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The Fichter Equation for Correcting Stereoscopic Slopes

Abstract: *In 1954 Fichter presented the following equation for deriving true slopes from stereoscopic slopes affected by height exaggeration and perspective distortion:* $\cot \delta = R \cot \epsilon + d/f \sin \beta$ *in which* δ *is the true angle;* R, *the relief exaggeration factor;* ϵ, *the apparent angle in the stereomodel;* d, *the distance from the stereocenter to the upslope point at which the slope is measured; and* β, *the angle between the radial from the stereo center and the strike at the point of slope measurement. Fichter, however, did not explain the derivation of the equation. The explanation herein presented has proved satisfactory in my own classes and may be of service to others involved in the training of photogeologists.*

In a stimulating paper on the geometry of the imaginary stereoscopic model, Fichter (1954), presented an equation for determining the true angle of slope from a stereoscopic slope affected by both height exaggeration and perspective distortion. Height exaggeration is the ratio of stereoscopic height to natural height at model scale. It is due largely to the smaller ratio of eye base to viewing distance (b_e/D in Figure 1) as compared to the ratio of air base to flying height $B/(H\text{-}h)$.

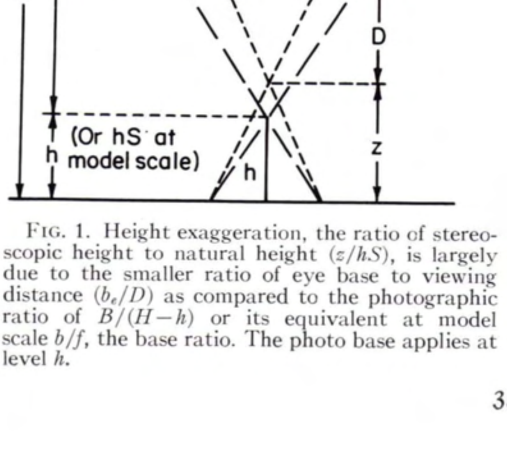

Fig. 1. Height exaggeration, the ratio of stereoscopic height to natural height (z/hS), is largely due to the smaller ratio of eye base to viewing distance (b_e/D) as compared to the photographic ratio of $B/(H-h)$ or its equivalent at model scale b/f, the base ratio. The photo base applies at level h.

The latter, at photo scale, is the base ratio, the ratio of photo base to focal length (b/f).

Height exaggeration is also influenced by the viewer's eyes. It is greatest for far-sighted people with narrow eye base. Both properties have the same effect as a reduction in the base-height ratio of the viewing triangle (b_e/D). It is as though in Figure 1, the eyes were shown closer together or at greater distance from the model. The net result is to increase the stereoscopic height, z. A third cause of height exaggeration is the stereoscope itself. Stereoscopes with lenses indirectly introduce the problems of near- and far-sightedness. This is because lenses present the eyes with parallel rays of light as though coming from an object at infinity, whereupon the viewer imagines the model to be at his personal distance of most distinct vision (Fichter, p. 137).

Fichter's equation for height exaggeration, R, is:

$$R = \frac{b}{f} \cdot K$$

in which b/f is the base ratio, and K a constant representing the amount of exaggeration that is due to the observer's eyes and the stereoscope used. K is determined with the aid of a stereodiagram of a pyramid in which the difference in parallax between top and bottom is one arbitrary unit (see Figure 3A, left foreground). The viewer estimates, from a horizontal scale involving the same units, the stereoscopic height caused by the one unit

