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A Study in Numerical Radial Triangulation

Abstract: *The study was suggested by recent articles by Turpin and by Wolf on the same subject. Radial triangulation differs from conventional analytic aerotriangulation in the sense that the former derives only the horizontal coordinates of points whereas the latter involves both horizontal and vertical. The formulas for the radial problem are accordingly much simpler to compute. The derivation provides a modern technique for establishing the mathematical principle, including the application of least squares and the analysis of redundant data. Two mathematical models are studied, both based on image coordinate measurement such as with a mono comparator; the models are basically equivalent. Five points per model gives more consistent results than three points, but both exhibit the characteristic bending of strips of photographs.*

Introduction

Most all of us who are engaged in photogrammetric work know the basic principles of radial triangulation by analog means. Analytical radial triangulation, on the other hand, does not seem to be widely accepted, particularly in the United States. However, some interest in this subject has been shown recently, as may be verified by two articles published in two successive issues, Nov. 1966 and Jan. 1967, of Photogrammetric Engineering.[1,2] These two articles, which are essentially the same, offered a new method for numerical radial triangulation using photo coordinates, and suggested its application to large-scale highway mapping.

Certain features of the proposed method, in this author's opinion, called for a more thorough treatment and led him to the writing of this paper, although he does not personally advocate the use of radial triangulation techniques. As a matter of fact, it is his belief that current analytical and semi-analytical methods of triangulation are generally preferable if the input data are measured plate coordinates. Nonetheless, for the benefit of those engaged in, or interested in, using the aforementioned method, this paper is written with the hope that it may prove to be of some value. Furthermore in the course of its development, the graduate students at Purdue University were offered an opportunity to apply simple, but fundamental, concepts of analytical photogrammetry using the computer facilities exclusively.

Background

Analytical radial triangulation methods may be divided into two categories depending on the type of input data used. The first of these utilizes *angles* or *directions* measured on instruments such as the Zeiss or Wild Radial Triangulators. Such methods are used mostly in Europe, and usually form chains of rhomboids from the measured directions. Strip coordinates are computed after the adjustment of rhomboids, and then the strips are transformed and adjusted to control.[3]

Unlike the first group, the second group of methods accepts measured *plate coordinates* as input. One such method was suggested by Hallert in 1957.[4] Another is the one proposed by Turpin[1] and Wolf[2] and is the subject of this paper. This method

* Presented at the Semi-Annual Convention of the American Society at St. Louis, Mo., October 1967.

differs from Hallert's by the fact that one does not need to determine the base line (or the trace of the flight line) on the photographs, which, in certain cases, may prove to be either difficult or inaccurate.

For the sake of completeness, a brief outline of the method follows:

1. Photo coordinates of pass points and available horizontal control points are measured by some means.
2. Using trigonometric functions, angles at, or directions from, the principal point are computed from the *measured* coordinates.
3. From given coordinates of three control points in the first overlap area (a minimum requirement of the method) and using techniques of resection the ground coordinates of the principal points of the first pair are determined.
4. By intersection technique, coordinates of three pass points in the triple overlap area are computed using the first two principal points. These three points will then serve as control for the following model and the preceding steps are repeated until a strip is completed.

One must agree that this method is both new and interesting. However, certain aspects of the method may be considered as undesirable and could be simplified and improved upon. For example: unlike Hallert's method, the measured plate coordinates are not retained in the mathematical model; the method of solution is given only for the unique case of three points, thus allowing for no overdetermination, and it would be rather difficult to adapt it to redundant situations using the method of least squares. These drawbacks indicate a rather stiff and limited mathematical model, and this paper is an attempt to alleviate these limitations.

Objective

The primary objective of this paper is not to propose a new method of analytical radial triangulation, but rather to modify the one under discussion. More specifically, we are seeking a mathematical model, or models, that exhibit the following characteristics:

i. retain the use of the original measurements, i.e. the photo coordinates.

ii. allow for the application of rigorous least squares adjustment to deal with the redundant case.

Two such mathematical models have been developed, programmed, and tested both on fictitious as well as actual data. We will first present the derivation of the two models and then the results obtained from the tests performed.

