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An Exact† Solution of Absolute Orientation

The system is linear, non-iterative, and simple enough to compute on a desk calculator.

(*Abstract on next page*)

Introduction

The problem of absolute orientation of a relatively oriented space model in photogrammetry is considered to be one of the determination of seven unknowns: scale, three shifts of one origin with respect to the other, and three independent parameters of an orthogonal rotation in three dimensions. The scale and the three origin shifts are usually considered easy to determine (Thompson 1964). The problem of obtaining the elements of orthogonal rotation is, however, regarded as non-linear, requiring an iterative approach for building up the exact solution. Nevertheless, two eminent photogrammetrists, have successfully developed exact linear equations in three parameters from which the relevant orthogonal matrix can be constructed. Either solution requires the measurement of the rectangular coordinates of three non-collinear points in both systems.

Thompson (1959) has based his equations on the rotation of a sphere of unit radius whose center coincides with one of the points as origin. The position vectors of the other points are intersected by this sphere and their stereographic projection is followed by transformation of the projection due to the rotation of the sphere about the origin. The resulting equations are quite simple, and are a great improvement upon the traditional nonlinear equations which can only be solved by an iterative process. There are two difficulties, however: firstly, the rotation must not equal or nearly equal 90°; secondly the coefficients of the parameters are non-linear. The equations are based on a somewhat lengthy theory of stereographic projection using complex planes.

Schut (1960) has derived his linear equations on the basis of a quaternion composed of three elements of a skew symmetric matrix and a fourth quantity determined by a relation that exists between the four. Schut's derivation is elegant and results in simple linear equations with linear coefficients. No restrictions are placed on the amount of rotation.

In either case the solution obtains three parameters from which the nine elements of the orthogonal rotation matrix are easily constructed by the use of the Rhodrigues form of orthogonal matrix.

In the following the authors have presented a linear solution of the absolute orientation problem which, in addition to being essentially simple, gives directly all the elements of the orthogonal rotation matrix. The other merits of the solution will be discussed after it has been described.

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† *Comments by the Publications Committee: Fundamentally, the solution is not exact in that the nine elements of the rotation matrix are solved independently. The consequence is that the rotation matrix is not necessarily orthogonal. The authors rectify this by orthogonalizing by means of their Equation 9.*

The same approach to finding the nine elements of the rotation matrix was applied by the late Professor Earl Church in his early publications. However, Professor Church did not subsequently orthogonalize the matrix. The same solution for nine elements was also employed by the Herget solution developed at Ohio State University and subsequently adapted by the U. S. Geological Survey and ACIC. Professor Herget, however, used a different technique for orthogonalizing the matrix. The derivation of the authors' Equation 9 is not provided.

The solution offered, while practical, is not exact and, because of the arbitrariness of the orthogonalization technique, should actually be an iterative solution.

ABSTRACT: *In this paper the authors have presented an exact linear solution of the problem of analytical absolute orientation. The solution using four points, obtains directly the nine elements of the orthogonal matrix and is in this regard an improvement on the solution by Thompson (1959) or Schut (1960). Full use of any extra control data may be made for obtaining a least-squares adjustment for the most probable values of the elements of absolute orientation. The advantages and limitations of the solution have been discussed. The slight modifications needed in planning analytical aerial triangulation when this method is utilized for model connection have been alluded to.*

THE METHOD

Let the model and the ground be measured in the orthogonal systems (x, y, z) and (X, Y, Z) respectively. The coordinates of a point in the two systems are, in general terms (x_p, y_p, z_p) and (X_p, Y_p, Z_p), respectively, and p may have integer values 1 to n, where n is the number of points measured in both systems.

The complete transformation is then described as

$$\begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} = \lambda^{-1} \boldsymbol{R} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} \tag{1}$$

where λ is the scale of the photogrammetric model. $\boldsymbol{R}$ is an orthogonal matrix defined as follows:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

and X_o, Y_o, Z_o are the shifts of the ground origin with respect to the model origin. The seven unknowns of absolute orientation are λ, three independent elements of $\boldsymbol{R}$, and X_o, Y_o, Z_o. As is well known, the six independent relations between the nine elements of $\boldsymbol{R}$ are,

$$\begin{aligned} \boldsymbol{R}_{*j}{}^T \boldsymbol{R}_{*j} &= 1, \qquad (j = 1, 2, 3) \\ \boldsymbol{R}_{*2}{}^T \boldsymbol{R}_{*1} &= 0 \\ \boldsymbol{R}_{*3}{}^T \boldsymbol{R}_{*2} &= 0 \\ \boldsymbol{R}_{*1}{}^T \boldsymbol{R}^{*}{}_3 &= 0. \end{aligned} \tag{2}$$

In (2), $\boldsymbol{R}_{1*}$ is the ith row of $\boldsymbol{R}$ and $\boldsymbol{R}_{*j}$ is the jth column.

