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Derivation of Parallax Equations

The coefficients in these general forms are independent of the sequence of the rotations

(*Abstract on page 1177*)

INTRODUCTION

AT THE OUTSET OF the derivation of the parallax equations, it is usual to choose a particular set of three angular parameters. This immediately limits the derivation to only those plotting instruments which incorporate these same parameters; for each different choice of angular parameters a new derivation is required. The derivation described in this article postpones the necessity of making a choice of parameters until after a *canonical* parallax equation is obtained. This *canonical* equation is equally applicable to every instrument whose point of rotation is the nodal point, and to every photographic orientation.

DERIVATION

The model coordinates may be written as a function of the photographic coordinates by the following formulas:

$$(X - X_0) = (Z - Z_0)\left[\frac{a_{11}x + a_{12}y + a_{13}f}{a_{31}x + a_{32}y + a_{33}f}\right]$$

$$(Y - Y_0) = (Z - Z_0)\left[\frac{a_{21}x + a_{22}y + a_{23}f}{a_{31}x + a_{32}y + a_{33}f}\right] \qquad (1)$$

where

X, Y, Z are the model coordinates of a point,
X_0, Y_0, Z_0, are the model coordinates of the nodal point, and
x, y, f are the photographic coordinates, and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is the matrix of the nine direction cosines relating the two coordinate systems.

By Definitions 2a, Equations 1 may be written in the form of Equations 2b:

$$\begin{bmatrix} m \\ n \\ q \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ f \end{bmatrix} \qquad (2a)$$

$$X - X_0 = (Z - Z_0)\frac{m}{q}; \qquad Y - Y_0 = (Z - Z_0)\frac{n}{q}. \qquad (2b)$$

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As the parallax equations are differential equations, Equations 2 must be differentiated with respect to three translations and nine direction cosines. Differentiating Equations 2 gives:

$$dX = dX_0 - \frac{m}{q}\,dZ_0 + (Z - Z_0)\left[\frac{(dm)q - m(dq)}{q^2}\right]$$

$$dY = dY_0 - \frac{n}{q}\,dZ_0 + (Z - Z_0)\left[\frac{(dn)q - n(dq)}{q^2}\right] \tag{3a}$$

where

$$\begin{bmatrix} dm \\ dn \\ dq \end{bmatrix} = \begin{bmatrix} \delta a_{11} & \delta a_{12} & \delta a_{13} \\ \delta a_{21} & \delta a_{22} & \delta a_{23} \\ \delta a_{31} & \delta a_{32} & \delta a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ f \end{bmatrix} \tag{3b}$$

The photographic coordinates in Equations 3b may be eliminated by taking the inverse of Equations 1;

$$x = f\left[\frac{a_{11}(X - X_0) + a_{21}(Y - Y_0) + a_{31}(Z - Z_0)}{a_{13}(X - X_0) + a_{23}(Y - Y_0) + a_{33}(Z - Z_0)}\right]$$

$$y = f\left[\frac{a_{12}(X - X_0) + a_{22}(Y - Y_0) + a_{32}(Z - Z_0)}{a_{13}(X - X_0) + a_{23}(Y - Y_0) + a_{33}(Z - Z_0)}\right] \tag{4}$$

which, by Definitions 5a, may be written in the form (5b):

$$\begin{bmatrix} M \\ N \\ Q \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix} \tag{5a}$$

$$x = f\frac{M}{Q}; \quad y = f\frac{N}{Q}; \quad f = f\frac{Q}{Q}\,. \tag{5b}$$

By substituting Equations 5b into 2a and 3b, and then substituting Equations 5a into the result, the following equations are obtained:

$$\begin{bmatrix} m \\ n \\ q \end{bmatrix} = \frac{f}{Q}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix}$$

$$\begin{bmatrix} dm \\ dn \\ dq \end{bmatrix} = \frac{f}{Q}\begin{bmatrix} \delta a_{11} & \delta a_{12} & \delta a_{13} \\ \delta a_{21} & \delta a_{22} & \delta a_{33} \\ \delta a_{31} & \delta a_{32} & \delta a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix}. \tag{6}$$

Equations 6 may be further reduced due to the fact that the nine direction cosines are subject to six conditions of orthogonality:

$$\begin{aligned} a_{11}^2 + a_{12}^2 + a_{13}^2 &= 1 & \qquad a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 &= 1 & \qquad a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} &= 0 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 &= 1 & \qquad a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0. \end{aligned}$$

