Eleventh Congress of the
International Society for Photogrammetry
Lausanne, Switzerland, July 8–20, 1968

Invited paper for Commission III

Review of Strip and Block Adjustment During the Period 1964-1967

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INTRODUCTION

THE PERIOD 1964–1967 is characterized by the further development and the successful completion of a number of computer programs for the simultaneous adjustment of aerial photographs in large blocks. Both the direct solution and the iterative solution of the resulting system of normal equations have proved to be entirely practical.

Block adjustments in which models, sections or strips are adjusted as units are still far more common. They can be divided into two main groups: adjustment of models or sections by means of similarity transformations, and adjustment of strips or parts of strips by means of polynomial transformations of higher than the first degree.

The proponents of the adjustment of photographs view this adjustment as the most *rigorous* solution and as the main trend in computational photogrammetry. Others regard the similarity transformation of models as the *true* or *rigorous* least squares adjustment and the ultimate solution, and they predict a decrease of interest in the polynomial adjustment. Nevertheless, the polynomial adjustment has many adherents and is much used because it is the easiest to program and to use, and because it gives a very satisfactory accuracy for topographic mapping.

Any interest that may still exist in the analog adjustment of input data has not resulted in more than one published paper.

SIMULTANEOUS ADJUSTMENT OF PHOTOGRAPHS

1. USE OF THE COLLINEARITY CONDITION

References [5] to [23] deal with the simultaneous adjustment of photographs in blocks or strips. References [24] to [31] discuss the solution of the large system of normal equations in this adjustment.

Already before the London Congress, H. H. Schmid and D. C. Brown used the condition of collinearity of image point, perspective centre, and object point for the simultaneous adjustment of a set of photographs. This condition leads to the linearized equation

$$\boldsymbol{v} + \boldsymbol{B}\boldsymbol{\delta} = \boldsymbol{\varepsilon} \tag{1}$$

in which

- $\boldsymbol{v}$ is the vector of corrections to the photograph coordinates,
- $\boldsymbol{\delta}$ is the vector of corrections to the parameters of camera orientation and to the coordinates of object points,
- $\boldsymbol{B}$ is a matrix of coefficients, and
- $\boldsymbol{\varepsilon}$ is the vector of residuals of the photograph coordinates in the non-linear condition equations.

The normal equations become

$$\begin{aligned} -\boldsymbol{P}\boldsymbol{v} + \boldsymbol{k} \quad &= \boldsymbol{0} \\ \boldsymbol{v} + \quad \boldsymbol{B}\boldsymbol{\delta} &= \boldsymbol{\varepsilon} \\ \boldsymbol{B}^t\boldsymbol{k} \quad &= \boldsymbol{0} \end{aligned} \tag{2a}$$

in which $\boldsymbol{P}$ is the weight matrix of the observed photograph coordinates and is computed as the inverse of the covariance matrix. Further, $\boldsymbol{k}$ is the vector of Lagrange multipliers or correlates, and the superscript t indicates the transpose. Elimination of $\boldsymbol{v}$ gives

$$\begin{aligned} \boldsymbol{P}^{-1}\boldsymbol{k} + \boldsymbol{B}\boldsymbol{\delta} &= \boldsymbol{\varepsilon} \\ \boldsymbol{B}^t\boldsymbol{k} \quad &= \boldsymbol{0}. \end{aligned} \tag{2b}$$

Subsequently, elimination of $\boldsymbol{k}$ gives

$$\boldsymbol{B}^t\boldsymbol{P}\boldsymbol{B}\boldsymbol{\delta} = \boldsymbol{B}^t\boldsymbol{P}\boldsymbol{\varepsilon}. \tag{2c}$$

2. TREATMENT OF CONTROL POINTS

Both Brown [11] and Schmid [5] to [7] have now introduced the use of a covariance matrix for the coordinates of the control points in a block of aerial photographs. Unknown coordinates are given a large variance and for the sake of simplicity, although unrealistically, one assumes that their estimated (*observed*) values are uncorrelated. If a rectangular coordinate system is used other than the one formed from planimetric coordinates and terrain heights, the proper covariance matrix in that system can be easily computed from the relation between the two coordinate systems.