The scale of the photogrammetric model is then given by,

$$\lambda = \left[\frac{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2} \right]^{1/2}, \tag{3}$$

points 1 and 2 being so chosen as to give the largest possible vector length. λ is redetermined from a few more pairs of points similarly chosen and a weighted mean value adopted as the most probable. The weights should be the vector lengths.

From (1), we may write, for n points measured in the two systems, an equal number of equations separately for X_p, Y_p, Z_p, as follows:

$$X_p = \lambda^{-1}(r_{11}x_p + r_{12}y_p + r_{13}z_p) + X_0 \tag{4}$$

$$Y_p = \lambda^{-1}(r_{21}x_p + r_{22}y_p + r_{23}z_p) + Y_0 \tag{5}$$

$$Z_p = \lambda^{-1}(r_{31}z_p + r_{32}y_p + r_{33}z_p) + Z_0. \tag{6}$$

Or, in the matrix form, these are,

$$\begin{aligned} X_p &= \lambda^{-1}(x_p y_p z_p)\boldsymbol{R}_{1*}{}^T + X_0 \\ Y_p &= \lambda^{-1}(x_p y_p z_p)\boldsymbol{R}_{2*}{}^T + Y_0 \\ Z_p &= \lambda^{-1}(x_p y_p z_p)\boldsymbol{R}_{3*}{}^T + Z_0. \end{aligned}$$

In each of the above sets of equations, four unknowns are involved: the three elements of a row of $\boldsymbol{R}$ and one shift. Hence a minimum number of four control points measured in both systems are required for the solution of all the three rows of $\boldsymbol{R}$ and the three shifts. If more than four control points are available, a least square solution based on the derived normal equations is worked out in the usual manner.

Suppose four control points, located approximately in the corners of the model, are available. By defining $X_{12} = X_1 - X_2$, and $X_{12} = x_1 - x_2$ etc., we have from (4),

$$\begin{bmatrix} X_{12} \\ X_{23} \\ X_{34} \end{bmatrix} = \lambda^{-1} \begin{bmatrix} x_{12} & y_{12} & z_{12} \\ x_{23} & y_{23} & z_{23} \\ x_{34} & y_{34} & z_{34} \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix}, \tag{7}$$

or briefly,

$$\boldsymbol{X} = \lambda^{-1}(\boldsymbol{xyz})\boldsymbol{R}_{1*}{}^T$$

Here $\boldsymbol{X}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, are (3×1) column matrices.

Similarly from (5) and (6), we have,

$$\begin{aligned} \boldsymbol{Y} &= \lambda^{-1}(\boldsymbol{xyz})\boldsymbol{R}_{2*}{}^T \\ \boldsymbol{Z} &= \lambda^{-1}(\boldsymbol{xyz})\boldsymbol{R}_{3*}{}^T. \end{aligned} \tag{7a}$$

The direct linear solution for the nine elements is obtained row by row as follows:

$$\begin{aligned} \boldsymbol{R}_{1*}^T &= \lambda(\boldsymbol{xyz})^{-1}\boldsymbol{X} \\ \boldsymbol{R}_{2*}^T &= \lambda(\boldsymbol{xyz})^{-1}\boldsymbol{Y} \\ \boldsymbol{R}_{3*}^T &= \lambda(\boldsymbol{xyz})^{-1}\boldsymbol{Z} \end{aligned} \quad . \qquad (8)$$

From (8) it is clear that only one 3×3 matrix need be inverted to give the nine elements of $\boldsymbol{R}$ which values are now adjusted by least squares for conditions given in (2). It may easily be shown that the corrections dR are given by,

$$\boldsymbol{dR} = \boldsymbol{R}\begin{bmatrix} \frac{1}{2}K_1 & \frac{1}{2}K_4 & \frac{1}{2}K_6 \\ \frac{1}{2}K_4 & \frac{1}{2}K_2 & \frac{1}{2}K_5 \\ \frac{1}{2}K_6 & \frac{1}{2}K_5 & \frac{1}{2}K_3 \end{bmatrix} \quad . \qquad (9)$$