Mathematical Model I

Each image on an aerial photograph is displaced due to relief referred to absolute datum. If the datum is chosen to pass through an arbitrary point, then such a point will not undergo relief displacement with respect to the selected datum. All other images on the photograph representing object points not lying on the datum plane will have relief displacement. Such relief displacement would need to be eliminated before the photographic positions of the datum points can be obtained. This simple geometry is depicted in Figure 1, where a, b, c and e are the images of the object points A, B, C and E, and a', b', c' and e' are the corresponding datum positions.

The datum position points can be transformed to the corresponding ground positions by simple two-dimensional linear transformation. As the datum position coincides with the image point of the reference point (e and e' in Figure 1), the measured coordinates may be used directly in the transformation. Hence, for the reference point we use:

$$X = px - qy + r$$
$$Y = qx + py + s \qquad (1)$$

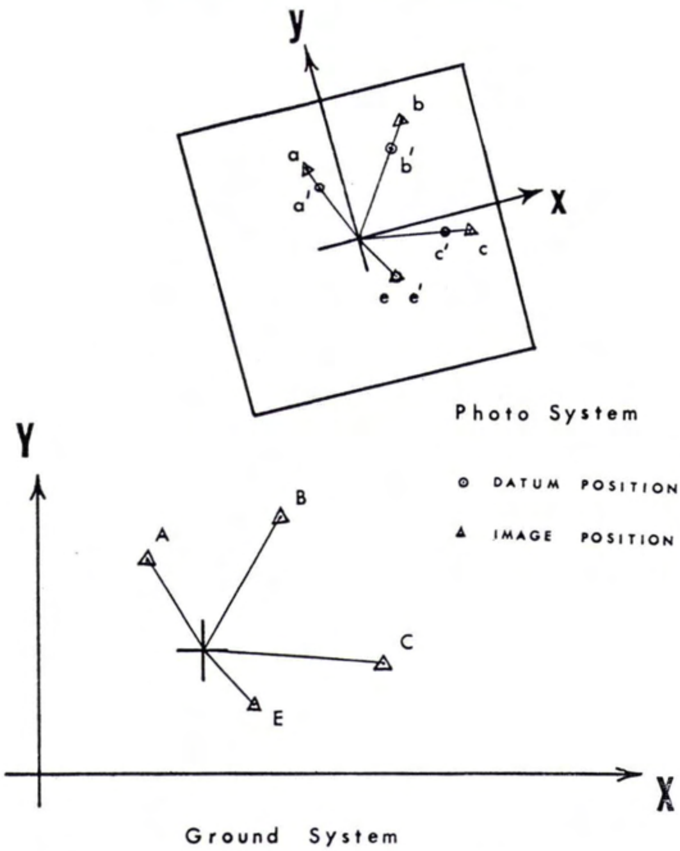

FIG. 1. Geometric representation of relief displacement.

where X, Y are the ground coordinates, x, y the measured image coordinates, and p, q, r and s are the four transformation coefficients. For each of the points other than the reference point, the coordinates of the datum position, x', y' must be used in the transformation. From the basic geometry of the vertical photograph, x', y' can be obtained from the measured x, y by

$$x' = x(1 - d/r) = xk$$
$$y' = y(1 - d/r) = yk \qquad (2)$$

where d is the relief displacement and

$$r = (x^2 + y^2)^{1/2}.$$

Substituting (2) into (1) we get

$$X = pxk - qyk + r = (px - qy)k + r$$
$$Y = qxk + pyk + s = (qx + py)k + s$$

or

$$(X - r) = (px - qy)k$$
$$(Y - s) = (qx + py)k$$

which, after eliminating k, reduces to

$$(Y - s)(px - qy) - (X - r)(qx + py) = 0. \tag{3}$$

Equation 1 represents the *pair of observation equations* for the reference point, whereas Equation 3 is the *single observation equation* for each of the additional points. As there are four transformation coefficients, p, q, r and s, *three control points* are necessary and sufficient for a unique solution.