With this procedure, the vector $\boldsymbol{\delta}$ contains also the corrections to the coordinates, known as well as unknown, of all control points. To retain the advantage that only one correction to an *observation* occurs in each condition equation, Equation 1 is supplemented with the equation

$$\boldsymbol{v}_g - \boldsymbol{\delta}_g = \mathbf{0} \tag{3}$$

in which $\boldsymbol{v}_g$ is the vector of corrections to *observed* coordinates of the control points and $\boldsymbol{\delta}_g$ is the vector of corrections to their estimated or given coordinates. The equation expresses that one identifies the *observed* values with the estimated or given values.

This procedure increases the size of the normal equations. However, it makes their formation simpler because now each terrain point receives three coordinate corrections, independent of whether it is completely known, partially known, or unknown.

3. PARTITIONING OF THE NORMAL EQUATIONS

The components of $\boldsymbol{v}$, $\boldsymbol{\delta}$, and $\boldsymbol{k}$ may be partitioned into groups. If the condition equations contain a set of constraint equations in which no observations occur, the corresponding v-components will be equal to zero. In addition, the sequence of the groups in the normal equations may be rearranged. This can lead to a variety of formulas.

It is customary to partition the vector $\boldsymbol{\delta}$ of Equation 2c into a vector $\boldsymbol{\delta}_1$ of corrections for improvement of the camera orientations and a vector $\boldsymbol{\delta}_2$ of corrections to approximate coordinates of terrain points. The attitude corrections included in $\boldsymbol{\delta}_1$ can be either the parameters of a correction matrix or corrections to approximate values of attitude parameters.

The corresponding partitioning of Equation 2c gives:

$$\boldsymbol{N}_{11}\boldsymbol{\delta}_1 + \boldsymbol{N}_{12}\boldsymbol{\delta}_2 = \boldsymbol{\varepsilon}_1$$
$$\boldsymbol{N}_{12}{}^t\boldsymbol{\delta}_1 + \boldsymbol{N}_{22}\boldsymbol{\delta}_2 = \boldsymbol{\varepsilon}_2$$

and elimination of $\boldsymbol{\delta}_2$ gives

$$(\boldsymbol{N}_{11} - \boldsymbol{N}_{12}{}^t\boldsymbol{N}_{22}{}^{-1}\boldsymbol{N}_{12})\boldsymbol{\delta}_1 = \boldsymbol{\varepsilon}_1 - \boldsymbol{N}_{12}{}^t\boldsymbol{N}_{22}{}^{-1}\boldsymbol{\varepsilon}_2. \tag{2e}$$

The components of $\boldsymbol{\delta}_1$ and of $\boldsymbol{\delta}_2$ can be partitioned into groups each of which contains only the corrections for one photograph or for one terrain point, respectively. The correspondingly partitioned matrices $\boldsymbol{N}_{11}$ and $\boldsymbol{N}_{22}$ contain non-zero submatrices along their main diagonal only, while the submatrix of $\boldsymbol{N}_{12}$ which corresponds to point i and photograph j has non-zero elements only if point i has been measured in photograph j.

Accordingly, the inverse of $\boldsymbol{N}_{22}$ can be computed by inverting its submatrices separately and Equation 2e can be computed directly from the Equation 1 by treating all the condition equations for one terrain point as a group and computing the contribution of this group to the normal equations separately. In this way, no space need be reserved for the matrices $\boldsymbol{N}_{22}$ and $\boldsymbol{N}_{12}$. If two photographs have no measured point in common, the corresponding off-diagonal submatrix in Equation 2e is equal to zero.

Each of the block adjustment programs that have been coded along these lines employs a *housekeeping routine*, *collapsing algorithm*, or *indexing technique* to avoid computing and storing zero-submatrices and to keep track of the locations of the non-zero ones.