In (9), $K_1 \ldots K_6$ are defined as the corrective amounts by which the six of Equations 2 fail to satisfy, and $\boldsymbol{R}$ is determined from (8). The improved orthogonal matrix is $\boldsymbol{R} + \boldsymbol{dR}$. The three shifts X_o, Y_o, Z_o are now obtained by substitution in (1). Four values of each are obtained and the mean values are accepted.

If additional points have been measured in both systems, there is no difficulty in using the redundant data in the mathematical process given above. Now the matrices $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ in (7A) will be $(n-1\times 1$ column matrices instead of 3×1. Equation 7a must therefore be normalized for solution which will now be

$$\boldsymbol{R}_{1*}^T = \lambda\{(\boldsymbol{xyz})^T(\boldsymbol{xvz})\}^{-1}(\boldsymbol{xyz})^T\boldsymbol{X} \qquad (10)$$

with similar expressions for $\boldsymbol{R}_{2*}^T$ and $\boldsymbol{R}_{2*}^T$. The nine elements of $\boldsymbol{R}$ are adjusted and X_o, Y_o, Z_o obtained as already indicated.

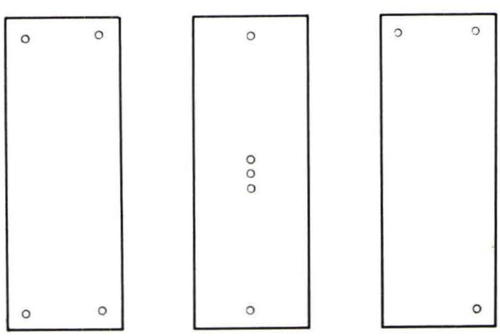
Advantages and Limitations

The system of equations being linear, the solution is direct. The values of the nine elements of the rotation matrix are directly determined and admit of least square adjustment. The minimum four control points required for solution include adequate redundancy for the rotation as well as for the determination of λ and the three shifts. The solution is capable of utilizing additional redundant data by the least-squares method. Further, the solution is unrestricted as regards the amount or rotation. The coordinates of the projection centers not being required, the solution remains entirely on ground and its accuracy does not require precise instrument calibration if used for computing model connections in independent model triangulation. Large shifts are no problem and no computations whatsoever are involved for preparing the linear equations as is the case in the methods of Thompson and Schut. The solution is economical in computations and the work can be easily carried out on a desk calculator.

For the absolute orientation of a single model, the best location for the four points is obviously in the corners of the model. It is important that no three points should be collinear. If connection of consecutive models in aerial triangulation is desired, the pattern of points 1, 2, 3 and 4 in the superlap should be as shown in Figure 1a, where Figure 1b shows the arrangement of wing and scale points for the usual methods. A solution with the help of points as in Figure 1a does not depend on the coordinates of the perspective center of the common photograph to be known. However, if this information is provided, as is the case in analytical aerial triangulation, its use reduces the number of observed points in the superlap to three, and the pattern should be as shown in Figure 1c.

For a superlap of 20 percent (45 mm at negative scale in photography which is 23 $\times$23 cm) the rectangle of four points in Figure 1a, or the triangle of three points in Figure 1c, is about 30 mm$\times$200 mm. It is obvious that no additional observations are necessary for the described solution; only a slight modification in the location pattern of pass points is required at the triangulation planning stage.

Sample computations by this method have produced excellent results. The solution for a hypothetical model using four points as in Figure 1a is shown in Tables 1a and 1b. The authors have worked out the same example by Thompson's and Schut's methods as well. Also, the method has been tested for pattern of Figure 1c. The solution given in Table 1 using the method described in this paper involves, in the author's opinion, the minimum computations and is therefore the most eco-



(1a) (1b) (1c)

FIG. 1. Arrangements of points for three applications of computational absolute orientation.

