$Y' + \Delta Y' = Y'' + \Delta Y'' = Y. \qquad (1.1)$$

The system (1.1) is of basic importance in solving the exterior orientation. It shows that each point with known coordinates (X, Y, t) yields four equations—three based on its own data and one due to its being common to both photographs. The coordinates in (1.1) should be expressed by the quantities measured on the photographs, the coordinates of the projection centres, and the law of refraction.

Consider the stereopair shown in Figure 1.2. O' is the origin of an auxiliary system parallel to that of the photographed object. The radius vector impinging on π at $(P)'$ is given by:

$$\overrightarrow{O'(P')} = \lambda' q'. \qquad (1.2)$$

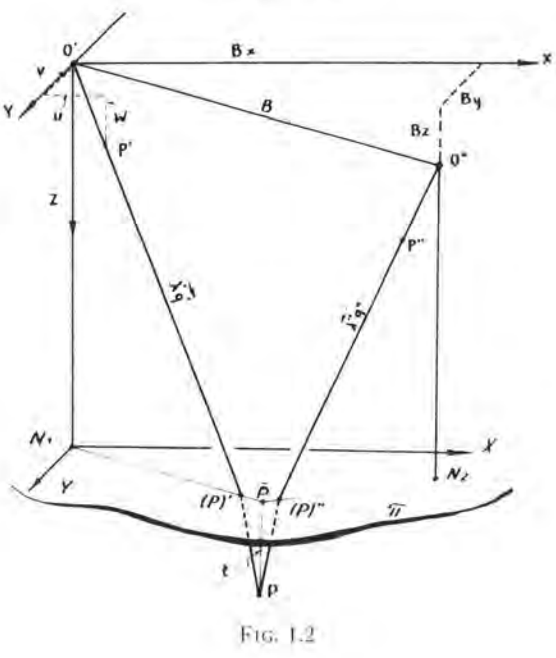

FIG. 1.2

Equation 1.2 is equivalent to three equations expressed in coordinate notation:

$$X' = \lambda' u'$$
$$Y' = \lambda' v'$$
$$Z' = \lambda' w'. \tag{1.3}$$

u', v' and w' are the coordinates of the point p' (on the left-hand photograph) given by the radius vector q', and obtainable from the measured values x', y', f. Using Equations 1.3, the coordinates of the point of incidence $(P)'$ on π are written as:

$$X_{(P)'} = Z' \frac{u'}{w'}$$
$$Y_{(P)'} = Z' \frac{v'}{w'}$$
$$Z_{(P)'} = Z'. \tag{1.4}$$

ABSTRACT: *The problem of exterior orientation is applied to a stereopair taken in two media. The approach to the refractive effect is based on comparison of the image formed by the actual rays with an imaginary orthographic projection on the interface. Formulas are derived on the assumption that the refracting surface is a horizontal plane. Orientation is determined in a single step, without the usual subdivision into relative and absolute stages; the relative orientation is, however, treated separately as mechanical resitution is preferably based on a numerical solution.*

For the coordinates of $(P)''$ in the same system, we have:

$$X_{(P)''} = Z'' \frac{u''}{w''} + B_x$$
$$Y_{(P)''} = Z'' \frac{v''}{w''} + B_y$$
$$Z_{(P)''} = Z'. \tag{1.5}$$

Considering that u, v, w are obtained from the measured values x, y, f by orthogonal transformation, the value $\Delta X'$ in (1.1) may be expressed as:

$$\Delta X' = X' \frac{\Delta R'}{R'} = \frac{u't}{\sqrt{|q'|\,2(n^2-1)+w'^2}} \cdot \tag{1.6}$$

where n is the coefficient of refraction and $|q|$ the modulus of the vector q.

The magnitudes $\Delta X''$, $\Delta Y'$ and $\Delta Y''$ can be similarly expressed. Introducing the auxiliary notation

$$F(w) = \sqrt{|q|^2(n^2-1)+w^2}$$

the system (1.1) is rewritten as:

$$Z' \frac{u'}{w'}\left(1 + \frac{t}{Z'} \frac{w'}{F(w')}\right) = B_x + Z'' \frac{u''}{w''}\left(1 + \frac{t}{Z''} \frac{w''}{F(w'')}\right) = X_p - X_0'$$

$$Z' \frac{v'}{w'}\left(1 + \frac{t}{Z'} \frac{w'}{F(w')}\right) = B_y + Z'' \frac{v''}{w''}\left(1 + \frac{t}{Z''} \frac{w''}{F(w'')}\right) = Y_p - Y_0'. \qquad (1.7)$$

X_0' Y_0' are the coordinates of the left-hand projection center with respect to the system of the object. The fundamental relations in (1.7) provide a system of equations for solving the orientation elements; conversely, they serve for determining the coordinates if the orientation is known.

