H. S. WILLIAMS
G. E. BELLING
Univ. of the Witwatersrand
Johannesburg, South Africa

Hybrid and Conformal Polynomials

A hybrid function is probably better for long aerotriangulated strips having a large number of control points.

(*Abstract on next page*)

ERROR CHARACTERISTICS IN AERIAL TRIANGULATION

ERROR PROPAGATION STUDIES over the past thirty years have provided the photogrammetrist with a volume of information relating to the behaviour patterns of systematic and random errors in aerial triangulation. Von Gruber's[1] early investigations into the propagation of systematic errors were followed by the theories of Roelofs, Gotthardt[2,3] and others in which the characteristics of random errors were derived, either empirically or by "rigorous" theory. With a knowledge of the form of these error patterns, adjustment procedures (which were rather primitive in the early stages) were developed, and aerial triangulation as a standard production procedure became a matter of course.

For many years, graphical methods predominated, with the Zarzycki[4] interpolation procedure offering the theoretically soundest approach to strip adjustment. Schermerhorn[5] and Zeller[6] advocated the extension of strip adjustment to blocks for purposes of controlling the odd behaviour of random errors. The possibility of numerical treatment was avoided however, and only became a reality with the advent of the electronic computer.

Switching from graphical to numerical procedures meant that photogrammetrists were faced with the task of formulating mathematical expressions which simulated the theoretically derived patterns generated by the systematic and random error propagation.

In 1946 W. Bachmann[7] proved for the first time that correction equations (formed according to a rigorous least squares approach) for the three coordinates were of third power in x, a fact substantiated by Roelofs[8] in 1951. The correction formulas derived by Roelofs were of a complicated nature and were subsequently simplified by van der Weele[9] to the following form

$$\begin{bmatrix} \Delta X_i \\ \Delta Y_i \\ \Delta Z_i \end{bmatrix} = \begin{bmatrix} a_1 & (b_1 - 3a_2y_i) & (c_1 - 2b_2y_i) & (d_1 - c_2y_i) \\ a_2 & (b_2 + 3a_1y_i) & (c_2 + 2b_1y_i) & (d_2 + c_1y_i) \\ a_3 & b_3 & (c_3 + b_4y_i) & (d_3 - c_4y_i) \end{bmatrix} \cdot \begin{bmatrix} x_i^3 \\ x_i^2 \\ x_i^1 \\ 1 \end{bmatrix} \tag{1}$$

The above expressions when considered as polynomials representing error or correction surfaces simplify the rigorous adjustment procedure considerably. In other words, the eight unknowns in the planimetric expressions, for example, could be solved for uniquely by reference to four ground control points suitably distributed throughout the strip. Control in excess of four produces redundancies and consequently dictates an adjustment solution for the unknowns. On the other hand redundant control will also allow for the solution of expressions containing more than eight unknowns. Higher order terms or hybrid forms of the lower order terms added to the above expressions may improve the flexibility of the correcting functions, although theoretical investigations by Ackermann[10] on this question prove the undesirability of higher order polynomials.

Reverting to the problem of numerical treatment, the early advocates of automation faced the problem of programming known, or devising new adjustment procedures which not only produced results to the required accuracy, but produced them economically. As a result, the natural tendency was to utilise polynomial expressions of various forms in a manner which yielded *acceptable* results and did not tax the computer beyond its storage, word length and other limits, as would have been the case with entirely rigorous procedures. Although the capacity of the modern computer possessing considerable back-up memory does not present any real problems nowadays, polynomial strip adjustment is still favored by many organisations, and

ABSTRACT: *Results of polynomial strip adjustment procedures appear to be comparable irrespective of the polynomial type used. Arguments produced to justify the use of one polynomial in preference to another are shown to be unconvincing particularly when dealing with transformations between reference frames of different status. Suggestions are made concerning empirically derived compositions which offer alternative solutions to the adjustment problem in photogrammetry. Other less-used polynomial forms are examined for use with strips and possibly for blocks as well.*

this fact alone is sufficient reason for analyzing and reviewing existing forms, and postulating on the properties of other less frequently used mathematical expressions.

