JOE M. ESTES*
USAF Aeronautical Chart & Information Center
St. Louis, Missouri 63118

The Anharmonic Method of Rectification

The mathematical solution of the photo system point by point derives positions from minimum control information

INTRODUCTION

THE ANHARMONIC METHOD provides a non-rigorous simultaneous solution to the classic problem of rectifying a photographic point and determining grid coordinates for the point in terms of a given geodetic or plane coordinate system. The anharmonic or cross ratio method removes tilt distortion from the photographic angles between known control points and points of unknown position either by

ABSTRACT: *This paper presents a mathematical solution to a photographic problem which was previously solved by a graphic technique (paper strip method). The theory of the anharmonic or cross ratio is contained in the Manual of Photogrammetry, Second Edition, pages 449 through 451, and Third Edition, pages 803 through 808. Therefore, this paper is devoted to the mathematical derivations of the formulation that was programmed and compiled at ACIC and used on an IBM 1620 electronic computer with 20K storage.*

graphical or mathematical techniques. The positions for the unknown points in the orthogonal plane are established by the intersection of rays from known control positions and the rectified photographic angles.

The principal advantage of the anharmonic method is the speed with which positions can be determined from minimal control data, a single aerial photograph, and no determination of the amount or direction of photographic tilt. The mathematical solution, utilizing a comparator and an electronic computer, has additional advantages in that it eliminates the inaccuracies from the graphical method, and groups of points can be established with very little more time than required to establish a single point.

MATHEMATICAL DERIVATIONS

Because the formulas are identical for each of the four corners of the quadrilateral generated by control points in the photographic and orthogonal planes (Figures 1 and 2), the formulation is derived about points A and a only.†

* Presented at the Semi-Annual Convention of the American Society of Photogrammetry, Los Angeles, Calif., September 1966 under the title "The Anharmonic Method of Point by Point photo Rectification (Mathematical Solution). (Those interested in the flow diagrams and Fortran program, which are omitted in this publication, should request them from the author.—*Editor*.)

† Photo coordinates in Figure 1 are designated X_a, Y_a; X_b, Y_b; etc. Orthogonal grid coordinates, Figure 2, are designated X_A, Y_A; X_B, Y_B; etc.

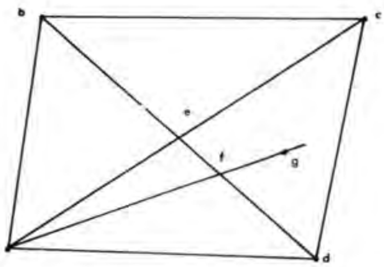

FIG. 1. Photographic point relationship.

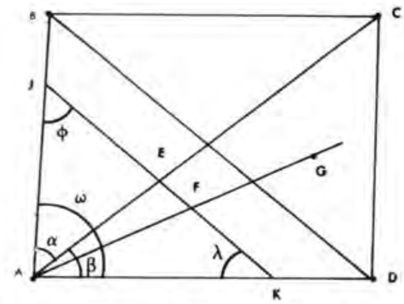

FIG. 2. Orthogonal plane point relationship.

Compute the slopes of ac, bd, and ag:

$$Mac = \frac{Y_a - Y_c}{X_a - X_c}; \quad Mbd = \frac{Y_b - Y_d}{X_b - X_d}; \quad Mag = \frac{Y_a - Y_g}{X_a - X_g}. \tag{1}$$

Solve the equations for ac and bd simultaneously to obtain X_e, Y_e:

$$X_e = \frac{Mac(X_a) - Mbd(X_b) + Y_b - Y_a}{Mac - Mbd}$$

$$Y_e = Mac(X_e) - Mac(X_a) + Y_a. \tag{2}$$

And solve the equations for bd and ag simultaneously for X_f, Y_f:

$$X_f = \frac{Mag(X_a) - Mbd(X_b) + Y_b - Y_a}{Mag - Mbd}$$

$$Y_f = Mag(X_f) - Mag(X_a) + Y_a. \tag{3}$$

Compute the distances eb, ed, bf, and bd:

$$eb = \left[(X_e - X_b)^2 + (Y_e - Y_b)^2\right]^{1/2}$$

$$ed = \left[(X_e - X_d)^2 + (Y_e - Y_d)^2\right]^{1/2}$$

$$bf = \left[(X_b - X_f)^2 + (Y_b - Y_f)^2\right]^{1/2}$$

$$bd = eb + ed. \tag{4}$$

Let lines JE, KE, and JF equal eb, ed, and bf respectively. Compute the slopes of AB, AC, and AD:

$$MAB = \frac{Y_A - Y_B}{X_A - X_B}; \quad MAC = \frac{Y_A - Y_C}{X_A - X_C}; \quad MAD = \frac{Y_A - Y_D}{X_A - X_D}. \tag{5}$$