To solve the exterior orientation, three given points are required. Where not enough coordinates are known and the number of equations of type (1.7) is less than 12, the system must be supplemented by orientation equations not dependent on given quantities, which are derived from the projective relations. Such expressions should be added to the system (1.7) even if sufficient given points are available, because of the extra observations they provide.

The corresponding equation is derived as follows. The first line of (1.7) yields the value t:

$$t = \frac{X_0'' - X_0' + Z'' \dfrac{u''}{w''} - Z' \dfrac{u'}{w'}}{\dfrac{u'}{F(w')} - \dfrac{u''}{F(w'')}} \,. \qquad (1.8)$$

Substituting (1.8) in the second line of (1.7), and introducing the base components $X_0'' - X_0' = B_x$, $Y_0'' - Y_0' = B_y$ and $Z_0' - Z_0'' = B_z$, we obtain:

$$B_x[v'F(w'') - v''F(w')] - B_y[u'F(w'') - u''F(w')] + B_z \frac{F(w'')}{w''} (u'v'' - u''v')$$

$$- Z'\left(\frac{u'}{w'} - \frac{u''}{w''}\right)[v'F(w'') - v''F(w')]$$

$$+ Z'\left(\frac{v'}{w'} - \frac{v''}{w''}\right)[u'F(w'') - u''F(w')] = 0. \qquad (1.9)$$

Formula (1.9) expresses the condition of proper relative orientation. It is seen that on substituting $n = 1$, the sum of the last two terms vanishes and the first three define the condition of intersection of corresponding rays, as known from single-medium photogrammetry.

2. Linearization of the Orientation Equations 1.7

A system of transcendental equations containing 12 unknowns leads to algebraical complications which may not be overcome. That is why the Equations 1.7 have to be linearized. Besides, any adjustment procedure requires that the observation and condition equation be represented in a linear form.

Linearizing the equations becomes possible if there is an approximate solution of the unknowns, which generally is the case. The present photography techniques assures a near-vertical photograph, so that in the first approximation the cosines of the tilt components equal unity and the sines are represented by the angles themselves. In this case the transformation matrix by which the triples of coordinates x,

y, f are transformed to the triples u, v, w becomes simple and the transformation assumes the well known form:

$$\begin{aligned} u &= x + y\Delta k + f\Delta\phi \\ v &= -x\Delta k + y + f\Delta\omega \\ w &= -x\Delta\phi + y\Delta\omega + f. \end{aligned} \tag{2.1}$$

The ratios u/w, v/w, neglecting terms of second and higher order are:

$$\begin{aligned} \frac{u}{w} &= \frac{1}{f}\left[x + y\Delta k + \left(f + \frac{x^2}{f}\right)\Delta\phi + \frac{xy}{f}\Delta\omega\right] \\ \frac{v}{w} &= \frac{1}{f}\left[y - x\Delta k + \frac{xy}{f}\Delta\phi + \left(f + \frac{y^2}{f}\right)\Delta\omega\right]. \end{aligned} \tag{2.2}$$

In order to complete the linearization of Equations 1.7 the ratio $w/F(w)$ must be expanded in a series. Substituting w in $F(w)$ according to Transformation (2.1) and expanding, we obtain:

$$\frac{w}{F(w)} = \frac{1}{K}\left[f - x\left(1 - \frac{f^2}{K^2}\right)\Delta\phi - y\left(1 - \frac{f^2}{K^2}\right)\Delta\omega\right] \tag{2.3}$$

where K is defined: $K^2 = |q|^2(n^2-1) \div f^2$.

Equations 1.7 can now be linearized. Substituting (2.2), (2.3), and $Z_0 = \overline{Z} + dZ$ into (1.7), we have for the left-hand photograph:

$$\begin{aligned} X_0' + \frac{x'}{f}dZ' + \left(\frac{\overline{Z}'}{f} + \frac{t}{K'}\right)y'\Delta k + \left[\overline{Z}'\left(1 + \frac{x'^2}{f^2}\right) + \frac{tf}{K'}\left(1 + \frac{x'^2}{K^2}\right)\right]\Delta\phi' \\ + \left[\overline{Z}'\frac{x'y'}{f^2} + \frac{tf}{K'}\frac{x'y'}{K^2}\right]\Delta\omega' + \left(\frac{\overline{Z}'}{f} + \frac{t}{K'}\right)x' = X_p \end{aligned} \tag{2.4}$$

