G. H. Schut*
National Research Council
Ottawa, Canada

Formation of Strips from Independent Models

Exact linear equations avoid the usual solutions of non-linear functions by iterative procedures

(Abstract on next page)

(Abstract on next page)

Introduction

In several photogrammetric instruments, the relative and absolute orientation of a pair of photographs can be performed very accurately, but strip triangulation is not possible because no provision is made for the base-in base-out interchange. Examples of this are, for instance, the Wild A8 and B8, the Thompson-Watts plotter, and the Stereosimplex Santoni Model III.

In organizations where these instruments are used, a considerable interest exists in the connection of the resulting independent models into strips. Evidence of this are the presentation of several papers on the subject at the *Symposium on Aerial Triangulation* at Delft in September 1964 and at the *Symposium on Spatial Aerotriangulation* at Urbana in February 1966.

These papers advance the view that the numerical formation of strips from independent models is a suitable alternative to the analogue triangulation on universal first-order plotters and to the purely analytical triangulation. Advantages over the latter procedures are the use of a simpler, less expensive, but not necessarily less accurate plotter, and the more easily manageable data, respectively.

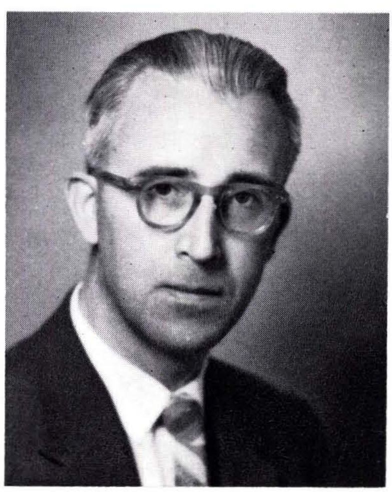
The strips obtained can be adjusted to ground control by means of polynomial transformations with results that can be "extremely good indeed." Even if the final adjustment is performed by a strip or block adjustment of the independent models, an initial formation and transformation of strips is very desirable for the detection of errors in ground-control points and tie points.

The present paper shows how the numerical formation of strips can be performed with the help of extremely simple formulas. The formulas have been incorporated in a small Fortran program. A listing of this program and operating instructions can be obtained from the Division of Applied Physics of the National Research Council of Canada upon request.

The Mathematical Concept

The mathematical formulation of the strip formation offers no great problems. The coordinate system of the first model of a strip can be retained as the coordinate system of the strip. Subsequently, each following model in succession can be transformed to that system by connecting it to the preceding model in three steps:

1. A translation which makes the coordinates of the common projection center the same as those in the preceding model;



* Presented at the Semi-Annual Convention of the American Society of Photogrammetry at St. Louis, Missouri, October 1967.

2. A rotation which makes the orientation of the vectors from the common projection center to the common points in the two models the same; and
3. A scaling which makes the length of these vectors in the two models the same.

The mathematical formulation of the rotation is of particular interest. The transformation formulas for a rotation in three-dimensional space give the coordinates after rotation as linear functions of the coordinates before rotation. Their coefficients are functions of three independent parameters.

Although the parameters can be chosen in many different ways,[1] the coefficients are always non-linear functions of the parameters. Therefore, it has long appeared necessary to compute the coefficients in an iterative procedure, with the help of linear equations derived from the transformation formulas by differentiation.

ABSTRACT: *This paper presents a set of simple formulas for the formation of strips from independent models. The formulas produce an exact fit at the projection centers and a least-squares fit, in a specified sense, at the tie points. The rotation which each model undergoes is computed from exact linear formulas derived from a four-dimensional transformation.*

Prof. E. H. Thompson[2] was the first to show that with a suitable choice of the three parameters it is possible to derive exact linear equations from which, consequently, the parameters can be solved directly. However, his derivation is complicated, and his equations are not suitable if the required rotation is 180°.