4. DIRECT SOLUTION OF THE NORMAL EQUATIONS

Even for large blocks, the direct solution of the normal equations has proved to be entirely practical provided that a block-elimination procedure is used. This means that instead of the matrix elements the above-mentioned submatrices of Equations 2d or 2e are used as units in the computations.

Both a Gauss-Cholesky type of elimination ([91] and [21]) and a Gauss-Jordan type [16] are used. If $\boldsymbol{S}$ is an on-diagonal submatrix which is to be used as pivot in the elimination, the Gauss-Cholesky elimination involves the simple computation of an upper triangular matrix $\boldsymbol{T}$ such that $\boldsymbol{T}^t\ \boldsymbol{T} = \boldsymbol{S}$ and of its inverse. The Gauss-Jordan elimination requires the computation of the inverse of $\boldsymbol{S}$. The two elimination procedures perform equally well.

5. ITERATIVE SOLUTION OF THE NORMAL EQUATIONS

The iterative solution of a system of equations has the basic disadvantage that it is difficult to know when a sufficiently good approximation of the exact solution has been reached. On the other hand, especially in the case of a large block, it has the advantages that during the solution the required storage space is not increased and that for a given absolute density of control the required computation time is only directly proportional to the number of photographs. Therefore, for very large blocks the iterative solution will be preferable.

The principal iterative method of solution is the Gauss-Seidel method. Although the proof of the convergence of this method can be found in mathematical textbooks, several writers have deemed it necessary to include the proof in their paper. The asymmetry inherent in the method can be avoided by alternately proceeding forth and back through the vector of unknowns.

The rate of convergence is sufficiently fast only if a block-iterative procedure is used. The simplest submatrices for this purpose are those obtained from the above partitioning of the matrices in the Equations 2d and 2e. For large blocks with sparse control, one may then require from 50 to a few hundred iterations.

The unknowns in the normal equations can be arranged in groups according to the natural or to artificial strips such that the on-diagonal submatrix of each group is connected via a non-zero off-diagonal submatrix to the on-diagonal submatrices of only the preceding and the following group. If these much larger submatrices are used as units in the block-iterative procedure, the required number of iterations is of the order of ten [85]. However, the computations in each iteration step are then much more complicated.

After a number of Gauss-Seidel iterations, one will observe that successive corrections to the obtained values of the unknowns tend to form a geometric series. This fact may be used to accelerate the convergence at that stage.

A simple and safe, but still rather slow, accelerated method consists in computing the ratio r of the corrections in two successive iterations and multiplying the set of Gauss-Seidel corrections in each iteration by $1+r$ or by $1+r+r^2$.

A much faster acceleration method consists in computing the sum of all following terms of the geometric series and using this sum as the correction. The set of Gauss-Seidel corrections is then divided by $1-r$. This is Luysternik's method [32]. If the Gauss-Seidel corrections do not form an exact geometric series, the Luysternik acceleration produces more or less random deviations from the exact solution. A number of ordinary Gauss-Seidel iterations is then required before an acceleration procedure can again be used.

A third acceleration method is called the block-successive overrelaxation method [33]. In theory, this method makes a sophisticated computation of an optimum acceleration factor possible. In practice, such a computation is too complicated. An educated guess as to the value of such a factor is therefore made and the sophistication consists mainly in the vocabulary that is used. This factor will be larger than one, and it must be smaller than two.

A very different type of iterative solution is obtained if an orthogonalization method or a gradient iterative method is used. At the ITC, Kubik [29, 30] has found the method of conjugate gradients [34] to be the most useful one of these methods. In theory, if no round-off errors are introduced, this method converges to the exact solution after as many iterations as there are unknowns. Either the matrix of coefficients and the constant terms of the correction equations or those of the normal equations are stored and used in every iteration. Experience with a test program showed that in general a satisfactory solution was obtained after some 60 iterations. Accordingly, the method appears to be competitive with the Gauss-Seidel block-iterative method.