The differentiation of these conditions gives:

$$a_{11}\delta a_{11} + a_{12}\delta a_{12} + a_{13}\delta a_{13} = 0$$

$$a_{21}\delta a_{21} + a_{22}\delta a_{22} + a_{23}\delta a_{23} = 0$$

$$a_{31}\delta a_{31} + a_{32}\delta a_{32} + a_{33}\delta a_{33} = 0$$

$$(a_{11}\delta a_{21} + a_{12}\delta a_{22} + a_{13}\delta a_{23}) + (a_{21}\delta a_{11} + a_{22}\delta a_{12} + a_{23}\delta a_{13}) = 0$$

$$(a_{11}\delta a_{31} + a_{12}\delta a_{32} + a_{13}\delta a_{33}) + (a_{31}\delta a_{11} + a_{32}\delta a_{12} + a_{33}\delta a_{13}) = 0$$

$$(a_{21}\delta a_{31} + a_{22}\delta a_{32} + a_{23}\delta a_{33}) + (a_{31}\delta a_{21} + a_{32}\delta a_{23} + a_{33}\delta a_{23}) = 0.$$

If the sum of products in the first set of parentheses of each of the last three equations be designated $-\mathrm{K}$, Φ, $-\Omega$, respectively, then those in the second set of parentheses are K, $-\Phi$, Ω (Planck 1957).

Abstract: *The parallax equations are derived in a manner which is independent of primary, secondary, and tertiary rotations, and of orientation. These equations take exactly the same form as the equations for the vertical case, yet are not so restricted. Three examples of the applicability to theoretical and computational problems are given.*

Because of these orthogonality conditions, Equations 6 reduce to:

$$\begin{bmatrix} m \\ n \\ q \end{bmatrix} = \frac{f}{Q}\begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix}$$

$$\begin{bmatrix} dm \\ dn \\ dq \end{bmatrix} = \frac{f}{Q}\begin{bmatrix} 0 & \mathrm{K} & -\Phi \\ -\mathrm{K} & 0 & \Omega \\ \Phi & -\Omega & 0 \end{bmatrix}\begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix}. \tag{7}$$

The substitution of Equations 7 into Equations 3a gives the parallax equations:

$$dX = dX_0 - \left(\frac{X - X_0}{Z - Z_0}\right)dZ_0 + (Y - Y_0)\mathrm{K} - (Z - Z_0)\left(1 + \left(\frac{X - X_0}{Z - Z_0}\right)^2\right)\Phi + \frac{(X - X_0)(Y - Y_0)}{(Z - Z_0)}\Omega$$

$$dY = dY_0 - \left(\frac{Y - Y_0}{Z - Z_0}\right)dZ_0 - (X - X_0)\mathrm{K} - \frac{(X - X_0)(Y - Y_0)}{(Z - Z_0)}\Phi + (Z - Z_0)\left(1 + \left(\frac{Y - Y_0}{Z - Z_0}\right)^2\right)\Omega. \tag{8}$$

If in Equations 8 the origin is taken at the nodal point, an especially simple form (which may be designated as the *canonical form* of the parallax equation) results:

$$dX = dX_0 - \frac{X}{Z}dZ_0 + Y\mathrm{K} + Z\left(1 + \left(\frac{X}{Z}\right)^2\right)\Phi + \frac{XY}{Z}\Omega$$

$$dY = dY_0 - \frac{Y}{Z}dZ_0 - X\mathrm{K} + \frac{XY}{Z}\Phi + Z\left(1 + \left(\frac{Y}{Z}\right)^2\right)\Omega. \tag{9}$$

Equations 9 are of the same form as the parallax equations usually derived under the assumption that the photograph is vertical. It is evident that the parameters K, Φ, Ω may be interpreted as infinitesimal rotations about an axis system which is parallel to the model system. However, as no assumption was made as to the numerical values of the elements of orientation, Equations 8 are equally applicable to any orientation. Also, as no primary, secondary, and tertiary rotations were incorporated in the derivation, the equations are applicable to every plotting instrument (provided the nodal point is the point of rotation of the projector).

Following are several examples of the use of Equations 9.