Having learned from Prof. Thompson's paper that exact linear equations are here possible, the present writer was able to derive a set of four such equations, which are homogeneous with respect to four parameters.[3] Because only the ratios of these parameters are defined, they represent three independent parameters. The use of these equations imposes no limit upon the size of the rotation. Their derivation requires only a few lines. They were derived in two ways: by means of the algebra of three-by-three matrices, and by means of quaternion algebra.

Since the coefficients in these equations are linear functions of the coordinates before and after rotation, they are as simple as possible. However, the derivation of the equations still leaves something to be desired. The derivation by means of matrix algebra leads directly to only three of the four linear equations, and the existence of the fourth equation is not obvious from an inspection of the other three. The derivation by means of quaternion algebra has the disadvantage that this discipline is little known.

A further look at the formulas has now led the writer to the conclusion that the transformation formulas can be most logically written as a transformation in a four-dimensional coordinate system by means of certain four-by-four matrices. In this way, the derivation of the transformation formulas and the linear equations requires only a few well-known theorems from matrix algebra.

A comparison of the new derivation with the earlier ones shows that, if the fourth coordinate is eliminated, the derivation by means of the algebra of three-by-three matrices is obtained. It also shows that quaternion algebra is nothing more than a set of algorithms for the manipulation of the four-by-four matrices. This explains why the linear equations could be derived by two such apparently heterogeneous disciplines.

DERIVATION OF TRANSFORMATION FORMULAS

Let the rectangular coordinate systems x, y, z in each of two consecutive models be referred to origins in the common projection center, and let a fourth variable t be associated with the coordinates x, y, and z.

These four variables will be arranged in a special way as the elements of a matrix $\mathbf{T}$, as shown in one of the Equations 1. In the same way, four real parameters a, b, c, and d will be arranged as the elements of a matrix $\mathbf{D}$:

$$\mathbf{D} = \begin{bmatrix} d & -c & b & a \\ c & d & -a & b \\ -b & a & d & c \\ -a & -b & -c & d \end{bmatrix} \text{ and}$$

$$\mathbf{T} = \begin{bmatrix} t & -z & y & x \\ z & t & -x & y \\ -y & x & t & z \\ -x & -y & -z & t \end{bmatrix}. \quad (1)$$

Any matrix which is constructed in this way has the following characteristics. Its column vectors, and also its row vectors, are mutually orthogonal and they have the same length. Its determinant is equal to the fourth power of the length of these vectors. Therefore, unless all four parameters are equal to zero, the matrix is non-singular.

The set of all matrices of this type forms a group, that is to say, the product of any two is also a matrix of this type and the inverse of any one is of this type. To show that the first of these requirements is satisfied, it is sufficient to form the matrix product **DT**. The second requirement is satisfied because the product of a matrix and its transpose is equal to the unit matrix multiplied by the square of the length of its vectors.

We shall restrict the matrices **D** to those matrices whose vectors have unit length. Therefore:

$$|\mathbf{D}| = (a^2 + b^2 + c^2 + d^2)^2 = +1 \qquad (2)$$

and

$$\mathbf{D}^{-1} = \mathbf{D}^t, \qquad (3)$$

where the superscript t indicates the transpose.

The four parameters of the matrix **T** can be regarded as the coordinates of a point in a four-dimensional space, and the matrix **D** can be used as a transformation matrix. With this interpretation, the elements in the fourth column of the product matrix

$$\mathbf{T}' = \mathbf{DT} \qquad (4)$$

are the coordinates after transformation:

$$\begin{aligned} x' &= +dx - cy + bz + at \\ y' &= +cx + dy - az + bt \\ z' &= -bx + ay + dz + ct \\ t' &= -ax - by - cz + dt. \end{aligned} \qquad (4a)$$

Because the determinant of **T′** is equal to the determinant of **T**, the vectors of **T′** have the same length as those of **T**. Therefore, the transformation preserves the lengths of all position vectors and the separation of all points. It is a rotation in the four-dimensional space of the coordinates x, y, z, and t.