6. THE ESSA-COAST AND GEODETIC SURVEY PROGRAM ([5]–[10])

From the beginning, the development of computer programs for strip- and block-adjustment at the Coast and Geodetic Survey has followed Schmid's approach.

In the present system, input for the block adjustment is provided by a set of programs or sub-programs for image coordinate refinement, strip triangulation, polynomial strip transformation, and resection of photographs.

In the block adjustment, the Equations 2d are formed. They are solved directly, by a procedure of Gaussian elimination (forward solution) and back substitution (back solution) which operates with the submatrices for each point and those for each photograph as units. With floating-point arithmetic and 14-decimal digit word size, no round-off difficulties are encountered even for the maximum size of block of about 180 photographs with six measured points across the centre of each photograph.

The solution of the Equation 2d consists in corrections to the approximate values of the photograph parameters (that is, to three coordinates and to three rotations) and to the provisional terrain coordinates. Although the adjustment has been written as an iterative procedure, in practice one pass through the adjustment is found to be sufficient.

Although computation costs will depend on the computer and on the time and care spent on the preparation of the program, it is of interest to notice that a small block required nine times more computer time when disk storage was used than when only core storage was used. The adjustment of a block of 180 photographs took about eight times longer than that of a block of 90 photographs.

7. THE PROGRAM OF D. BROWN ASSOCIATES, INC., ([11]–[13])

Here, also, pre-edited input data for the block adjustment can be provided by other programs, and the unknowns in the normal equations are corrections to approximate coordinates and rotations. The block adjustment program computes directly the normal Equations 2e. An iterative solution is employed which, judging by the number of iterations that is required, operates on small submatrices.

On the basis of an investigation of iterative solutions, the method of successive overrelaxation is considered to be the only one that converges sufficiently fast. The acceleration factor $2/(1+\sqrt{(1-r)})$ is used, r being here the ratio between the largest corrections in two successive iterations. Especially if r is close to unity, this leads to an even slower convergence than is obtained with the factor $1+r$. Luysternik's method is rejected because its use in every iteration causes divergence.

Equation 3 is used for control points and for photographs with one or more known parameters. In any second or following pass through the adjustment, the second part of this equation is replaced by the sum of the vectors of corrections obtained earlier. This makes the formation of the normal equations more complicated. It is meant to avoid the complications of the correlation that the preceding adjustment introduces if corrected approximations are used as new *observations*. However, considering the facts that with a properly organized initial positioning one pass through the adjustment can be sufficient and that the correlation of the initial estimates of unknown parameters of different points or photographs is already neglected, this modification of Equation 3 can be dispensed with.

The most recent version of the program [116] has been used to adjust a block of 162 photographs on a computer with only 8,000 words of core storage and four magnetic tape units and to adjust a block of 1,000 photographs (5×200) on a large and fast computer.

8. PROGRAMS EVOLVED FROM 'THE HERGET METHOD' ([14]–[19])

The *Herget method* was initiated in 1954 at Ohio State University under contract with the Aeronautical Chart and Information Centre. It has since gone through a sequence of modifications most of which were sponsored by the U. S. Army Engineer research organization at Fort Belvoir (see [14] and [16]).

Herget used only one type of condition equation for all measurements: that of coplanarity of vectors. These vectors are the vectors from the projection centres to the image points and unit vectors through control points. One photograph at a time was envisaged to be relaxed in an iterative procedure.

In 1956, separate condition equations for partial control points and a rather unusual *scale constraint equation* for three conjugate rays were added by McNair. Subsequently, condition equations for two equal-height points in the same model and for known airbase were added and a simple simultaneous solution of the complete set of normal equations was introduced. At this point, the method became known as the U. S. Geological Survey's *Direct Geodetic Restraint Method*. A new program for the adjustment of up to 22 photographs and with undisclosed further modifications was completed in 1965 [17].

A further series of modifications was initiated in 1961. Weighting of the observations was made possible and a search for an optimum pivotal element was introduced in the direct solution of the normal equations.