The least squares solution for the mathematical model represented by Equations 1 and 3 is quite interesting inasmuch as it entails two different conditions, one linear and the other nonlinear, in the transformation coefficients. In order that a rigorous least squares may be applied, one must differentiate between observations, parameters, and constants. (Of course, it is possible to consider all elements as observations and make the classification by varying the weights,[5] but it is not necessary for the present simple mathematical model.) Here, we will consider x, y as observations, X, Y as constants because their errors may be assumed to be relatively small, and p, q, r and s as the unknown parameters. Moreover, as both conditions are used in the same adjustment procedure, general nonlinearity in the parameters will have to be assumed.

Referring to the approximate values of the parameters by p^0, q^0, r^0, s^0, the linearized form of Equation 1 will be

$$\begin{bmatrix} p^0 & -q^0 \\ q^0 & p^0 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} x & -y & 1 & 0 \\ y & x & 0 & 1 \end{bmatrix}\begin{bmatrix} \delta_p \\ \delta_q \\ \delta_r \\ \delta_s \end{bmatrix} = \begin{bmatrix} X - p^0x + q^0y - r^0 \\ Y - q^0x - p^0y - s^0 \end{bmatrix} \tag{4}$$

or more concisely

$$\underset{2,2}{\mathbf{A}}\ \underset{2,1}{\mathbf{V}} + \underset{2,4}{\mathbf{B}}\ \underset{4,1}{\mathbf{\Delta}} = \underset{2,1}{\mathbf{F}^0} \tag{5}$$

Similarly, the linearized form of Equation 3 is

$$\underset{1,2}{\overline{\mathbf{A}}}\ \underset{2,1}{\mathbf{V}} + \underset{1,4}{\overline{\mathbf{B}}}\ \underset{4,1}{\mathbf{\Delta}} = \underset{1,1}{\overline{\mathbf{F}^0}} \tag{6}$$

where,

$$\underset{1,2}{\overline{\mathbf{A}}} = [(Y - s^0)p^0 - (X - r^0)q^0 \qquad -(Y - s^0)q^0 - (X - r^0)p^0]$$

$$\underset{1,4}{\overline{\mathbf{B}}} = [(Y - s^0)x - (X - r^0)y \qquad -(Y - s^0)y - (X - r^0)x \qquad q^0x + p^0y \qquad -(p^0x - q^0y)]$$

$$\underset{1,1}{\overline{\mathbf{F}^0}} = -(Y - s^0)(p^0x - q^0y) + (X - r^0)(q^0x + p^0y).$$

Again the reader is reminded that Equation 5 is written only once for the reference point, whereas Equation 6 is written as many times as there are additional points. Both equations are, furthermore, in the same form of combined observations and parameters. The formation of the normal equations from these are well known and therefore no derivation is included here.

The contribution of Equation 5 to the normal equations will be

$$[B^t(AQA^t)^{-1}B]\Delta = B^t(AQA^t)^{-1}F^0. \quad (7)$$

If the cofactor matrix Q is assumed to be the unit matrix (i.e. the observations are uncorrelated and of equal weight), the matrix inverse shown in Equation 7 can be reduced to a scalar form as follows:

$$\begin{aligned}(AQA^t)^{-1} &= \left(\begin{bmatrix} p^0 & -q^0 \\ q^0 & p^0 \end{bmatrix} I \begin{bmatrix} p^0 & q^0 \\ -q^0 & p^0 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} (p^{02}+q^{02}) & 0 \\ 0 & (p^{02}+q^{02}) \end{bmatrix}^{-1} \\ &= (p^{02}+q^{02})^{-1} I = \alpha I.\end{aligned}$$

Hence Equation 7 simplifies to

$$[\alpha B^t B]\Delta = \alpha B^t F^0 \quad (8)$$

In a similar manner, the contribution of one condition Equation 6 to the normal equation set can be shown to be

$$[\beta \bar{B}^t \bar{B}]\Delta = \beta \bar{B}^t \bar{F}^0 \quad (9)$$

where

$$\beta = \{(p^{02}+q^{02})[(X-r^0)^2 + (Y-s^0)^2]\}^{-1}. \quad (10)$$

Thus, for a total of n known control points, the combined set of normal equations becomes:

$$\left[\alpha B^t B + \sum_{i=2}^{n} (\beta \bar{B}^t \bar{B})_i\right]\Delta = \left[\alpha B^t F^0 + \sum_{i=2}^{n} (\beta \bar{B}^t \bar{F}^0)_i\right]. \quad (11)$$