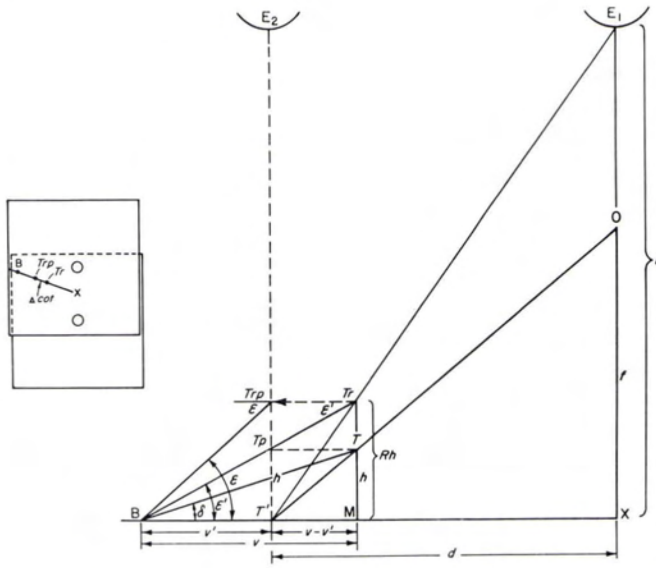

FIG. 2. Combined effects of height exaggeration and perspective distortion on slopes facing away from the stereocenter X. Because of height exaggeration R, point T at the top of the true slope BT, appears at Tr, at height Rh. The resultant slope angle ϵ' is thus greater than the true angle δ. When the stereoscope is moved from E_1 over the flight line to E_2, over the slope to be measured, Tr is drawn to position Trp. This lateral shift is perspective distortion. The ultimate angle of slope ϵ thus embodies the effects of both height exaggeration and perspective distortion.

of parallax. The result varies from individual to individual and is the viewer's stereo constant, K. The base ratio is multiplied by K to give the height exaggeration factor, R. In addition to the pyramid diagram, Fichter presents a series of diagrams each of which consists of three intersecting lines. The diagram in which the three axes appear mutually perpendicular under the stereoscope provides the viewer's stereo constant. Hackman (1956) has proposed a different type of stereo diagram involving inclined planes for the determination of R.

In Figure 2, a true slope is shown as BT. The image of T at the top of the slope, however, appears at Tr in the model as a result of relief exaggeration. If the eyes could be placed in the same position with respect to the model as were the original aircraft positions, also at the scale of the model, the model would appear in natural scale. In Figure 1, for example, if the eyebase could be broadened or the viewing distance to the model shortened so that the eyes were positioned along the lines of vision from aircraft to target, the lines of vision from target to eyes would restore the target to natural scale in the model. It is because the base ratio of the viewing triangle is less than the base ratio of the triangle of photography that the relief is exaggerated. The net result, then, is that the image of T appears at Tr. It is as though the viewing position was at E_1 in Figure 2, some multiple of the focal length above the stereocenter, X. The stereocenter is the point midway between the principal points. The height of E_1 above X is actually Rf.

If the stereoscope is above the flight line, and centered over the stereocenter, the model is subject only to relief exaggeration and determination of the true slope is simple: we measure the exaggerated angle of slope (ϵ') in the model and determine the true angle (δ) by:

$$\tan \delta = \frac{\tan \epsilon'}{R}. \qquad (1)$$

The exaggerated angle in the stereo model is either (1) estimated by eye if one has the ability to do this within a few degrees, (2) measured with the aid of a miniature clinometer placed against the slope in the model (Figure 3A), or (3) with the Stereo-slope Comparator (Figure 3B). If, using Equation 1, the height exaggeration is found to be 2, and the apparent slope angle is 30°, the true angle is:

$$\tan \delta = \frac{0.57735}{2} = 0.28868$$

$$\delta = 16°.$$

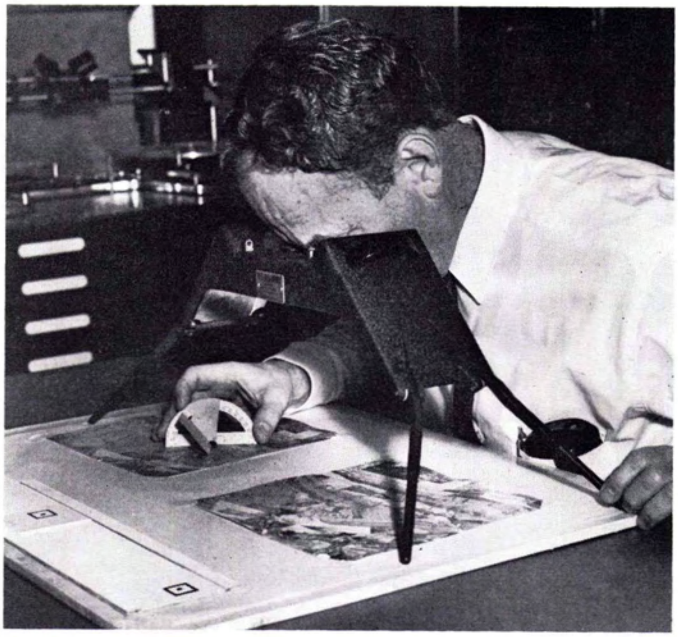

FIG. 3A. Devices for measurement of apparent slopes in the stereoscopic model. Homemade clinometer. A disadvantage is that the device is seen by only one eye. Note the adjustable Fichter pyramid diagram (at top of board from observer) used for determination of the stereoscopic constant K under pocket or mirror stereoscopes.

