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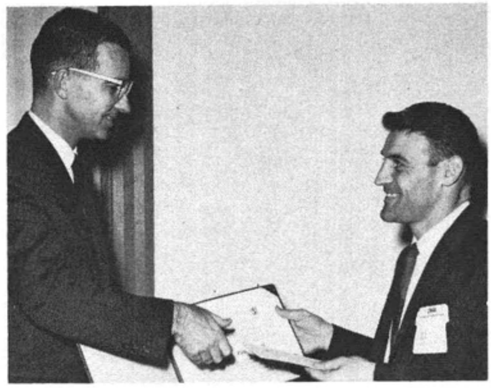
Analytical Radial Triangulation

The numerical solution of the classical graphic problem avoids the identification of the flight line.



Mr. Wolf (right) receives Graduate Student Award from Mr. Corwin Brumley of Bausch & Lomb for this paper at the Annual Convention of the American Society of Photogrammetry, Washington, D. C., March 1966.

(*Abstract on page 111*)

INTRODUCTION

TRADITIONALLY RADIAL TRIANGULATION has been accomplished by graphical procedures; it can also be done by strictly analytical methods. In the past the solution by graphics has been more popular with photogrammetrists because it was fast and easy to perform. With the advent of the high speed electronic computer, photogrammetrists have found that many problems involving computations too lengthy to be practical in the pre-computer era are now rapidly and simply handled. Radial triangulation is one particular photogrammetry problem which may be revolutionized by the computational relief provided through high speed electronic computers.

Definite benefits can be derived as a result of determining planimetry by analytical radial triangulation rather than by graphical means. Discrepancies in the graphical solution result from such things as scaling limitations, the inability to construct a perfectly straight line, the thickness of pencil lines, and other errors. With analytical radial triangulation. some of these errors are eliminated, thereby increasing the accuracy of the computed planimetry.

In an attempt to increase the accuracy of radial triangulation further, a new analytical procedure has been developed which does not require the location of conjugate principal points and which requires no graphical con struction. Rather, the procedure requires only comparator measurement of the photographic coordinates of images, followed by computation. Because it is difficult in some cases to locate conjugate principal points with accuracy, this new analytical radial triangulation procedure should yield more accurate results than existing analytical methods. This new analytical technique is herein referred to as *Analytical Three-Point Resection Radial Triangulation*.

DISCUSSION OF THE PROCEDURE

With the analytical three-point resection radial triangulation, it is necessary that three horizontal control points appear in the overlap area of a pair of photographs. Figure 1 illustrates such a situation. On the overlapping photographs No. 101 and No. 102 shown, there appear the images a, b, and c of the three horizontal control points A, B, and C. The figure provides some detail of the geometry involved in the analytical three-point resection radial triangulation procedure.

Briefly, the analytical three-point resection radial triangulation method consists of resecting on the three horizontal control points to determine the coordinates of each of the ground principal points of the overlapping pair of photographs. This problem is analogous to the Plane Surveying problem of calculating the location of an unknown transit station, having measured from that transit station the two angles formed between the

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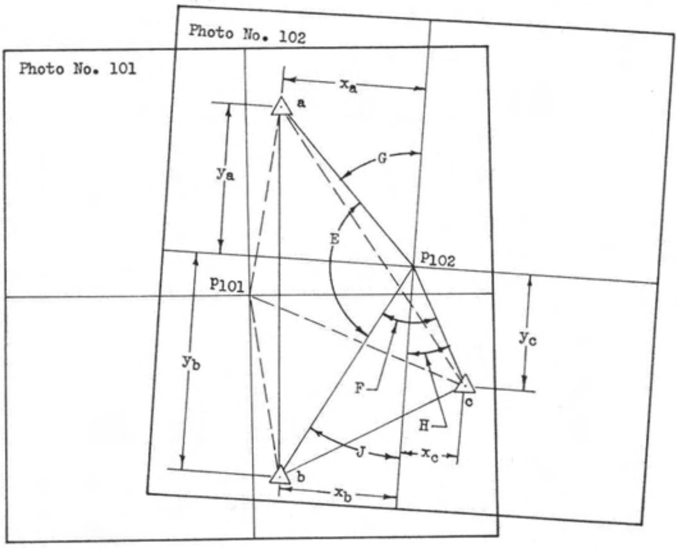

FIG. 1. Geometry of the Three-Point Resection Procedure.

rays from the transit station to three points whose positions are known.