The most general transformation which is possible by means of matrices **D** is:

$$\mathbf{T}' = \mathbf{D}_1\mathbf{T}\mathbf{D}_2. \qquad (5)$$

This equation cannot be reduced to Equation 4 and, consequently, this transformation contains six independent parameters. Because it, also, is a rotation in four-dimensional space, and this rotation is defined by only six independent parameters, Equation 5 represents all possible rotations.

Equation 4 has the undesirable property that, as Equation 4a shows, the transformed coordinates x', y', and z' are functions not only of x, y, and z, but also of the as yet undefined coordinate t. Therefore, we will not use this equation but, instead, derive suitable transformation formulas from Equation 5.

Pre-multiplication of both sides of Equation 5 by $\mathbf{D}_1^t$ gives, with Equation 3:

$$\mathbf{D}_1^t\mathbf{T}' = \mathbf{T}\mathbf{D}_2. \qquad (6)$$

As each of the two product matrices belongs to the same group as the matrix **T**, this equality signifies that four relations exist between the parameters of $\mathbf{D}_1$, $\mathbf{T}'$, $\mathbf{T}$, and $\mathbf{D}_2$. These relations are found by performing the matrix multiplications and equating the elements in the fourth column of the two product matrices:

$$\begin{aligned} +d_1x'+c_1y'-b_1z'-a_1t' &= +ta_2-zb_2+yc_2+xd_2 \\ -c_1x'+d_1y'+a_1z'-b_1t' &= +za_2+tb_2-xc_2+yd_2 \\ +b_1x'-a_1y'+d_1z'-c_1t' &= -ya_2+xb_2+tc_2+zd_2 \\ +a_1x'+b_1y'+c_1z'+d_1t' &= -xa_2-yb_2-zc_2+td_2. \end{aligned} \qquad (6a)$$

From the Transformations 5 we now wish to select only those for which x', y', and z' are independent of t. For those transformations, the linear functions of x', y', and z' in Equation 6a are then also independent of t. A comparison with transformations in three-dimensional space, where the requirement that x' and y' must be independent of z implies a rotation about the z-axis, shows that here the transformation must be a rotation about the t-axis. Therefore, we require that $t'=t$. Inspection of the Equations 6a shows that this makes the linear functions of x', y', and z' independent of t if further $\mathbf{D}_2=\mathbf{D}_1^t$.

These requirements can be derived also, but in a more laborious way, from Equation 5. For that purpose, it is necessary to perform the matrix multiplications in that equation and to write the parameters of **T′** as linear functions of the parameters of **T**.

With these requirements, Equation 5 reduces to the transformation

$$\mathbf{T}' = \mathbf{DTD}^t \qquad (7)$$

and Equation 6a reduces to the linear equations

$$\begin{aligned} &\qquad\qquad -(z'+z)b+(y'+y)c+(x'-x)d=0 \\ &+(z'+z)a \qquad\qquad -(x'+x)c+(y'-y)d=0 \\ &-(y'+y)a+(x'+x)b \qquad\qquad +(z'-z)d=0 \\ &+(x'-x)a+(y'-y)b+(z'-z)c \qquad\qquad =0. \end{aligned} \qquad (8)$$

This reduces the number of independent parameters from six to three and, since the t

coordinate remains unchanged, the rotation is restricted to the x, y, z subspace of the four-dimensional space.

Consequently, Equation 7 will be used for the rotation of a model and the Equations 8 will be used for the computation of the parameters of the rotation.

An Algorithm for the Computations

The connection of one model of a strip to the preceding model requires that the coordinates of the common projection center and those of at least two model points be known in both models.

For the computation of the rotation, in both models the origin of the coordinate system is temporarily shifted to the common projection center and the vectors from this projection center to common model points are reduced to unit length. From the thus reduced coordinates of each of these model points, four linear Equations 8 can now be obtained. Therefore, more equations become available to compute the parameters a, b, c, and d, than the minimum number that is required.