Incidentally, with the stereoscope over the stereo center, R can be determined without the aid of stereo diagrams. If, for example, there is a known slope in the model, one that was determined in the field, or measured from a reliable map, or computed from parallax measurements, one need only compare the tangent of the relief-exaggerated slope to the tangent of the known slope to derive R:

$$R = \frac{\tan \epsilon'}{\tan \delta}$$

where ϵ' is the relief-exaggerated slope and δ is the true slope. Generally, however, a true slope will not be known, so this simple method of obtaining R will not be possible. Although true slopes may be determined with the parallax bar (a procedure which is recommended where only one or two slopes are to be determined), the process is too slow if many slopes are to be measured. The Fichter and Hackman methods involve the determination of R with the aid of diagrams and the rapid derivation of true slopes by Equation 1.

If the eyes of the observer, with or without a stereoscope, are shifted away from the stereo center to a position directly above the slope to be measured (as from E_1 to E_2 in Figure 2), the top of the slope is dragged in the same direction, to the position Trp. This has the effect of further exaggerating the declivity of slopes which face away from the stereo center and of decreasing the declivity of those that face the stereocenter. This *perspective distortion* (Fichter) has increased the angle of slope in Figure 2 from ϵ' to ϵ. It is this doubly-exaggerated slope that the viewer almost always sees in the stereomodel and from which the true angle must be determined.

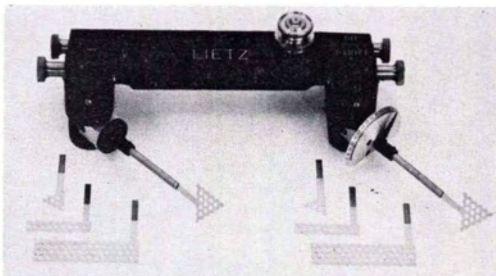

FIG. 3B. The Hackman Stereo Slope Comparator which provides paired inclined targets for more precise determination of apparent slopes.

Fichter's complete equation for determination of true slopes is:

$$\cot \delta = R \cot \epsilon + \frac{d}{f} \sin \beta \qquad (2)$$

in which ϵ is the apparent angle of slope in the stereo model; d, the distance from the stereo-center to the upper point of the slope to be measured; and β, the angle between the radial from the stereo center and the strike at the point of slope measurement. Neglecting for the moment why ϵ rather than ϵ' is used, it will be noted that the first member on the right is the familiar tangent equation for relief exaggeration expressed in cotangent form, that is:

$$\left[\tan \delta = \frac{\tan \epsilon}{R}\right] = \left[\cot \delta = R \cot \epsilon\right].$$

The last expression in Equation 2, $d/f \sin \beta$, is the correction for perspective distortion.

Although angle ϵ includes the effect of perspective distortion as well as relief exaggeration, division of its tangent by R provides the same reduction in height as does division of $\tan \epsilon'$ by R. Thus, in Figure 2,

$$\tan \epsilon = Rh/v'$$
$$\tan \epsilon' = Rh/v.$$

If each value is divided by R, the vertical side of each triangle is reduced to h, the unexaggerated height. Inasmuch as angle ϵ is the angle visible in the stereomodel, it is the one used in the equation.

In correcting the tangent (or cotangent) of ϵ to arrive at the correct value of δ we can either correct for height exaggeration first and perspective distortion second, or vice versa. In Figure 2 this would amount to first dropping point Trp to Tp and then shifting laterally to T; or shifting Trp laterally to Tr and then dropping down to T. In Fichter's equation, the height exaggeration correction is listed first. In Figure 2 this is equivalent to dropping point Trp to Tp, thereby reducing angle ϵ to angle ϵ'.

The tangent of the doubly affected angle ϵ is Rh/v' (Figure 2). If we remove the relief exaggeration by dividing by R, we get h/v' which is the tangent of ϵ' the angle affected only by perspective distortion. In brief:

$$\tan \epsilon' = \frac{\tan \epsilon}{R},$$

or, expressed in cotangent form:

$$\cot \epsilon' = R \cot \epsilon.$$

The correction for perspective distortion is as follows, again referring to Figure 2:

$$\cot \delta = \frac{v}{h}, \quad \text{and} \quad \cot \epsilon' = \frac{v'}{h}$$