Using the slope-tangent formula, find $\tan \alpha$, $\tan \beta$, $\tan \omega$:

$$\tan \alpha = \frac{MAB - MAC}{1 + (MAB)(MAC)}$$

$$\tan \beta = \frac{MAC - MAD}{1 + (MAC)(MAD)}$$

$$\tan \omega = \frac{MAB - MAD}{1 + (MAB)(MAD)}. \tag{6}$$

Compute the angles φ and λ;

$$AE = \frac{eb \sin \varphi}{\sin \alpha} = \frac{ed \sin \lambda}{\sin \beta}$$

$$\sin \lambda = \frac{eb \sin \varphi \sin \beta}{ed \sin \alpha} \cdot$$

Let

$$R = \varphi + \lambda = 180° - \omega$$

$$\lambda = R - \varphi$$

$$\sin \lambda = \sin R \cos \varphi - \cos R \sin \varphi$$

$$\frac{eb \sin \varphi \sin \beta}{ed \sin \alpha} = \sin R \cos \varphi - \cos R \sin \varphi$$

$$\frac{eb \sin \beta}{ed \sin \alpha} = \sin R \cot \varphi - \cos R$$

$$\cot \varphi = \frac{eb \sin \beta}{ed \sin \alpha \sin R} + \cot R$$

$$\sin R = \frac{\cos R}{\cot R}$$

$$\cot \varphi = \cot R\left(1 + \frac{eb \sin \beta}{ed \sin \alpha \cos R}\right)$$

$$\cot \lambda = \cot R\left(1 + \frac{ed \sin \alpha}{eb \sin \beta \cos R}\right) \tag{7}$$

where

$$\cot R = -1/\tan \omega$$
$$\cos R = (\cot R)/(1 + \cot^2 R)^{1/2}$$
$$\sin \alpha = 1/[1 + (1/\tan \alpha)^2]^{1/2}$$
$$\sin \beta = 1/[1 + (1/\tan \beta)^2]^{1/2} \tag{8}$$

Compute the distances AJ and AK:

$$AJ = bd \sin \lambda/\sin \omega$$
$$AK = bd \sin \varphi/\sin \omega \tag{9}$$

where

$$\sin \lambda = 1/(1 + \cot^2 \lambda)^{1/2}$$
$$\sin \varphi = 1/(1 + \cot^2 \varphi)^{1/2}$$
$$\sin \omega = 1/[1 + (1/\tan \omega)^2]^{1/2} \tag{10}$$

Compute the ground distances AB and AD:

$$AB = [(X_A - X_B)^2 + (Y_A - Y_B)^2]^{1/2}$$
$$AD = [(X_A - X_D)^2 + (Y_A - Y_D)^2]^{1/2}. \tag{11}$$

The ground coordinates X_J, Y_J; X_K, Y_K are derived:

$$X_J = X_A + \frac{AJ}{AB}(X_B - X_A)$$

$$Y_J = Y_A + \frac{AJ}{AB}(Y_B - Y_A)$$

$$X_K = X_A + \frac{AK}{AD}(X_D - X_A)$$

$$Y_K = Y_A + \frac{AK}{AD}(Y_D - Y_A). \quad (12)$$

And the ground coordinates X_F and Y_F:

$$X_F = X_J + \frac{bf}{bd}(X_K - X_J)$$

$$Y_F = Y_J + \frac{bf}{bd}(Y_K - Y_J). \quad (13)$$

Compute the slope of line AG:

$$MAG = \frac{Y_A - Y_F}{X_A - X_F}. \quad (14)$$

Develop the equation for line AG:

$$Y_G = MAG(X_G) - MAG(X_A) + Y_A. \quad (15)$$

Equations for lines BG, CG, and DG are developed:

$$Y_G = MBG(X_G) - MBG(X_B) + Y_B \quad (16)$$

$$Y_G = MCG(X_G) - MCG(X_C) + Y_C \quad (17)$$

$$Y_G = MDG(X_G) - MDG(X_D) + Y_D. \quad (18)$$

Solve Equations 15 and 16 simultaneously for X_G and Y_G*:

$$X_G = \frac{MBG(X_B) - MAG(X_A) + Y_A - Y_B}{MBG - MAG}$$

$$Y_G = MAG(X_G) - MAG(X_A) + Y_A. \quad (19)$$

* Computations for X_G and Y_G are checked by solving any two of Equations 15, 16, 17, and/or 18 simultaneously.