The parameters should be solved from these equations by the method of least squares, and all available equations should be used. This has the two advantages that it is not necessary to select independent equations from the available ones and that the coefficients of the resulting normal equations become very simple functions of those of the Equations 8.

As the first parts of the Equations 8 are homogeneous linear functions of the parameters and their second parts are equal to zero, this is also the case with the normal equations.

Denoting the elements of the matrix of the normal equations by s_{ij}, where i and j indicate the row and column of the element and denoting a sum of values over all points by square brackets, the elements become:

$$\begin{aligned} s_{11} &= [(x'-x)^2+(y'+y)^2+(z'+z)^2] \\ s_{22} &= [(x'+x)^2+(y'-y)^2+(z'+z)^2] \\ s_{33} &= [(x'+x)^2+(y'+y)^2+(z'-z)^2] \\ s_{44} &= [(x'-x)^2+(y'-y)^2+(z'-z)^2] \\ s_{12} &= -2[yx'+y'x] = s_{21} \\ s_{13} &= -2[xz'+x'z] = s_{31} \\ s_{23} &= -2[zy'+z'y] = s_{32} \\ s_{14} &= 2[zy'-z'y] = s_{41} \\ s_{24} &= 2[xz'-x'z] = s_{42} \\ s_{34} &= 2[yx'-y'x] = s_{43}. \end{aligned} \tag{9}$$

From the normal equations only the ratios of the four parameters can be determined. One of the parameters can be made equal to 1, and the others can be solved from the resulting equations.

In the case of aerial photographs, if d is made equal to 1, the parameters a and b which are related to transversal and longitudinal tilt[1] are always finite. However, the parameter c is related to the rotation in the horizontal plane and can vary from zero in the case of no rotation to infinitely large in the case of a 180° rotation. Consequently, the choice of $d=1$ is necessary if the three rotations are extremely small, and this choice makes an accurate computation of the other three parameters possible unless the rotation in the horizontal plane is equal to or very close to 180°. In the latter case it is necessary to make the choice $c=1$.

Therefore, to solve the four normal equations, one can start with the elimination of a and b, using s_{11} and s_{22} as pivotal elements. Then, if the reduced value of s_{44} is greater than that of s_{33} (both are positive), put $c=1$ and solve d from the reduced fourth equation. Otherwise, put $d=1$ and solve c from the reduced third equation. Finally, solve the parameter b from the reduced second equation and solve the parameter a from the unreduced first equation.

This gives the following simple algorithm for the solution of the normal equations:

$$\begin{aligned} s_{22}' &= s_{22} - s_{12}s_{12}/s_{11} \\ s_{23}' &= s_{23} - s_{13}s_{12}/s_{11} \\ s_{24}' &= s_{24} - s_{14}s_{12}/s_{11} \\ s_{33}' &= s_{33} - s_{13}s_{13}/s_{11} - s_{23}'s_{23}'/s_{22}' \\ s_{34}' &= s_{34} - s_{14}s_{13}/s_{11} - s_{24}'s_{23}'/s_{22}' \\ s_{44}' &= s_{44} - s_{14}s_{14}/s_{11} - s_{24}'s_{24}'/s_{22}' \\ &\text{Either } d = 1 \text{ and } c = -s_{34}'/s_{33}' \\ &\text{or } \quad c = 1 \text{ and } d = -s_{34}'/s_{44}' \\ b &= -(s_{24}'d + s_{23}'c)/s_{22}' \\ a &= -(s_{14}d + s_{13}c + s_{12}b)/s_{11}. \end{aligned} \tag{10}$$

The matrix **D** can now be formed, its vectors can be reduced to unit length, and Equation 7 can be used to apply the rotation to all model points. In addition, a scale factor, computed as the ratio of lengths in the two models, can be applied to the rotated coordinates. At the same time, the origin of the x', y', z'-system can be shifted back from the common projection center to its original position.