In the present version of the program [16], for control points the collinearity equations and Equation 3 are used. For pass points (non-control points), the linearized coplanarity equation and a scale constraint equation have been retained. The latter equation specifies that the distance from ground point to projection centre along the second of three rays must be the same for the two pairs of rays. No approximate coordinates of pass points are needed here. Because such coordinates can be easily computed from the coordinates of any two image points, this is only a small advantage. The need to select combinations of points for the formation of the observation equations and the resulting occurrence of a coefficient matrix for the vector $\boldsymbol{v}$ in Equation 1 which differs from the unit matrix is a slight disadvantage.

Instead of corrections to rotational parameters of the photographs, here parameters of a correction matrix are computed by which the matrix of the approximate orientation is premultiplied. This leads to a slightly simpler formulation of the elements of the matrix $\boldsymbol{B}$. A direct solution of the normal Equations 2e with a block elimination technique is employed. Subsequent to the adjustment, ground coordinates are computed from the corrected image coordinates in two photographs. The program, developed by the Raytheon Co., is called the *Simultaneous Multiple Station Analytical Triangulation Program*.

9. BLOCK ADJUSTMENT AT THE FRENCH INSTITUTE GÉOGRAPHIQUE NATIONAL ([20]–[22])

At the I.G.N., also, the simultaneous adjustment of photographs (bundles) has long been a subject of investigation. De Masson d'Autume [22] describes now a method in which, after an initial positioning of the photographs, the observations are reduced to quasi-observations valid for exactly vertical photographs. The sum of squares of the corrections to these quasi-observations is minimized. This simplifies the computation of the condition equations and of the ground coordinates without, for approximately vertical photography, perceptibly affecting the results. The collinearity Equations 1 and the normal Equations 2e are used.

A direct solution of the normal equations with efficient use of fast-access storage is envisaged by arranging the bundles in groups, each of which has points in common with only the preceding and the following group. In this way, the submatrices of no more than two groups need be in fast-access memory at the same time.

In addition, a procedure is described to correct photograph coordinates for systematic deformation before the block adjustment is performed. In this way, the complications which arise if deformation parameters are introduced as unknowns in the normal equations are avoided. The procedure consists in computing suitable polynomial corrections from the residuals of the adjustment of a block with sufficient ground control. The corrections have been designed to eliminate the various systematic deformations which may occur in a triangulated strip. They are then applied to any other strip to block in which the same conditions apply.

Because different film rolls can have very different systematic distortions, it may be advisable to compute such corrections from measurements of the fiducial marks, separately for each roll or even for each photograph. Further, there is room for disagreement as to whether the derived corrections are the simplest and most suitable ones for the purpose.

LINEAR ADJUSTMENT OF MODELS AND OF SECTIONS

In the case of the triangulation of independent models and of strips, a strip- or block-adjustment by similarity transformation of individual models, or of two-model sections, can be performed. Such adjustments are treated in ref. [20], [21], and [35] to [42] for three dimensions, in ref. [43] to [50] for planimetry only, and in ref. [51] to [54] for heights only.

The Equations 1, 2, and 3 are used here too, but with an appropriate redefinition of the unknowns. Here, $\boldsymbol{v}$ is the vector of corrections to the measured model coordinates, $\boldsymbol{\delta}_1$ is the vector of orientation parameters of the models or of corrections to such parameters, and $\boldsymbol{\delta}_2$ is again the vector of corrections to the approximate coordinates of terrain points. Consequently, the patterns that occur in the matrix of normal equations and the possible methods of solution of these equations are in general the same as in the adjustment of photographs. However, the size of the normal equations can be much reduced by various specifications as well as by the separate adjustment of planimetry and of heights.

An alternative to the solution of the set of simultaneous equations consists in the transformation of model after model in an iterative procedure. King [35] and the present writer [39] have programmed this procedure but compute transformation formulas for all models of one strip simultaneously. King shows that one step in this procedure and the corresponding step in the iterative solution of the complete set of normal equations give the same result. The transformation which follows the computation of the transformation formulas corresponds to an updating of the coefficients of the normal equations. Although that updating (*a Newton iteration*) is sometimes advocated [27], it is of little or no importance if one starts from a reasonably good positioning.