$$\begin{aligned} Y_0' + \frac{y'}{f}dZ' - \left(\frac{\overline{Z}'}{f} + \frac{t}{K'}\right)x'\Delta k + \left[\overline{Z}'\frac{x'y'}{f^2} + \frac{tf}{K'}\frac{x'y'}{K'^2}\right]\Delta\phi' \\ + \left[\overline{Z}'\left(1 + \frac{y'^2}{f^2}\right) + \frac{tf}{K'}\left(1 + \frac{y'^2}{K'^2}\right)\right]\Delta\omega' + \left(\frac{\overline{Z}'}{f} + \frac{t}{K'}\right)y' = Y_p. \end{aligned} \tag{2.5}$$

$\overline{Z}$ denotes an approximate value of Z_0. Similar expressions exist for the right-hand photograph as well. Assuming $n=1$, the Expressions 2.4, 2.5 yield the well-known formulas for coordinate increments in single-medium photogrammetry.

Formulas 2.4, 2.5 imply that the air base is oriented nearly along the X-axis of the photographed object. This is based on an assumption that the angle κ may be replaced by $\Delta\kappa$, but this assumption is very rarely valid. In these circumstances, an approximation to the azimuth of the airbase should be used in transforming the given (or measured) coordinates, and the transformed values substituted in the above expressions.

3. Linearization of the Condition of Relative Orientation

Equation 1.9 may be linearized along the same lines. Supposing that the angular elements of orientation are small, and the base components b_y, b_z in the model system are small compared with b_x, (1.9) becomes:

$$a_0 + a_1\Delta k' + a_2 k'' + a_3 f\Delta\omega' + a_4 f\Delta\omega'' + a_5 f\Delta\phi' + a_6 f\Delta\phi'' + a_7\frac{b_y}{b_x} + a_8\frac{b_z}{b_x} = 0 \tag{3.1}$$

with the coefficients

$$a_0 = y' - y'' + \frac{\Delta K}{K'}\left(y' + \frac{y'x'' - y''x'}{b_x}\right)$$

$$a_1 = -x' - \frac{\Delta K}{K'}\left(x' + \frac{y'y'' + x'x''}{b_x}\right)$$

$$a_2 = x'' + \frac{\Delta K}{K'}\left(\frac{y'y'' + x'x''}{b_x}\right)$$

$$a_3 = 1 + \frac{y'y''}{K'^2} + \frac{\Delta K}{K'}\left(1 + \frac{x''}{b_x}\right) + \frac{K'^2 - f^2}{K'^2}\,\frac{x'y'' - x''y'}{f^2}\cdot\frac{y'}{b_x}$$

$$a_4 = -1 - \frac{y'y''}{K'K''} - \frac{\Delta K}{K'}\left(\frac{x'}{b_x} + \frac{x'y'' - y'x''}{f^2}\cdot\frac{y''}{b_x}\right) - \frac{K'K'' - f^2}{K'K''}\,\frac{x'y'' - y'x''}{f^2}\,\frac{y''}{b_x}$$

$$a_5 = \frac{x'y''}{K'^2} - \frac{\Delta K}{K'}\,\frac{y''}{b_x} + \frac{x'y'' - y'x''}{f^2}\cdot\frac{K'^2 - f^2}{K'^2}\,\frac{x'}{b_x}$$

$$a_6 = -\frac{y'x''}{K'K''} + \frac{\Delta K}{K'}\left(\frac{y'}{b_x} - \frac{x'y'' - y'x''}{f^2}\,\frac{x''}{b_x}\right) - \frac{K'K'' - f^2}{K'K''}\cdot\frac{x'y'' - y'x''}{f^2}\,\frac{x''}{b_x}$$

$$a_7 = -(x' - x'') - \frac{\Delta K}{K'}\,x'$$

$$a_8 = \frac{x'y'' - y'x''}{f} + \frac{\Delta K}{K'}\,\frac{x'y'' - y'x''}{f}.$$

The value b_x in (3.2) is the base component in the x-direction in the photograph scale, and ΔK is defined as $\Delta K = K'' - K'$. The coefficients according to (3.2) may be simplified by means of an additional approximation. Substituting $y'' - y' = p_y$ and $x' - x'' = b_x - p_x$, the term $x'y'' - y'x''$ becomes $y'b_x + x'p_y - y'p_x$. This expression occurs in most of the coefficients and is usually divided by f^2. For the whole model, $(x/f) < 1$ and $(y/f) < 1$, while p_x and p_y are small compared with f, so that the following approximation is justified:

$$\frac{x'y'' - y'x''}{f^2} \cong \frac{y'b_x}{f^2}.$$

We now introduce a coordinate system for both photographs with its origin at the principal point of the left-hand photograph, as is usually done in single-medium photogrammetry. Thus $y' \cong y'' \cong y$ and $x'' = x' - b_x$. All coefficients in (3.2), except a_0, are multiplied by differential values of the orientation elements. For this reason, the orientation elements are only slightly affected if p_y and p_x are neglected.