The length and azimuth of the line joining the ground principal points of the overlapping pair can be calculated from the ground principal point coordinates. In locating a new pass point in the overlap area, this line then becomes the known side of a triangle in which all three angles are known. The triangle is solved by employing the law of sines, from which the length and directions of the sides are calculated. This is followed by the calculation of the coordinates of the pass point. This procedure of locating new pass points is referred to herein as *intersection*. Figure 2 illustrates the geometry of the intersection triangle for locating pass point *D*, whose image *d* appears on the overlapping pair of photographs. Intersection may be applied to compute the ground coordinates of as many points as are desired whose images appear in the overlap area of the pair of photographs. Coordinate

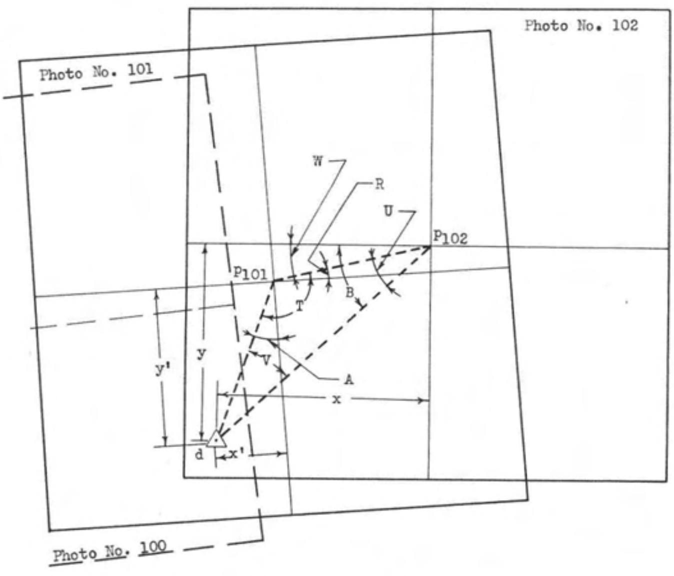

FIG. 2. The Intersection Triangle of an Overlapping Pair.

computation of three pass points whose images appear on the third photograph of the flight strip will allow extension of the radial triangulation. On Figure 2, for example, note that point d appears on photograph No. 100 and it may be subsequently used in resecting for the ground principal point coordinates of that photograph.

Referring to Figure 1, the analytical three-point resection procedure is demonstrated in more detail. First of all, the x and y photographic coordinates of the images of the three control points are measured. The angles G, H and J may be calculated by utilizing the following equations:

$$G = \tan^{-1}(x_a/y_a)$$
$$H = \tan^{-1}(x_c/y_c)$$
$$J = \tan^{-1}(x_b/y_b)$$

Angle F is equal to angle J plus angle H, and the value of angle E equals 180° minus the sum of the angles J and G. If the photography is truly vertical, the angles E and F are the true horizontal angles that would be measured to the control points from the ground principal point, if the ground principal point were occupied with a transit. These angles E and F are shown in the ground space coordinate system illustrated on Figure 3.

ABSTRACT: *A new analytical technique in radial triangulation has been developed. This procedure eliminates the need to locate the flight line on the photographs. Because it is difficult in some cases to locate flight lines with accuracy, this new analytical radial triangulation procedure should yield more accurate results than existing analytical methods. The procedure consists of resecting on three horizontal control points to determine the coordinates of the ground principal points of a pair of overlapping photographs. Pass points are then located by numerical intersection. The new technique has been tested on photography of the University of Wisconsin Summer Surveying Camp area. This particular photography included a dense system of second-order horizontal control which was premarked so that it appeared on the photography. This control provided excellent checks on the accuracy of the calculated planimetry. The test results indicated that the new procedure is both accurate and easy to perform.*