EXAMPLE 1

In order to adapt Equations 9 to a specific instrument and orientation, it is only necessary to determine the order of rotations for the instrument. As an example, suppose that the formula for a convergent photograph is desired, and that the orientation matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By differentiating the above and evaluating at $\phi=\phi$, $\kappa=o$, $\omega=o$, one obtains:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$\begin{bmatrix} \delta a_{11} & \delta a_{12} & \delta a_{13} \\ \delta a_{21} & \delta a_{22} & \delta a_{23} \\ \delta a_{31} & \delta a_{32} & \delta a_{33} \end{bmatrix} = \begin{bmatrix} -\sin\phi d\phi & \cos\phi d\kappa & \cos\phi d\phi \\ -(d\kappa + \sin\phi d\omega) & 0 & \cos\phi d\omega \\ -\cos\phi d\phi & -(\sin\phi d\kappa + d\omega) & -\sin\phi d\phi \end{bmatrix}.$$

The product of the above two matrices gives:

$$\begin{aligned} \mathrm{K} &= \cos\phi d\kappa \\ \Phi &= d\phi \\ \Omega &= \sin\phi d\kappa + d\omega. \end{aligned} \tag{10}$$

Substituting Equations 10 into 9 and rearranging:

$$dX = dX_0 - \frac{X}{Z}dZ_0 + \left[Y\cos\phi + \frac{XY}{Z}\sin\phi\right]d\kappa + Z\left(1 + \frac{X^2}{Z^2}\right)d\phi + \frac{XY}{Z}d\omega$$

$$dY = dY_0 - \frac{Y}{Z}dZ_0 + \left[-X\cos\phi + Z\left(1 + \frac{Y^2}{Z^2}\right)\sin\phi\right]d\kappa$$

$$+ \frac{XY}{Z}d\phi + Z\left(1 + \frac{Y^2}{Z^2}\right)d\omega. \tag{11}$$

EXAMPLE 2

It has been proven (Zeller 1952) that for a vertical photograph the critical surface of relative orientation is in general a second degree surface. This is now shown to hold for any values of the parameters, for from Equations 9 when $dY=0$,

$$dY = 0 = \Omega Y + \Omega Z + \Phi XY - \mathrm{K}XZ - dZ_0Y + dY_0Z. \tag{12}$$

Equation 12 is of the same form as that given by Zeller for a vertical photograph; however Equation 12 is not restricted to the vertical case.

EXAMPLE 3

If Equations 8 or 9 are to be used in an analytical solution for relative or absolute orientation, there is no need to choose three rotations as in Example 1. An orthogonal correction matrix can be constructed directly from K, Φ, and Ω, (Schut 1958–59).

$$\boldsymbol{C} = \frac{1}{\boldsymbol{D}}\begin{bmatrix} 1 + \Omega^2 - \Phi^2 - \mathrm{K}^2 & 2(\Omega\Phi + \mathrm{K}) & 2(\Omega\mathrm{K} - \Phi) \\ 2(\Omega\Phi - \mathrm{K}) & 1 - \Omega^2 + \Phi^2 - \mathrm{K}^2 & 2(\Phi\mathrm{K} + \Omega) \\ 2(\Omega\mathrm{K} + \Phi) & 2(\Phi\mathrm{K} - \Omega) & 1 - \Omega^2 - \Phi^2 + \mathrm{K}^2 \end{bmatrix}$$

$$D = 1 + \Omega^2 + \Phi^2 + \mathrm{K}^2. \tag{13}$$

This C-matrix may be multiplied by the first approximation to the A matrix so as to obtain a new approximation.

The schedule of computations is:

1. Calculate X, Y from Equations 1.
2. Calculate the orientation corrections from Equations 8.
3. Calculate the correction matrix from Equations 13.
4. Calculate new approximations to parameters.

The complete absence of trigonometric functions makes this particularly attractive for a desk calculator.

REFERENCES

Planck, M., 1957, Introduction to Theoretical Physics, Vol. 1; *General Mechanics*, Mac Millan Company, New York, p. 177.

Schut, G. H., 1958–1959, Construction of Orthogonal Matrices and their Application in Analytical Photogrammetry, *Photogrammetria*, Vol. XV, Nr. 4, p. 156.

Zeller, M., 1952, *Text Book of Photogrammetry*, H. K. Lewis & Co. Ltd., London, p. 202.

Errata

In the Membership list of the July 1967 issue, Mr. Morris M. Thompson was inadvertently credited with being a registered land surveyor (LS), which he informs us is incorrect, although the PE (Professional Engineer) citation *is* valid.

Because of the numerous difficulties encountered in compiling and reproducing the list of more than 4,000 entries, it is understandably improbable to prepare an errorless product. We hope that members will inform us of all of the discrepancies so that we can maintain current records.