Equation 11 represents a set of 4×4 normal equations whose solution yields a correction vector Δ. This vector is then added to the approximate values to render new approximations and the procedure repeated until no significant improvement occurs. When this situation is reached, the final values of r and s would be the X and Y ground coordinates of the principal point of the photograph. To compute the ground coordinates of a pass point, Equation 3 is written twice, once for each of the two resected photographs. This leads to two linear equations which, when solved, yield the values of the required coordinates X and Y of the pass point.

Mathematical Model II

In this model the mathematics is even simpler than for the previous one because the basic equation (and there is only one) used is that for a straight line in a plane, i.e.,

$$Y = mX + c \quad (12)$$

where

X, Y are the coordinates of the point in the ground system,
m is the slope of the radial line with respect to the ground coordinate system, and
c is the corresponding Y-intercept.

Let X_o, Y_o denote the as-yet-unknown coordinates of the principal point in the ground system as shown in Figure 2. If X, Y are the known coordinates of a ground

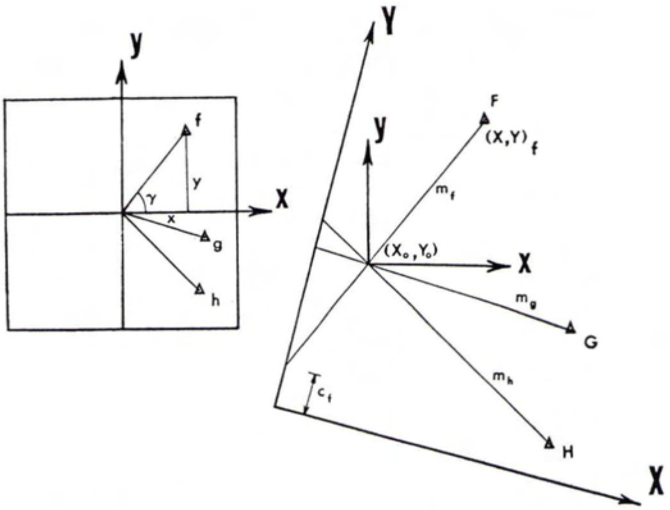

FIG. 2. Geometric basis for the second mathematical model.

control point, then each radial line on the ground would yield the following two equations

$$Y = mX + c$$

and

$$Y_0 = mX_0 + c.$$

Subtracting,

$$(Y - Y_0) = m(X - X_0). \tag{13}$$

To obtain a value for the slope m, we refer to Figure 3; hence

$$m = \tan(\gamma + \theta)$$

or

$$m = \frac{\tan\gamma + \tan\theta}{1 - \tan\gamma\tan\theta} \tag{14}$$

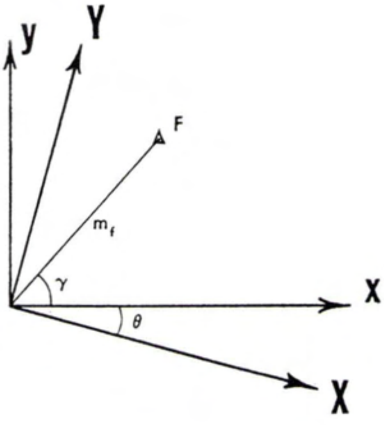

FIG. 3. The relation of slope m to the angles γ and θ.

where γ is the angle between ray and photo x-axis, and θ is the angle between the x,y-photographic system and the X,Y-ground system, which is unknown. It is obvious that $tan\ \gamma = y/x$, and if we let $tan\ \theta = b$, Equation 14 becomes

$$m = \frac{y/x + b}{1 - (y/x)b}. \tag{15}$$

Substituting Equation 15 into 13 and clearing the fraction, we get

$$(Y - Y_0)[1 - (y/x)b] - (X - X_0)[(y/x) + b] = 0. \tag{16}$$

Equation 16 is the only condition equation involved in the present mathematical model. Similar to those used in the previous model, it retains the original observations and therefore is easily used in a least squares adjustment. It is pertinent to note here that there are only three unknown parameters involved, namely X_0, Y_0, and b.