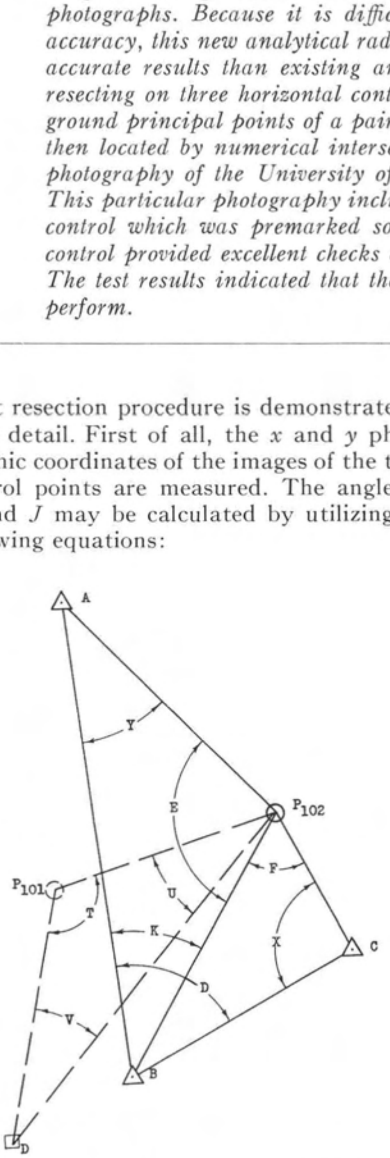

FIG. 3. Ground Space Coordinate System.

Knowing the horizontal ground coordinates of the control points A, B and C, the ground lengths and directions of the sides AB and BC may be computed. The equations for these calculations are of the following form:

Length of AB

$$= [(\text{Latitude of } AB)^2 + (\text{Departure of } AB)^2]^{1/2}$$

Bearing of AB

$$= \tan^{-1}(\text{Departure of } AB/\text{Latitude of } AB)$$

The angle D may then be computed from the two known directions of AB and BC. The angles X and Y are calculated next using the following three-point problem equations:

$$S = 180° - \tfrac{1}{2}(D + E + F)$$
$$M = \tan^{-1}(AB \sin F/BC \sin E)$$
$$N = \tan^{-1}[\text{Cot}(M + 45°) \tan S]$$
$$X = S + N$$
$$Y = S - N$$

Angle K equals 180° minus the sum of the angles Y and E. Because one side and all of the angles of the triangle ABP_{102} are known at this point, the side AP_{102} is computed using the law of sines. The bearing AP_{102} is also readily calculated by combining the bearing of AB properly with the angle Y. The latitude and departure of the line AP_{102} are then calculated and the coordinates of P_{102} are computed. As a check on the computation, one could calculate through the triangle BCP_{102} and compare the resulting coordinates of P_{102} with those obtained through the triangle ABP_{102}.

At this point it is possible to establish the azimuth of the positive photographic x-axis of photograph No. 102, by simply subtracting the angle $(90° - G)$ from the azimuth of the line AP_{102}. (Refer to Figure 1) The positive x-axis is herein considered as being the direction to the right from the principal point. It will be seen shortly that this azimuth is necessary for extending the radial triangulation.

Referring to Figure 2, the intersection procedure for locating new pass points is presented in more detail. Assume that the coordinates of the ground principal points of the overlapping photographs No. 101 and No. 102 have been calculated by the three-point resection method. Therefore the length and direction of the line joining these ground principal points can be calculated by employing the same equations as those previously presented for calculating the length and direction of the line AB. The direction of this line is the true direction of $p_{101}p_{102}$ shown on Figure 2. Next the photographic coordinates x and y on photograph No. 102, and the photographic coordinates x' and y' on photograph No. 101 are measured to the image d of the pass point D, a new point whose location is desired. On Figure 2, angle A equals $\tan^{-1}(x'/y')$ and angle B equals $\tan^{-1}(y/x)$. As the direction of the line $p_{101}p_{102}$ and the direction of the positive x-axis of photograph No. 102 are known, the angle W can be computed.