Standard and Hybrid Polynomials

Expressions (1) above, although subject to the limitations of polynomial correcting functions, do nevertheless correspond to a more exact adjustment procedure (for a certain control point distribution than do numerous hybrid forms. The question which inevitably arises when comparing adjustment procedures is *What is the accuracy of the adjusted coordinates?* Generally as far as is known, little difference exists between the results from various polynomial adjustment procedures although tests currently being conducted indicate a possible superiority of composed polynomials over conformal polynomials insofar as height adjustment is concerned. Some details of these tests are given below.

The possibility of the adjustment being executed in two stages (i.e. separation of planimetry and height) raises the question as to whether this technique is valid and theoretically or otherwise justifiable. Arguments exist for and against this procedure. The main consideration is clearly whether or not such a development vitiates the end result. This is generally not the case. Furthermore, different accuracy requirements are very often specified for planimetry and height, a factor which strongly favors separate treatment. In this event, the polynomial for height adjustment could possibly be improved by the addition of hybrid terms as was done by Nowicki and Born.[11]

Discretion and rationalism must prevail however, for the addition of numerous terms (some of higher order) merely to improve the *goodness of fit* is indeed questionable. Too often an *exact* fitting of the error surface to the available ground control is the aim of the procedure, i.e. the strip adjustment, is judged by the standard deviation of the residuals at control points. Excessive flexibility in curve fitting has the obvious shortcoming of accommodating the random *jumps* known to exist in aerial triangulation, particularly over long strips. It is not suggested, however, that hybrid forms of polynomials should be abandoned. On the contrary, empirical derivations of suitable polynomial forms based on analyses of either systematic errors resulting from different triangulation procedures, or "trial and error" adjustments of fully controlled test strips often produce highly satisfactory results.

A further variation possible is the use of either several polynomials on a single strip, i.e. a different set of coefficients between bands of control, or composed polynomials[12,13] in which a variety of composition restraints have been imposed. The latter of these two suggestions has both theoretical and practical advantages, and results of recent tests, particularly with height adjustment have been most encouraging.

Conformal and Orthogonal Polynomials

Among the polynomial forms which are currently being used in the adjustment of spatial strip coordinates are those which satisfy the Cauchy Riemann relations. These conformal forms may be derived directly in terms of complex numbers[15] from:

$$(X + iY) = \sum_{j=1}^{n} (a_j + ia_{j+1})(x + iy)^{j-1} \tag{2}$$

or, alternatively, they may be derived from normal polynomials upon which conformality conditions are imposed.[14] Justification for the use of conformal polynomial forms, mainly in the adjustment of planimetry has been represented as being because they are *convenient* to use and furthermore, because they transform any small area from the machine or strip system to the terrain coordinates without deformation. Whether or not these criteria are sufficient cause for the application of conformal polynomials is possibly a matter for conjecture.

The authors are of the considered opinion however that conformality alone does not provide an adequate reason for selecting a conformal transformation for converting coordinates from one reference frame to another. In fact, the foregoing reason may be severely criticized when transformation occurs between reference systems of unequal status as is usually the case in photogrammetry. In general, if good ground control surveys are based on a sound national geodetic framework, the photogrammetric systems will understandably have the lower status while it would not be difficult to envisage situations where the ground control system could very well prove to be the one of inferior status. For proper flexibility, transformations between different status systems must be expected to display *affine* characteristics to a greater or lesser extent.

Although it may be desirable to retain basic geometrical properties of the reference frames it is in no way essential that this be done. Clearly, the adjustment polynomial that should be chosen is the one that gives the best result when tested against known and acceptable criteria, such as σ_z and σ_p, the absolute standard errors in height and position. These criteria (as was stated earlier) are unfortunately only too often applied to the residuals obtained at the control points alone to which the polynomial is fitted.