If $X_0{}^0$, $Y_0{}^0$, and b_0 denote approximate values, the linearized form of Equation 16 is similar to Equations 5 and 6 and is given by

$$\underset{1,2}{\boldsymbol{A}}\ \underset{2,1}{\boldsymbol{V}} + \underset{1,3}{\boldsymbol{B}}\ \underset{3,1}{\boldsymbol{\Delta}} = \underset{1,1}{\boldsymbol{F}^0} \tag{17}$$

where

$$\underset{1,2}{\boldsymbol{A}} = \left[\left(\frac{y}{x^2}\{(Y - Y_0^0)b^0 + (X - X_0^0)\}\right), \left(-\frac{1}{x}\{(Y - Y_0^0)b^0 + (X - X_0^0)\}\right)\right]$$

$$\underset{1,2}{\boldsymbol{V}^t} = [v_x \quad v_y]$$

$$\underset{1,3}{\boldsymbol{B}} = \left[\frac{y}{x} + b^0, \quad -\left(1 - \frac{y}{x}b^0\right), \quad -\frac{y}{x}(Y - Y_0^0) - (X - X_0^0)\right]$$

$$\underset{1,3}{\boldsymbol{\Delta}^t} = [\delta X_0 \quad \delta Y_0 \quad \delta b]$$

$$\underset{1,1}{\boldsymbol{F}^0} = -(Y - Y_0^0)\left(1 - \frac{y}{x}b^0\right) + (X - X_0^0)\left(\frac{y}{x} + b^0\right).$$

The general form of the normal equations based on Equation 17 is given by

$$[\boldsymbol{B}^t(\boldsymbol{AQA}^t)^{-1}\boldsymbol{B}]\Delta = \boldsymbol{B}^t(\boldsymbol{AQA}^t)^{-1}F^0. \tag{18}$$

Again if it is assumed that $\boldsymbol{Q} = \boldsymbol{I}$, the normal equations due to n points would be

$$\left[\sum_{i=1}^{n} (\psi \boldsymbol{B}^t\boldsymbol{B})_i\right]\boldsymbol{\Delta} = \left[\sum_{i=1}^{n} (\psi \boldsymbol{B}^t\boldsymbol{F}^0)_i\right] \tag{19}$$

where

$$\psi = x^4\{(x^2 + y^2)[(Y - Y_0^0)b^0 + (X - X_0^0)]^2\}^{-1}. \tag{20}$$

Equation 19 represents a set of 3×3 normal equations whose solution yields the vector of corrections $\boldsymbol{\Delta}$. As in the case of the first model, the solution is iterative and at the end the final values of X_0, Y_0 and b are obtained.

Once the resection of the first two photographs is performed, the ground coordinates of any pass point in the overlap area can be obtained by intersection. Two equations of the type (16) may be written with the pass point coordinates X, Y as un-

TABLE 1. DISCREPANCIES IN RESECTION OBTAINED BY THE WRITER AND BY REFERENCE 2

Control Point	*Mathematical Model I*		*Mathematical Model II*		*Results from Reference 2*	
	ΔN	ΔE	ΔN	ΔE	ΔN	ΔE
TTR—5	−1.40	−0.83	−1.40	−0.83	−1.4	−0.9
TTR—64	−2.79	+0.55	−2.78	+0.56	−2.7	+0.5
TTR—65	−2.72	−0.29	−2.70	−0.28	−2.6	−0.2
TTR—55	−3.21	−4.68	−3.12	−4.60	−3.0	−4.8
TTR—11	−2.17	−8.37	−1.21	−8.16	−0.9	−8.4
TTR—54	+2.34	−11.62	+2.57	−10.42	+3.6	−11.4
TTR—53	+5.41	−11.44	+7.13	−11.05	+6.8	−11.4

knowns; when solved simultaneously the equations yield the required coordinates. The procedure is then repeated in the order: resection, intersection, resection, etc., until an entire strip is completed.