TABLE 1a. COMPUTATION OF A HYPOTHETICAL MODEL: COORDINATE DIFFERENCES

Point	X	Y	Z	X_{i4}	Y_{i4}	Z_{i4}	x	y	z	x_{i4}	y_{i4}	z_{i4}
1	46.000	60.000	−110.000	−20.000	120.000	−10.000	−218.474	153.810	−190.448	−210.762	111.344	52.750
2	66.000	60.000	− 90.000	0.000	120.000	10.000	−182.996	184.374	−158.712	−175.284	141.908	84.486
3	46.000	−60.000	−110.000	−20.000	0.000	−10.000	− 36.952	11.120	−255.946	− 29.240	−31.346	−12.748
4	66.000	−60.000	−100.000	0.000	0.000	0.000	− 7.712	42.466	−243.198	0.000	0.000	0.000

TABLE 1b. COMPUTATION OF A HYPOTHETICAL MODEL: SOLUTION FOR **R**

x_{i4}'	y_{i4}'	z_{i4}'	X_{i4}	r_{1j}	Y_{i4}	r_{2j}	Z_{i4}	r_{3j}
−105.381	55.672	26.375	−20.000		120.000		−10.000	
−14.620	−15.673	−6.374	−20.000		0.000		−10.000	
−87.642	70.954	42.243	0.000		120.000		10.000	
−1.00000000	0.52829250	0.25028230	−0.18978750	0.57503332	1.13872510	−0.75633328	−0.09489380	0.31189688
1.00000000	1.07202460	0.43597810	1.36798900		0.00000000		0.58399450	
1.00000000	−0.80958900	−0.48199490	0.00000000		−1.36920650		−0.11410050	
	1.60031710	0.68626040	1.17820150		1.13872510		0.58910070	
	−0.28129650	−0.23171260	−0.18978750		−0.23048140		−0.20899430	
	−1.00000000	−0.42882776	−0.73623003	0.80305842	−0.71156216	0.59451036	−0.36811498	−0.03894031
	1.00000000	0.82373083	0.67468845		0.81935396		0.74296801	
		0.39490307	−0.06154158	−0.15583971	0.10779180	0.27295761	0.37485303	0.94922794
	Mean $X_0=32.216$			Mean $Y_0=-42.348$			Mean $Z_0=17.455$	

nomical. It is also believed that, involving the minimum arithmetical operations, it would give results of as high precision as is possible from the given observations.

A least-squares adjustment of the elements of the rotation matrix according to (9) produces the following values for the matrix elements:

0.57504	−0.75634	0.31190
0.80313	0.59453	−0.03902
−0.15592	0.27294	0.94931

the assumed values in the hypothetical example being,

0.57505	−0.75634	0.31190
0.80312	0.59456	−0.03898
−0.15594	0.27291	0.94932

and the unadjusted computed values having been given in Table 1b.

Appendix I

For analytical photogrammetric triangulation, the orientation matrix as determined above is sufficient. Subsequently for plotting however, it is more appropriate to know the rotation elements in terms of Ω, Φ, K. For a right handed system with Ω, Φ, K as the rotation sequence, the following relations are well known:

$$\boldsymbol{X} = \boldsymbol{R}\mathbf{x} = \mathrm{K}\Phi\Omega x \tag{11}$$

The elements of $\boldsymbol{R}$ are as follows

$$\begin{aligned}
r_{11} &= \cos\varphi\cos\kappa \\
r_{12} &= \cos\omega\sin\kappa + \sin\omega\sin\varphi\cos\kappa \\
r_{13} &= \sin\omega\sin\kappa - \cos\omega\sin\varphi\cos\kappa \\
r_{21} &= -\cos\varphi\sin\kappa \\
r_{22} &= \cos\varphi\cos\kappa - \sin\omega\sin\varphi\sin\kappa \\
r_{23} &= \sin\omega\cos\kappa + \cos\omega\sin\varphi\sin\kappa \\
r_{31} &= \sin\varphi \\
r_{32} &= -\sin\omega\cos\varphi \\
r_{33} &= \cos\omega\cos\varphi
\end{aligned} \tag{12}$$

From (12), we immediately have the solution for ω, ϕ, and κ.

$$\begin{aligned}
\varphi &= \sin^{-1} r_{31} \\
\omega &= \cos^{-1}(r_{33}\sec\varphi) = \sin^{-1}(-r_{32}\sec\varphi) \\
\kappa &= \cos^{-1}(r_{11}\sec\varphi) = \sin^{-1}(-r_{21}\sec\varphi)
\end{aligned} \tag{13}$$

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