By so doing a spurious impression of accuracy is obtained from the resulting small standard errors. This is not surprising because a judicious selection of the number of polynomials' coefficients, n, equal to three times the number of control points, m, will reduce σ_p and σ_z to zero at the control points. In other words the observational redundancies $(m-n)$ are reduced to zero when a simultaneous adjustment in X, Y and Z is undertaken. Clearly as $(m-n)$ tends to infinity, σ_z and σ_p will tend to appropriate maxima partly under the control of n, to which some practical limit must apply. Undoubtedly this limit will be governed by the fact that error propagates through the strip of aerial triangulation as a function of X^3.[7,16]

Truncating the number of polynomial terms in this way would introduce a remainder summation term the effect of which must be expected. Maximum standard errors would thus provide a theoretically sounder basis for judging the merits of the various polynomial forms, even although the implementation of such an ideal would not be practicable, because there is no means of readily determining $(\sigma_s)_{max}$ and $(\sigma_p)_{max}$ for any particular strip adjustment problem. Approximations to these quantities are possibly best determined from the appropriate *absolute* standard errors derived from check points distributed uniformly throughout the strip, and not used as control points for the actual polynomial fit. To include discrepancies at the control points in the calculation of standard errors, especially when these points tend to exceed the number of check points, serves only to suppress the reality of the situation by giving too low estimates of the relevant standard errors.

From what has been written, strips provided with sparse control distributions should not be used when judging the merits or demerits of one polynomial type over another. Special test areas, densely controlled, and hence having an abundance of check data are the only units capable of providing reliable comparisons. Whether these test areas should be surveyed and photographed to some preconceived standard of accuracy, or not, or be a theoretical test[17] made to simulate an actual practical example is, in the authors' opinion, a matter of personal preference. The important thing is that secondary discontinuities resulting from error propagation are in fact manifested and, in consequence, are accommodated by the fitted polynomial.

After all, the criteria used in discriminating between the effectiveness of one polynomial compared with another are solely relative and it could well be argued that a purely mathematical model, quite unrelated to things practical, would be adequate for the above tests. Such a proposition might be sound enough in itself; nonetheless a theoretically perfect model could not be expected to test the relative flexibility of polynomials, unless the kinds of discontinuities encountered in practice were *built-in*.

It is highly desirable therefore that a suitable common test *model* be used by researchers and others when examining polynomial forms for adjustment and transformation purposes. Recent tests using the I.T.C. Test Block[17], one such test *model* form, and triangulations with grid-plates on a Wild Autograph A7 have proved most rewarding.

Different applications of conformal polynomials in the adjustment of strips of aerial triangulation are published. These range from the adjustment of X and Y simultaneously, using second and even higher order terms[15] and Z separately, to the adjustment of X and Y, and X and Z,[14] jointly, using third-order terms. The choice of one-pass or two-pass processing systems remains almost solely a matter of limitation of computer core-storage. All non-linear methods so far devised provide only simultaneous two-space adjustment procedures and therefore the retention of conformality between each of the dimensions of three space is an unattainable ideal at the present time.

Mikhail[18] has shown using quarternions that simultaneous three dimensional transformations of degree higher than the first are not likely to exist and has suggested the application of conformality in the projection planes as an acceptable compromise. It is felt, however, that this suggestion is of limited value only as the latter concept is found to be tenable for polynomials comprising terms of order less than the third. Arthur[19] disputes Mikhail's, and also Schut's[14], contentions but, it is felt, produces no substantial evidence to refute the conclusions obtained. Recent papers by Baetslé[24] and Schut[25] shed further light on the question of transformations in three dimensions and the unlikelihood of these existing in second- or higher-order forms. How does this affect the argument for using conformal polynomials?