Angle U equals angle B minus angle W. Because the direction of the positive x-axis of photograph No. 101 is also known from previous calculations, the angle R may be readily calculated. Angle T then equals 90° plus angle A plus angle R. Angle V therefore equals 180° minus the sum of angles J and T, because the sum of the angles in a triangle equal 180°.

Now again, if the photography is truly vertical, the angles U, T and V of the triangle $p_{101}dp_{102}$ are true horizontal angles. This triangle is illustrated in the ground space coordinate system of Figure 3. In the triangle, the length of the line $P_{101}P_{102}$ is known, as are the three angles. Therefore the side $P_{102}D$ can be calculated by applying the law of sines. The direction of this line may also be calculated by properly combining the angle U and the direction of the line $P_{101}P_{102}$. The latitude and departure of $P_{102}D$ are then calculated, followed by the computation of the coordinates of the point D. As a check on the computations, one could calculate the coordinates of D through the line $P_{101}D$.

Radial triangulation by this procedure may be accomplished either parallel with, or transverse to, the flight line; the procedure is basically the same. In employing analytical three-point resection radial triangulation, one should exercise judgment in selecting pass points in order to secure mathematically strong triangular figures. Accuracy diminishes if triangles are used which contain very large or very small angles. In addition, it should be noted that the three-point resection problem becomes indeterminate if the three control points and principal point lie on a common circular arc.

Testing the Procedure

In order to determine the feasibility and expected accuracy of this new method of analytical radial triangulation, an actual test using the method was carried out through nine overlapping aerial photographs of the University of Wisconsin Summer Surveying Camp area in northern Wisconsin. This area is dense in ground control as a result of several years of continuous second order traversing and triangulation carried out by the students and carefully checked by the staff. The ground control was marked prior to photographing with crosses made of white cheese cloth so that it appeared distinctly on the photographs. A portion of an index mosaic showing the area of the test is presented on Figure 4. The ground control appearing in the nine photographs is labelled.

The photography used in the test was at a scale of 1:6,000. A glass scale having a least reading of 0.01 millimeter was used to measure the photographic coordinates of the images. A correction for paper shrinkage was applied to the measured photographic coordinates. The angle of tilt was computed for some of the photographs included in the test and a maximum of about three degrees was found to exist. No correction was made for this tilt. The calculations were performed on an IBM 1620 computer.

Beginning with photographs No. 52-102 and No. 52-101, the test was extended westward, parallel with the flight line through seven photographs. The triangulation was then directed southward, transverse to the flight line through two more photographs. The