$$\cot \delta - \cot \epsilon' = \frac{v}{h} - \frac{v'}{h} = \frac{v - v'}{h}$$

Now, $\Delta TT'M \sim \Delta OT'X$

$$\therefore \frac{v - v'}{h} = \frac{d}{f}.$$

Substituting d/f for $(v\text{-}v')/h$ in Equation 2:

$$\cot \delta - \cot \epsilon' = \frac{d}{f}$$

or

$$\cot \delta = \cot \epsilon' + \frac{d}{f}$$

But $\cot \epsilon' = R \cot \epsilon$ (see above)

$$\therefore \cot \delta = R \cot \epsilon + \frac{d}{f}.$$

The correction for perspective distortion is d/f, however, only if T and B (Figure 2, small diagram) lie along the radial from the stereocenter, that is, if the strike of the slope is normal to the radial, d (Figure 4A). In this particular case, point T is displaced the maximum distance toward B, that is, the maximum foreshortening of the slope occurs. $(Tr - Trp)$ represents the amount of cotangent shortening. To determine the amount of foreshortening of all slopes regardless of orientation, we must, following Fichter, introduce the effect of the angle between the strike line and the radial, d. In Figure 4A, angle B is 90° and its sine is 1. Multiplying d/f by $\sin \beta$, therefore, does not change the value of the correction which is at its maximum. In Figure 4B, the strike at Tr coincides with the radial d and angle β and its sine are zero. Multiplying d/f by $\sin \beta$, therefore, means no change in the distance between the strike lines through Tr and B; the slope has simply been sheared into a parallelogram, but not foreshortened. The angle of slope is therefore unchanged and may be determined by simply applying the height-exaggeration correction:

$$\tan \delta = \tan \epsilon / R.$$

This procedure is also valid at the stereocenter where $d = 0$ and $d/f \sin \beta = 0$.

For all other orientations, where β is other than 0 or 90 (Figure 4C), $\sin \beta$ will be some value between 0 and 1. This value, when applied to d/f, the value for maximum distortion, will give the amount by which the

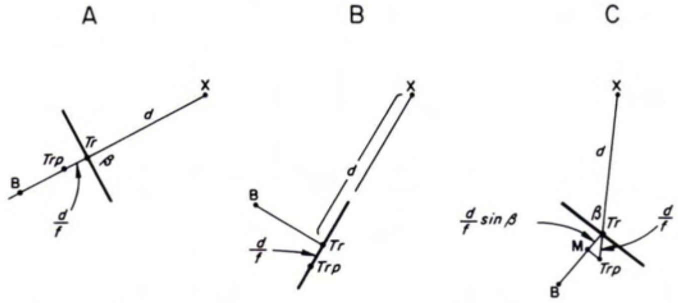

FIG. 4. Effect of orientation on amount of slope foreshortening. *A*—Strike of slope normal to radial *d* from stereocenter *x*. Angle $\beta=90°$. *Tr* is upper point on slope affected only by relief exaggeration. *Trp* is the upper point displaced by perspective distortion directly toward *B*, the lower point. The amount of slope foreshortening, $d/f \sin \beta$, is the maximum possible, d/f, because $\sin \beta=1$. *B*—Strike of slope coincides with radial *d*. Angle $\beta=0$. Point *Tr* is displaced to *Trp* by the amount d/f. The displacement, however, is along the strike, hence causes no foreshortening of the slope, that is, no change in the distance between *B* and the strike line through the upper point. Thus, when $\beta=0°$, the expression of $d/f \sin \beta$ is zero and there is no correction for perspective distortion. *C*—Strike of slope oblique to radial *d*. Angle β between 0° and 90°. *Tr* displaced obliquely on slope to *Trp* by the amount d/f. The new position of the strike line through *Trp* is closer to *B* by the amount *TrM*, or $d/f \sin \beta$.

strike line through *Tr* has been displaced toward *B*, that is, the amount by which the cotangent has been shortened.

If slopes are inclined toward the stereocenter, shifting of the stereoscope away from the stereocenter flattens the slope thereby increasing the cotangent of the angle. Fichter proposes that to avoid confusion in signs, slopes facing *away* from the stereocenter, as well as their cotangents, be regarded as positive, and those facing *toward* the stereocenter, as negative, with the perspective distortion correction always regarded as positive. Thus, for slopes facing *away* from the stereocenter:

$$\cot \delta = + R \cot \epsilon + d/f \sin \beta,$$

and for slopes facing *toward* the stereocenter:

$$\cot \delta = - R \cot \epsilon + d/f \sin \beta.$$

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