The methods of adjustment can be divided into three groups: (*i*) adjustment of independent models or sections with seven parameters for each unit, (*ii*) adjustment with enforced coordinate connection in points at or near the principal points, (*iii*) the same adjustment with in addition correction for systematic errors. A detailed description of these methods can be found in ref. [38] and [39].

Most of the authors use a method that belongs to the first group. Especially simple is the method of block adjustment given by Roelofs [47]. Here, an internal block adjustment is performed in which scale, azimuth, and shifts of the sections are adjusted separately. Thompson [49] and Van der Weele [50] describe the use of base lines in this adjustment. Thompson's paper is of additional interest because it exposes the often read fallacy that some errors in strip triangulation are by their nature of the third degree in the x-coordinate.

References [38] and [39] describe a method of the second group in which the coordinate connections between models are enforced by choosing the transformed coordinates of the connecting points (or, rather, corrections to their approximate values) as parameters. This reduces the number of parameters from seven to just over four per model.

References [41], [42], [52], and [53], too, enforce the coordinate connection but they use the well-known double summation of the effect of transfer errors. Although this reduces the number of parameters to those of one model of a strip, it produces condition equations for each control point and for each tie point between strips in which corrections to the transfers of scale, azimuth, and tilts occur as corrections to quasi-observations. Jerie, in the latter two references, has reduced the complications which this causes in the formation of the normal equations by the use of smoothing procedures and fictitious points.

Especially in the case of sparse ground control, a provision for the elimination of systematic errors in the strip triangulation should be included. In [38], [52], and [53] this is achieved by including second-degree terms in the transformation. If one wishes to avoid this contamination with the idea of polynomial adjustment, the procedure in ref. [39] can be followed. Here, the conditions that the transfer errors should be equal to zero are replaced by the conditions that, at least in the case of equal model widths, the transfer errors at each two successive connections should be the same.

Strip Triangulation

1. STRIP FORMATION FROM INDEPENDENT MODELS

At several centres, the triangulation of independent models is followed by strip formation and polynomial strip- and block-adjustment. Ref. [55] to [64] treat the strip formation for that purpose.

The strip formation consists in connecting each model to the preceding one by means of a similarity transformation. In most cases, an exact coordinate connection is made at the common perspective centre. Very simple formulas for this purpose are given by Thomson [59], [60], and Schut [57].

Reference [62] gives the standard procedure for determining the model coordinates of the perspective centres from grid measurements made at two heights. Ref. [55] describes the computation of these coordinates by resection, using measurements made at one height. The latter computation requires pre-calibration of the projection cameras.

Inghilleri and Galetto [55] perform only an approximate relative orientation in the analog instrument. The adjustment of the relative orientation, based upon recorded parallaxes, is included in the strip formation.

2. TRIPLETS IN STRIP TRIANGULATION

References [65] to [68] describe two methods of analytical strip triangulation based upon the orientation of triplets. Anderson and McNair perform independent orientations of the triplets. The triplets are joined into strips by making the orientation of the centre photograph of a triplet and the b_X of its first model equal to those obtained for this photograph and for this base component in the preceding triplet. Keller and Tewinkel perform the triplet orientation while enforcing the orientation of its first photograph and the strip coordinates of the points whose images lie across the centre of this photograph.

Consequently, with both methods the manner in which two successive triplets are connected and as a result the strip deformation caused by errors and by deformation of the photographs depends upon the direction of triangulation. This can be avoided by following McNair's recommendation [66] to connect successive independent triplets by similarity transformations using all common points.