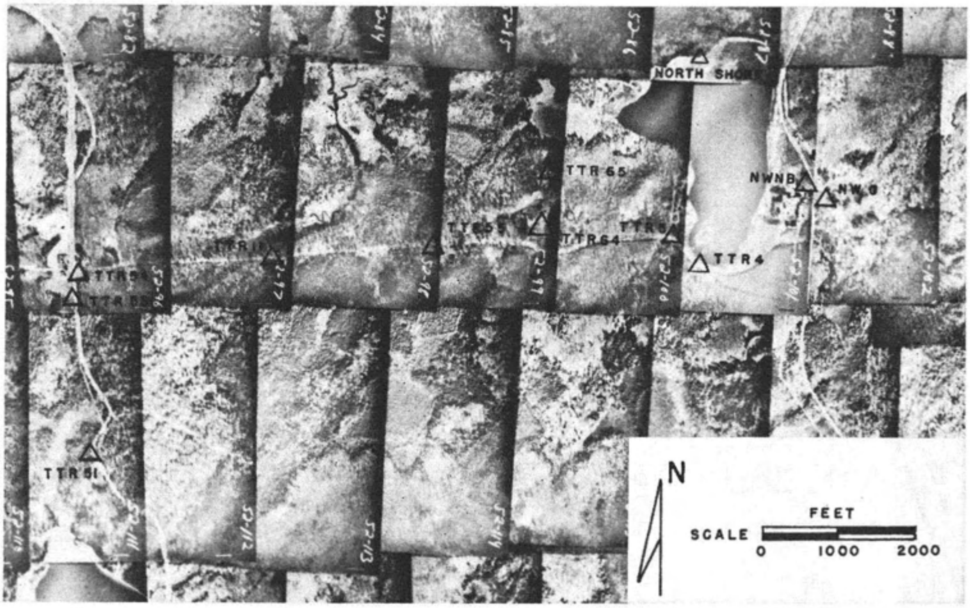

FIG. 4. The Area of the Triangulation Test.

control points *North Shore*, *TTR4*, and *NWG* were resected upon to determine the coordinates of the ground principal point of photograph No. 52-102. The same control points were used in resecting for photograph No. 52-101, except that *NWNB* was substituted for *NWG*. These points then served as the basic control points from which the triangulation was propagated.

The other control points that appear intermediately along the triangulation route were used as pass points in the network. They are *TTR5, TTR64, TTR65, TTR55, TTR11, TTR54, TTR53, and TTR51*. Since these intermediate ground control points served only as pass points, their ground coordinates were calculated in the triangulation process and these calculated coordinates, although differing slightly from the control coordinates, were used uncorrected in the continuation of the triangulation. The intermediate control points therefore did not serve as correction points but served rather as check points on the accuracy of the triangulation as it progressed.

The results of the test are tabulated in Table 1. It is recognized that in current analytical photogrammetric procedures one would incorporate the intermediate control point coordinates in order to increase the accuracy of the computed planimetry. This method of computation was intentionally by-passed in order that a determination of the nature of the error propagation in the test could be made. Examination of the data of

TABLE 1. Results of Analytical Radial Triangulation

Point	*Calculated Coordinates*		*Control Coordinates*		*Difference*		*Length of Closure*
	N	*E*	*N*	*E*	*N*	*E*	
(*1*)	(*2*)	(*3*)	(*4*)	(*5*)	(*6*)	(*7*)	(*8*)
TTR5	99,768.5	98,489.0	99,769.9	98,489.9	−1.4	−0.9	1.7
TTR64	99,909.8	96,669.9	99,912.5	96,669.4	−2.7	0.5	2.7
TTR65	100,649.0	96,759.1	100,651.6	96,759.3	−2.6	−0.2	2.6
TTR55	99,679.1	95,209.8	99,682.1	95,214.6	−3.0	−4.8	5.7
TTR11	99,539.0	93,243.5	99,539.9	93,251.9	−0.9	−8.4	8.4
TTR54	99,423.2	90,496.9	99,419.6	90,508.5	3.6	−11.4	12.2
TTR53	99,082.2	90,445.6	99,075.4	90,457.0	6.8	−11.4	13.3
TTR51	96,968.3	90,555.3	96,965.2	90,567.8	3.1	−12.5	12.9

column 8, Table 1, illustrates that the errors accumulated parabòlically. These accumulated errors may be adjusted accordingly.

Conclusions

A new analytical procedure in radial triangulation has been presented. This procedure does not require that flight lines be located for its operation, which should yield more accurate results than existing analytical radial triangulation techniques. The procedure consists of resecting on three horizontal control points to determine the coordinates of the ground principal points of a pair of overlapping photographs. Pass points, whose images appear in the overlap area of the pair, are then located by numerical interesction.

The procedure has been tested on photography which was dense with marked ground control. This provided excellent checks on the computed planimetry. The results of the test could be considered when establishing the maximum number of photographs tolerable in cantilevered radial triangulation control extensions wherein equipment and procedures similar to those of this test are planned.

The only equipment required for the procedure was a glass scale which could be read to the nearest 0.01 millimeter. It is true that the computations were handled on a 1620 computer, which may not be ordinary office equipment for every photogrammetrist. Computer services are readily available nowadays, however, and at a fairly reasonable rate. Of course, the computations could also be done with an ordinary desk calculator.

Acknowledgement

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