Although Equation 7 is of an extremely simple form, its use for the transformation of the coordinates of all model points requires a rather large number of multiplications. Instead, the matrix multiplications in the equation can be performed once for all. This gives the transformed coordinates as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \lambda \mathbf{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x_P' \\ y_P' \\ z_P' \end{bmatrix} \qquad (11)$$

Here, λ is the scale factor, the matrix $\mathbf{R}$ is

$$R = \begin{bmatrix} d^2+a^2-b^2-c^2 & 2ab-2cd & 2ac+2bd \\ 2ab+2cd & d^2-a^2+b^2-c^2 & 2bc-2ad \\ 2ac-2bd & 2bc+2ad & d^2-a^2-b^2+c^2 \end{bmatrix} \cdot \frac{1}{d^2+a^2+b^2+c^2}, \qquad (12)$$

x, y, and z are the untransformed coordinates, reduced only for the initial shaft of origin, and x_P', y_P', and z_P' are the coordinates of the common projection center in the model to which the connection is made.

The Nature of the Transformation

The parameters x, y, z, and t have been interpreted as the coordinates in a rectangular Cartesian coordinate system in a four-dimensional Euclidean space.

It is of interest to note that the role of the parameter t differs from the roles of the other three. It occurs exclusively on the main diagonal of the matrix $\mathbf{T}$ and, besides, only the t-parameter of a matrix product is formed from the products of corresponding parameters.

A similar case occurs in the theory of relativity, where in the four-dimensional continuum of space and time the nature of the time coordinate is different from that of the three space coordinates.

If we wish to make the coordinates x, y, z, and t represent coordinates in this four-dimensional continuum, we must equate x, y, and z with the space coordinates. We can then let t represent the time-coordinate by means of the equation.

$$t = ict \qquad \text{(i)}$$

in which i is the imaginary unit, c is the velocity of light in vacuum, and $\bar{t}$ is the time in conventional units.

The theory of relativity concerns the Lorentz transformations and the invariance of the relations between the mathematical representations of physical entities under these transformations. These transformations are those for which

$$x^2 + y^2 + z^2 - c^2\bar{t}^2$$

is invariant. In other words, they leave the lengths of vectors in the x, y, z, t-system unchanged. Therefore, they are the rotations and inversions in this system.

It has already been shown that Equation 7, with real values of the parameters, represents the rotations in the three-dimensional subspace of x, y, and z. These rotations are a subgroup of the group of all possible Lorentz transformations.

An attempt to represent all these transformations by means of matrices $\mathbf{T}$ and $\mathbf{D}$ does not meet any difficulties. It can be shown that the rotations which involve the t-axis can be represented by

$$\mathbf{T}' = \mathbf{DTD} \qquad \text{(ii)}$$

in which the parameters a, b, and c are purely imaginary and d is real. In three-dimensional space, this equation represents the uniform relative motion of the coordinate systems x, y, z and x', y', z' as a function of time. Combining the two transformation formulas, one obtains the general transformation formula

$$\mathbf{T}' = \mathbf{D}_2\mathbf{D}_1\mathbf{T}\mathbf{D}_1'\mathbf{D}_2 = \mathbf{DT}\bar{\mathbf{D}}' \qquad \text{(iii)}$$

in which the parameters a, b, c, and d are complex numbers, the determinant of $\mathbf{D}$ is still equal to unity, and the postfactor is the transpose of the conjugate of $\mathbf{D}$.

The matrices $\mathbf{T}$ and $\mathbf{D}$ are not used in the textbooks on the special theory of relativity and its applications. Instead, tensors and spinors of different variances and valences are used. The manipulation of those awkward entities requires much space and absorbs much of the attention of the reader.

An investigation has shown that the special theory of relativity can be presented with the help of these matrices and that this leads to very clear and concise expositions. It makes it possible to eliminate the use of tensors and spinors completely and to avoid present tedious derivations and restrictions.