It has been claimed that triplet triangulation results in a *stronger* or *more rigid* strip than triangulation by independent relative orientation and scaling of successive models. However, Moellman [69], using C&GS programs, reports that after second- or third-degree polynomial strip adjustment there is no way to distinguish between the results of the two triangulations. McNair [66] has obtained better results from his triplet triangulation than from a model-by-model triangulation. However, for the latter he employs the *modified Herget method* in which the strip coordinates of points in the preceding model are enforced during the orientation of a photograph. Besides, his conclusion is based upon the size of systematic errors and not upon the residuals after polynomial strip adjustments.

It is claimed as an advantage of triplet orientation that it is here possible to recognize and eliminate errors in the x-coordinates of points in the triplet overlap. In the two-photo orientation recognition is possible by comparing the heights of these points in the two models in which they occur. Errors are more evident than in the triplet orientation because here the least squares adjustment does not minimize them. Points with such errors are here simply eliminated from use in the scale transfer and, if the x-error causes Y-parallax, also from the relative orientation.

On balance, the triangulation by means of independent relative orientation of each two successive photographs and scaling of the resulting models has the advantage that it uses simpler formulas and requires only half the computation time [69] of the triplet triangulation. The latter, in the version used by Tewinkel and in the version recommended by McNair, has the advantage that a smoother fit between models is obtained automatically.

3. RADIAL TRIANGULATION

Numerical radial triangulation is discussed in ref. [70] to [76]. Roelofs and Timmerman describe results and the influence of errors if the classical rhomboid triangulation is used. Van den Hout describes the use of triangles in a block adjustment by the *Anblock* method. The lack of general availability of the extensive literature on the rhomboid triangulation may explain recent interest in the numerical formulation of the old graphical radial triangulation by means of alternate resection and intersection (ref. [74] to [76]).

Polynomial Adjustment of Strips

References [17], [36], [38], [39], [69], and [77] to [94] deal with the adjustment of strips, individually or in blocks, by means of polynomial transformations. The adjustment serves to correct the strips for deformation and to obtain a reasonable fit of the transformed strips at the ground-control points and at the tie points.

Reference [84 a to f] have made it gradually clear that conformal transformations by means of polynomials of higher than the first degree do not exist. Therefore, in practice the adjustment of a strip is performed by means of various nearly conformal transformations. It can be performed as a sequence of two-dimensional transformations (see e.g. [85], [86]) or of transformations of planimetry and of heights (see e.g. [82], [88]), in each case with appropriate correction of the remaining coordinate or coordinates. Alternatively, after an initial positioning, it can be performed as one three-dimensional transformation (see e.g. [36] and [84a]). A proper correction for strip deformation cannot be guaranteed if independent polynomials are used for all three coordinates [99] or for planimetry and heights [69].

Before a block adjustment is performed, the strips must first be positioned approximately. In this positioning, a second-degree correction for longitudinal curvature should be included. This makes it possible to perform the block adjustment of planimetry and that of heights separately.

The transformation of planimetric ground coordinates of control points to the axis-of-flight system of a strip is rather common (see e.g. [82]) but is rather awkward if a block adjustment is performed. It can be avoided either by using the known parameters in the formulas for the initial positioning directly in the correction equations and applying the polynomial transformations to axis-of-flight coordinates [36] or by applying a conformal transformation to the coordinates obtained after the initial positioning [85, 86].

As in the cases of block adjustment of photographs and of models, normal equations can be formed for the simultaneous adjustment of all strips. The direct solution of the equations is here rather simple [85], [87]. Alternatively, the iterative procedure can be used in which strip after strip is transformed and tie points from overlapping strips are used as additional control points [36], [78], [85]. The iterative procedure is simpler to program than the simultaneous solution but it consumes more computer time. With the amount of control that is commonly available in topographic mapping, as a rule a sufficient convergence of the iterative procedure is obtained after about ten iterations of the planimetric adjustment and five iterations of the height adjustment [35], [85]. Tewinkel [9] and Jacobs [80] even state that after careful positioning of the individual strips there is often little need for a block adjustment.

Practice as well as theory have long since shown that it is advisable to keep the degrees of the polynomials as low as possible. Restriction to the second degree is possible by dividing long strips into sections. Such sections can be transformed by means of either independent polynomials or composed polynomials.