As there is no way of maintaining point-geometry in three dimensions after nonlinear transformations it would appear that no argument can be produced as a consequence to justify any insistence on the retention of conformality in high-order adjustment polynomials in two dimensions, based solely on the conformal property. It may, of course, be argued that as different precision criteria are generally specified for planimetry and height the separation of the adjustment of the third dimension from the other two is justified in practice, and hence the application of conformal polynomials for the adjustment of X and Y is more or less axiomatic.

Such an argument could only have weight if the results of a transformation using conformal polynomials produced smaller σ_p's and σ_z's than transformations via any other polynomial forms. Further, this contention is unlikely to be proved correct as, if experience so far gained is any indication, conformal polynomials show themselves to be more inflexible than some other types. Second generation high speed digital computers also reduce the influence of the *convenience* factor of the conformality property to a level which can often be neglected.

The determination of the fitted polynomial coefficients is carried out by well-known adjustment procedures which reduce to the solution of systems of linear equations (often large) by elimination of matrix inversion techniques. By judiciously selecting the kind of polynomial, assuming that arguments concerning accuracy are equal, it is possible to reduce all off-diagonal terms of the matrix of coefficients of the normal equations to zero. Such polynomials are referred to as orthogonal, examples of these being those of Laguerre, Legendre, Chebyshev, Gegenbauer, Jacobi and Hermite, all of which satisfy the orthogonality conditions.

$$\int_a^b \omega(x)\Phi_m(x)\Phi_n(x)dx = 0, \qquad m \neq n$$

$$\int_a^b \omega(x)\Phi_n^2(x)dx \neq 0,$$

for a set of functions $\Phi_0(x)$, $\Phi_1(x)$, $\cdots \Phi_m(x)$ in the interval $a \leq x \leq b$ where the weighting function $\omega(x)$ is non-negative. Taking advantage of these properties, fitting a surface to known $Z_{jk} = F(x_j, y_k)$, $(j, k = 0, 1, 2, \cdots n)$ reduces to a linear combination of known orthogonal polynomials in accordance with the form

$$Z = \sum_{i=0}^{n} a_i(y)T_i(x) \tag{3}$$

in which $a_i(y)$, $(i = 1, 2 \cdots n)$ are representable in the form

$$a_i(y) = \sum_{e=0}^{n} a_{ie}T_e(y),$$

the a_{ie} all being constants and the $T_e(y)$ being the corresponding orthogonol polynomial approximations to y. For any given y_i,

$$a_i(y) = \frac{\sum_{k=1}^{s} T_i(x_k)f(x_k y)}{\sum_{k=1}^{s} T_i^2(x_k)},$$

the $f(x_k, y)$'s being the appropriate values of Z at the y, x_k intersections. This then presupposes the $a_i(y)$ are also known for values of Z within an assigned x, y framework. Within these imposed restrictions the polynomial form may be written[23]:

$$Z = \sum_{i=0}^{n} T_i(x) \left[\sum_{e=0}^{n} \left(\frac{\sum_{k=1}^{s} T_i(x_k) \sum_{j=1}^{m} T_e(y_j) f(x_k y_j)}{\sum_{k=1}^{s} T_i^2(x_k) \sum_{j=1}^{m} T_e^2(y_j)} \right) T_e(y) \right]$$

$$= \sum_{i=0}^{n} T_i(x) \sum_{e=0}^{n} a_{ie} T_e(y) \tag{4}$$

by normalizing the initial values of x to the interval $(-1, 1)$, existing orthogonal function tables may be used in evaluating the coefficients of the polynomial terms for Z. The y-limits are proportionately scaled.

For research purposes, using theoretical models or formalised test areas with Equation 4 is very satisfactory, particularly if repetitive measurements and adjustments are involved. Its general application, however, is not completely straightforward since the orthogonal polynomials have to be evaluated at non-uniformly spaced (x, y), a procedure which cannot be handled efficiently without access to an automatic computer. The possibility of using orthogonal polynomials for block adjustment is well-worth investigating and tests have already been commenced. According to Vlcek[21] the method can be extended to problems involving three or more independent variables.