With the above approximations, the values in (3.2) assume the following form:

$$A_0 = -p_y - \frac{\Delta K}{K'}\,\frac{xp_y - yp_x}{b}$$

$$A_1 = -x_1 - \frac{\Delta K}{K'}\,\frac{x^2 + y^2}{b}$$

$$A_2 = (x - b) + \frac{\Delta K}{K'}\left(\frac{x^2 + y^2}{b} - x\right)$$

$$A_3 = \left(1 + \frac{y^2}{f^2}\right) + \frac{\Delta K}{K'}\,\frac{x}{b}$$

$$A_4 = -\left(1 + \frac{y^2}{f^2}\right) - \frac{\Delta K}{K'}\left(\frac{y^2}{f^2} + \frac{x}{b}\right)$$

$$A_5 = \frac{xy}{f^2} - \frac{\Delta K}{K'}\,\frac{y}{b}$$

$$A_6 = -\frac{(x-b)y}{f^2} + \frac{\Delta K}{K'}\left(\frac{y}{b} - \frac{(x-b)y}{f^2}\right)$$

$$A_7 = -\left(1 + \frac{\Delta K}{K'}\,\frac{x'}{b}\right)$$

$$A_8 = \frac{y}{f}\left(1 + \frac{\Delta K'}{K'}\right)$$

Note that no approximation was applied for the coefficient a_0, as its second term represents the effect of refraction and may not be neglected.

In general, the relative orientation in two-medium photogrammetry is given by eight elements: three additional elements, defining the refraction plane, have to be taken into consideration, as this plane is a decisive factor in the projection process. In the case where the interface is a horizontal plane, as accepted above, five elements suffice.

Solution of the relative orientation may be based on Equation 3.1, in which the coefficients from (3.3) are substituted. To solve the coorientation, the following equation is used:

$$-\left(1 + \frac{\Delta K}{K'}\,\frac{x}{b}\right)b_y'' + \frac{y}{f}\left(1 + \frac{\Delta K}{K'}\right)b_z'' + \left[(x-b) + \frac{\Delta K}{K'}\left(\frac{x^2+b^2}{b} - x\right)\right]\Delta k''$$
$$-\left[\left(1 + \frac{y^2}{f^2}\right) + \frac{\Delta K}{K'}\left(\frac{y^2}{b} + \frac{x}{b}\right)\right]f\Delta\omega''$$
$$-\left[\frac{(x-b)y}{f^2} - \frac{\Delta K}{K'}\left(\frac{y}{b} - \frac{(x-b)y}{f^2}\right)\right]f\Delta\phi'' - p_y - \frac{\Delta K}{K'}\,\frac{xp_y - yp_x}{b} = 0. \qquad (3.4)$$

All quantities in (3.4) are given in the model (or photograph) scale.

4. Adjustment Procedures for Solving the Exterior Orientation

Two alternatives exist for solving the exterior orientation. The first is based on given data only. Here (2.4), (2.5) are regarded as observation equations, and the adjustment procedure applied consists in solving for unknowns (indirect observations). As is seen from the above formulas, each photograph is separately oriented. The solved linear elements are added to the corresponding initial values, and with the aid of the angular elements the measured quantities are transformed into u, v, w. New coefficients for (2.4), (2.5) are calculated, and another iteration step is taken. Iteration is discontinued either when the corrections to the elements become small, or when the residuals of the adjustment are smaller than a predetermined criterion.

In the second alternative, (2.4) and (2.5) are combined with the equation of coorientation (3.4). Adjustment becomes complicated as the elements derived from (2.4), (2.5) should also satisfy the condition of proper orientation (3.4). One means of overcoming this drawback consists in representing the solution in the form $X = X_0 + \delta X$, X being the vector representing the unknowns, X_0—an initial solution found by adjustment of a set of observation equations derived from (2.4) and (2.5) and δX representing the corrections obtained by adjustment based on (3.4). Details of this approach are given in (3).

It should be noted that Equation 3.4 includes differences of the orientation elements. Moreover, the base components must be represented in the model (or photograph) scale.

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