An investigation of the present writer [39] has shown that with fairly sparse control the block adjustment of models does not give a better absolute accuracy than the block adjustment of strips. Soehngen [87] has obtained better results with the height adjustment of models. However, he uses a larger number and well-located control points.

Subblocks, and External Block Adjustment

Anderson [95, 96] describes the computation of subblocks of 3×3 or $m \times n$ photographs with 60 percent longitudinal and lateral overlap and their assembly into a block by means of similarity transformations. For the internal adjustment of such an assembly, the method of Roelofs [47] would seem to be very suitable.

For the adjustment to ground control of an internally adjusted block, polynomial transformations of the block coordinates could be used. However, with a low degree of the transformations one can hardly expect to obtain a good fit at all ground-control points and with high degrees one may obtain too large errors in uncontrolled areas. A second-degree transformation may be suitable for an initial positioning of the block and to enable the final block adjustment to be performed separately for the three coordinates.

Arthur [97] describes an interpolation method for such an adjustment. However, this method does not give a solution in the case of four ground-control points situated at the corners of a square. Since this should be a well-defined case, the suitability of the method in other cases requires careful examination.

Vlcek [98] and Wainauskas [99] describe the use of orthogonal polynomials. The only advantage of their use appears to be that it may be possible to identify and reject terms that do not contribute significantly to an improvement of the fit at the control points.

Accuracy of Strip and Block Adjustment

Ackermann and Jerie, [100] to [107], have investigated the theoretical accuracy of strip and block adjustment, assuming that systematic errors are either absent or have been eliminated. They deal with the adjustment by means of similarity transformation of models and, for the height adjustment, also the ITC-Jerie analog adjustment.

It is of particular interest that Ackermann [100] finds that in his investigations the similarity transformation of models and the second- or third-degree polynomial strip transformation give about equivalent results.

In a practical test an adjustment of a block of 180 photographs with 60 percent longitudinal and lateral overlap by the U. S. Coast and Geodetic Survey [10] has produced root-mean-square values of the residuals at check points of only nine microns at photograph scale.

The polynomial adjustment of strips with normal lateral overlap gives values that usually vary between 25 and 60 microns. [39]. Jacobs [81], using analytical triangulation and the Coast and Geodetic Survey's corrections for film deformation, obtains values of around 15 microns.

Conclusion

Prof. Schermerhorn's statement that the methodical development of mathematics and programming procedures in analytical triangulation seems to be complete [2] can now be extended to strip and block adjustment. Still, further refinements, modifications, and simplifications of present procedures will un-undoubtedly continue to appear.

For instance, one can expect that more work will be done on the construction of economical direct and iterative solutions of the large systems of normal equations which occur in the adjustment of photographs and of models. In this field a more than four-year-old claim by members of the ITC that an exceptionally economical direct solution is possible by a suitable arrangement of the unknowns still awaits clarification. Brown [11b] has recently made a rather similar claim concerning the iterative solution. In addition, the use of the method of conjugate gradients and of related methods may warrant further investigation.

In the field of analytical triangulation, the simultaneous triangulation of all photographs of a strip in an arbitrary system should not be much more complicated than the triplet triangulation and could with advantage replace the latter.

The adjustment of internally adjusted strips and blocks to ground control should be further investigated. This may provide a very suitable procedure especially for small computers and where the utmost in accuracy is not needed.

Of particular interest is the further development of methods which serve to eliminate the systematic errors which may occur in triangulated strips by means of corrections applied to the photograph coordinates.

A major problem that still remains to be solved is how to satisfy the need of especially the many small photogrammetric organizations for suitable computer programs. Many of these organizations have neither the necessary experience nor the access to a sufficiently large computer to use large sophisticated programs and they would be best served by small programs for specific purposes. As yet, only the U. S. Coast and Geodetic Survey and the National Research Council of Canada have made it their policy to publish their programs.

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