Statistics of Test Adjustments

In Table 1 the standard errors in planimetry and height are given for two tests using the first twenty models of Strip 1 of the I.T.C. Test Block.[17] The first adjustment example was conducted using a two-section composed polynomial of second order for height and a separate third order polynomial form, after van der Weele for planimetry. The second example cited followed Mikhail's[18] suggestion of conformality in the cardinal projection planes, and because of this is restricted to the use of polynomials of second degree only. Nineteen control points were used in each of the tests, the distribution of control being identical in each case. No special restraints were imposed on individual point locations and therefore no model in the strip was fully controlled.

The tests were repeated with varying numbers and distributions of control; that for fourteen control points randomly distributed is reflected appropriately in Table 1. The standard errors are computed from 63 points in each case, including 44 and 49 check points, respectively, the results indicating that a small variation in the number and relative positions of control points does not appear to affect the results.

Table 1: Statistics of Test Adjustments

Metres	*(i) Composed Polynomial*		*(ii) 3rd Order Polynomial*		*(iii) 2nd Order Polynomials conformal in the Projection Planes*	
No. of Control Points	19	14	19	14	19	14
Scale of Theoretical Photography: 1:43,500						
X	—	—	2.13	2.17	2.77	2.93
Y	—	—	2.60	2.54	2.75	2.63
P	—	—	3.37	3.34	3.91	3.93
Z	2.16	2.12	—	—	3.19	3.20
‰ H	0.32	0.32	—	—	0.48	0.48

TABLE 2: STANDARD ERRORS FROM PRACTICAL EXAMPLES

Adjustment Function	*Standard Errors (ft)*			
	x	y	z	z *as* °/₀₀ H
Composed Polynomials	—	—	12	0.66
3rd Order Polynomials	—	—	40	2.22
2nd Order Polynomials	—	—	59	3.27
3rd Order Polynomials after v.d. Weele	11.2	12.6	—	—
2nd Order Conformal Polynomials	35.6	8.2	—	—

An actual strip of 1:36,000 photography, of 35 models, flown with a Wild RC8 camera and triangulated on a Wild Autograph A7 was adjusted using (*i*) and (*ii*). In all 14 control points were used distributed evenly at ten model intervals. Unfortunately, only ten independent check points were available for the calculation of the *absolute* standard errors, consequently it would be unwise to base any conclusion on such a small sampling. Nonetheless the standard error comparisons in Table 2 with more conventional polynomials are worth mentioning.

The quality of this last example is somewhat doubtful and the tabulated standard errors should not therefore be taken as indicative of the ultimate precisions obtainable using the various functions mentioned. Table 2 is, however, valid as a means of showing the comparative superiority of the composed polynomial for height adjustment over the other forms, and of the third order hybrid polynomial over the second conformal polynomial for adjustment of planimetry.

CONCLUSIONS

Investigations being conducted by the authors indicate that no one adjustment function can be relied upon to provide the smallest standard deviations, the precision criterion currently used, for every photogrammetric strip triangulation adjustment. Every polynomial form can, however, be expected to have its own distinctive *maximum performance* limits beyond which it is not possible to go. It is believed therefore that some optimum number and distribution of control in the strip must exist for each adjustment polynomial. How this opinion may be proved or disproved remains a matter for further investigation. A superficial illustration of the apparent lack of effect of a change in number and distribution of ground control points in a strip adjustment is provided in Table 1.

The tendency of many photogrammetrists to favor the use of conformal polynomials regardless of the length of the strips triangulated is in the opinion of the authors unjustified. For short strips and few control points the conformal function, with its restraints, possibly has some advantages. For long strips, having a large number of control points, where the status of the control and photogrammetric reference systems differ greatly, the composed hybrid type of function is probably better. Certainly, tests so far completed show this to be so. Even for the adjustment of short strips, the latter polynomial type may well prove to be the most effective.

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