Having studied the two mathematical models presented above, the reader might be interested in the relationship between them. Of primary interest would be the answer to the following question: why are *four* unknown parameters involved in the first model whereas the second model involves only *three?* The answer to this question. as well as a discussion of the relationship between the two mathematical models, is given in the Appendix.

Before we present some test results, it is important to point out that the triangulation procedure we are dealing with is of the cantilever extension type. Furthermore, the triangulation is performed directly in the ground coordinate system. It is natural to expect that the farther the triangulation progresses from the original control the more deformed the strip becomes. (As will be seen subsequently, this fact has been verified by the tests we ran, and is in agreement with Wolf's results.[2] However, Turpin[1] seems to have a contrary opinion which, in our view, is rather surprising.) Therefore, as is common practice, a strip adjustment would be subsequently needed to effect a fit to the ground. However, this fitting has not been done here as it is not considered to be a part of the triangulation method under investigation.

RESULTS OF TESTS PERFORMED

The first test was carried out using the same data used in Reference 2. The photography was at a scale of 1:6,000 and the length of the strip was seven photographs. The input data in the form of photo coordinates of pass points as well as the control data were supplied by the author of the article cited in the reference. The methods of both mathematical models were used on the data, and corresponding results as well as those from Reference 2 are given in Table 1. Three control points were used in the resection of the first two photographs of the strip, then three pass points in the triple overlap area are used thereafter to extend the triangulation. Table 1 shows no significant differences, which is to be expected because the basic concept behind all three methods is essentially the same.

The second test was performed using fictitious data having the following characteristics: scale 1:5,000; length of strip, 8 photographs; maximum tilts, $\omega = 2°\ 18'$, and $\phi = 1°\ 38'$. As the two mathematical models are equivalent (see Appendix) the method of mathematical Model I was used. To evaluate possible increase in geometric strength through the use of more points for resection, two tests were conducted, one using three points and the other using five points. A summary of the results obtained is given in Table 2A. These results indicate that the build-up of strip deformation progresses more rapidly in the case of three points than in the case of five points. Consequently, it may be implied that the geometric strength of the triangulated strip

TABLE 2A. DISCREPANCIES IN RESECTION OBTAINED USING THREE POINTS AND FIVE POINTS

Point Number	Using 3 Points		Using 5 Points	
	ΔX	ΔY	ΔX	ΔY
3	− 0.46	+ 0.10	− 0.46	+ 0.10
6	+ 0.27	+ 0.32	+ 0.26	+ 0.32
13	+ 1.98	+ 2.27	+ 1.13	+ 2.22
16	+ 3.24	− 1.12	+ 1.48	− 1.46
25	+ 8.20	+ 2.02	+ 5.81	− 0.60
27	+ 4.29	−10.11	+ 2.38	− 9.09
32	+ 6.36	−12.46	+ 4.29	− 9.74
34	+13.94	−11.23	+ 9.61	− 8.95
36	+ 2.68	−20.54	+ 1.83	−15.92
39	+14.53	−21.32	+ 9.38	−16.66
40	+22.61	−15.72	+15.58	−12.34

increases as the number of points used for resection is increased. To verify this fact further, the two strips were subjected to a linear conformal transformation to the ground control system using one point at the beginning and one point at the end. This way, no attempt is made to correct for possible strip deformations. The results obtained after the said transformation are shown in Table 2B. They indicate that the strip which is triangulated using three points exhibits larger deformations than that using five points. Hence, the increase in the number of points used affords a *stiffer* strip.

The last test was performed on data obtained from the Bureau of Public Roads. This data consisted of photo coordinates for a strip of 10 photographs at a scale of 1:4,800, and the ground coordinates of 14 control points well distributed along the strip. The results obtained are summarized in Table 3.

CONCLUSION AND RECOMMENDATION

The derivations and test results included in this paper indicate that the mathematical models developed afford more flexibility than the one originally used. For those who are using, or planning to use, the method of radial triangulation considered in this paper, it is recommended that they apply the suggested mathematical model. As given it would allow for the application of rigorous least squares adjustment to handle the inclusion of redundant data.