A very simple example of the application of these matrices, still within the range of our present subject of transformations, may conclude this excursion into the theory of relativity. Let one coordinate frame in three-dimensional space move with a uniform velocity with respect to a second frame that itself moves with a uniform velocity with respect to a third frame, and let at the time $t=0$ the axes of the three frames coincide. Then, each of the two movements can be expressed by the Equation ii. If the matrices of these two transformations are called $\mathbf{D}_1$ and $\mathbf{D}_2$, respectively, it follows that the resulting motion of the first frame with respect to the third one can be expressed by

$$\mathbf{T}' = \mathbf{D}_1\mathbf{D}_2\mathbf{T}\mathbf{D}_2\mathbf{D}_1. \qquad \text{(iv)}$$

This motion is again a translation as expressed by Equation ii if and only if $\mathbf{D}_1\mathbf{D}_2 = \mathbf{D}_2\mathbf{D}_1$. Since in general two matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ do not commute, the resulting transformation is in general not a pure translation. Therefore, it contains a rotation. This means that in general the coordinate axes of the first frame appear to be rotated with respect to those of the third frame.

REFERENCES

1. G. H. Schut, Construction of Orthogonal Matrices and their Application in Analytical Photogrammetry. *Photogrammetria*, Vol. XV, No. 4, 1958/59.
2. E. H. Thompson, An Exact Linear Solution of the Problem of Absolute Orientation. *Photogrammetria*, Vol. XV, No. 4, 1958/59.
3. G. H. Schut, On Exact Linear Equations for the Computation of the Rotational Elements of Absolute Orientation. *Photogrammetria*, Vol. XVII, No. 1, 1960/61.

Book Reviews

The Reproduction of Colour, by R. W. G. Hunt, Kodak Ltd., Harrow, England. Published by John Wiley and Sons, Inc., in 1967. 500 pages, 13 color plates, $16.00.

This second edition of Dr. Hunt's book on color is directed primarily toward the fields of television, motion pictures, and studio and amateur photography. The scope of the material and the book differs considerably from that of the ASP Manual of Color Photography even though some subject headings (including terminology, chemistry and processing, printing techniques, and sensitometry) are common to both texts. One important aspect of color, that concerned with the taking of color photographs, was evidently beyond the scope of Dr. Hunt's text.

Part One of the book, entitled Fundamentals, reviews the basic concepts of color formation and reproduction and provides the reader with the background that is necessary for understanding the technical discussions in the rest of the book.

Part Two, Color Photography, describes subtractive methods and materials for color reproduction and is reasonably up-to-date.

Part Three, Color Television, may have been the primary reason for putting out a second edition. The subject is treated in complete detail and, in the reviewers' opinion, the chapter might well have served as the basis for a separate publication. On the other hand, Part Four, Color Printing, treats the important subject of photomechanical printing with inconsistent brevity.

The author is to be commended for his clarity of thought and orderly presentation of material. The book is an excellent reference work for those who are interested in the particular aspects of color technology that are covered.

—*A. B. Clarke* and *James Halliday*

Aerial Discovery Manual

by Carl H. Strandberg, John Wiley & Sons, Inc., 1967

Two major factors have made publication of the Aerial Discovery Manual appropriate and timely. First are advances in aerial photography, photogrammetry, and remote sensors; and the recognition that these tools can play an important part in resources evaluation. The second is the recognized need by resource scientists for means of becoming more efficient in their studies, of covering more terrain in less time, and of collecting more significant data on their problems. With courses in aerial photographic interpretation and remote-sensor interpretation becoming more widespread in many universities, the Aerial Discovery Manual fills the need for a basic introductory text on the subject suitable for an undergraduate course. The Manual is divided into three sections—Aerial Photographic Interpretation, Photogeology, and Photohydrology. The author assumes that the student who begins to use this book is not at all familiar with aerial photography, remote sensing, or photointerpretation but has an interest in learning these techniques and applying them to his particular field.

The first section provides the student with basic information on aerial reconnaissance, how to obtain aerial photographs, the instru-