TABLE 2B. THE RESULTS SHOWN IN TABLE 2A AFTER BEING SUBJECTED TO A LINEAR CONFORMAL TRANSFORMATION

Point Number	Using 3 Points		Using 5 Points	
	ΔX	ΔY	ΔX	ΔY
3	0	0	0	0
6	− 1.75	+3.18	−0.95	+2.48
13	− 5.47	+5.61	−4.00	+4.96
16	− 3.73	+7.67	−3.05	+5.21
25	− 7.02	+8.67	−4.79	+5.13
27	− 8.78	+3.33	−6.38	+1.22
32	− 9.18	+3.96	−6.51	+2.83
34	− 4.60	+2.77	−3.09	+1.98
36	−13.90	0	−9.20	−0.32
39	− 6.41	−4.30	−4.94	−3.75
40	0	0	0	0

TABLE 3. DISCREPANCIES IN RESECTION OBTAINED WITH DATA FROM THE BUREAU OF PUBLIC ROADS

Point Number	*Using 3 Points*	
	ΔX	ΔY
16	0	+0.18
15	−0.11	−0.04
14	+0.07	−0.04
13	+0.28	−0.29
12	+0.16	−0.75
11	+0.60	−0.84
9	+1.64	−1.46
8	−0.88	+0.30
7	+1.70	−0.03
5	−0.08	+4.26
4	−4.59	−3.01
3	+6.05	−0.01
2	+17.84	−6.59
1	+16.02	+7.68

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APPENDIX

The question was raised on page 365 of the text as to why the number of unknown parameters is not the same for both mathematical models. The answer to this question would be best obtained if we could reduce one mathematical model to the other. We will attempt to do so in this Appendix.

The four parameters of the first model are p, q, r and s, which appear in Equations 1 and 3. The first two are actually functions of two other parameters, a scale factor u and a rotation θ. The relations between these are

$$p = u \cos \theta$$
$$q = u \sin \theta. \qquad \text{A.1}$$

The angle θ is the same as that used in the second mathematical model. Furthermore, the two shifts r and s represent, as mentioned in the text, the coordinates of the principal point in the ground coordinate system, which were denoted in the second model by X_0 and Y_0, respectively. Hence,

$$r = X_0$$
$$s = Y_0. \qquad \text{A.2}$$

From A.1 and A.2, it becomes clear that the parameter missing from the second model is the scale factor u. Consequently, we will now seek to find out if Equations 1 and 3 would each reduce to Equation 16 when the scale factor u is eliminated from them. Let us start with Equation 1.

Substituting A.1 and A.2 into Equation 1, we get

$$X = (u \cos \theta)x - (u \sin \theta)y + X_0$$

$$Y = (u \sin \theta)x + (u \cos \theta)y + Y_0.$$

Rearranging and dividing the first equation by the second, the scale factor u drops out and we get

$$\frac{(X - X_0)}{(Y - Y_0)} = \frac{x \cos \theta - y \sin \theta}{x \sin \theta + y \cos \theta}. \qquad \text{A.3}$$

Dividing the numerator and denominator of the right hand side by $x \cos \theta$, Equation A.3 becomes

$$\frac{(X - X_0)}{(Y - Y_0)} = \frac{1 - (y/x) \tan \theta}{\tan \theta + (y/x)}. \qquad \text{A.4}$$

Recalling that $\tan \theta = b$, and clearing the fractions we get

$$(Y - Y_0)[1 - (y/x)b] - (X - X_0)[(y/x) + b] = 0 \qquad \text{A.5}$$

which is identical to Equation 16.

Following the same procedure, the substitution of Equation A.1 and A.2 into Equation 3 yields

$$(Y - Y_0)[(u \cos \theta)x - (u \sin \theta)y] - (X - X_0)[(u \sin \theta)x + (u \cos \theta)y] = 0.$$

Dividing this equation by $(u \cos \theta)\, x$, and realizing that $\tan \theta = b$, we get

$$(Y - Y_0)[1 - (y/x)b] - (X - X_0)[(y/x) + b] = 0 \qquad \text{A.6}$$

which is the same as Equation 16. Thus, it may be concluded that the two mathematical models are equivalent.

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