J. A. M. WOLTERS*
University of Tokyo and
Geographical Survey Institute, Japan

Accuracy of Analytics by Computer Simulation

15 points instead of 6 for relative orientation reduced position errors 25% and height errors 10%.

ABSTRACT: *Twenty-five independent strips comprising ten models each were analyzed to determine the accuracy of analytical aerotriangulation for flat terrain, for valleys, and for side-sloping terrain. These are the terrain types usually encountered in highway construction. The height differences average 10 percent of the flying height for valley-type terrain and 15 percent for side-sloping terrain. Six points as well as 15 points were used for relative orientation. A regular pattern of points was chosen on the ground and projected on the photographs (25 points per photograph). A random normal variable generator, internally programmed, was used to give displacements to the image coordinates. After the strip was bridged, a second-degree conformal transformation was applied to transform the strip coordinates to ground coordinates. The investigations were continued for flat terrain, using six points for relative orientation, and simulating stereocomparator measurements. Simulations were also conducted for conditions where the accuracy of the photocoordinates decreased towards the edges as well as when the measurements of corresponding points in the different photographs were correlated. Finally, the triplet method was simulated.*

INTRODUCTION

WITH THE INTRODUCTION of analytical photogrammetry it has become possible to eliminate or reduce many of the systematic errors which adversely effect the aerotriangulation results of analog methods. Using analog methods, the instruments are more complicated and it is not possible to correct known systematic errors numerically. It is, therefore, of practical significance to investigate the propagation of accidental errors in triangulated strips.

In the case of flat terrain and using the six-point method for relative orientation, the theoretically expected accuracy has been derived for flat terrain. For mountainous terrain, and using more than six points for relative orientation, a theoretical derivation becomes laborious. If it is assumed that the accuracy of the image coordinates varies over the photo plate, or that the observations of corresponding points in different photographs are correlated and a transformation of the strip coordinates to ground coordinates is involved, then a computer simulation is a practical solution to evaluate the accuracy.

This paper can be subdivided into two parts. In the first part the accuracy of triangulated strips in mountainous terrain is reported. In the second part some tests are described assuming a decreasing accuracy of the photo coordinates towards the edges, as well as correlated observations of corresponding points in different photographs. Also a comparison has been made using a monocomparator and a stereocomparator. Finally, triangulation results with the triplet method are reported.

The simulation programs were written in ALGOLIP, which is a simplified version of ALGOL 60. The computations were done on the OKITAC 5090 of the Institute of Industrial Science, University of Tokyo; the OKITAC is a medium-size, slow-speed computer.

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In all the cases investigated, 25 independent strips of ten models each were analyzed. The focal distance was assumed to be 15 cm and the base-height ratio 2/3. The photographs were assumed to be taken vertically with equal base length. The only significant effect of nonvertical photographs was the increased number of iterations necessary for relative orientation. The terrain points were chosen in a regular pattern spaced at distances of half a base length. This means 105 points were covered by a strip of ten models. All terrain points were used to transform the strip coordinates to ground coordinates. With this procedure, variations in the accuracy could not be attributed to a specific choice in the location of ground control.

After projecting the terrain points on the photographs (25 points per photograph), randomly distributed displacements in both x and y were applied to the fictitious plate coordinates. In the following it has been assumed that the errors in the measurements of the plate coordinates were normally distributed. To introduce displacements to the photo coordinates, use was made of a random normal variable generator, which was internally programmed.

This generator can be explained as follows. If U_1 and U_2 are independent uniform deviates, then

$$X = (-2LnU_1)^{1/2} \cos 2\pi U_2$$

is a random normal variable with mean 0 and variance 1. To generate independent uniform deviates, use was made of the pseudorandom number generator $R_i+1=177147\ R_i$. The starting value for R_o was 78125. To go from random numbers to uniform deviates, the random number was divided by the wordlength of the computer. In this case $U=R/10^{12}$. For different kinds of computers, there exist several methods which generate normal deviates faster (see ref.), but this method was selected because the memory space required is small. Mathematically this approach has the attractive advantage that the transformation for going from uniform deviates to normal deviates is exact. The errors assigned to the photo coordinates have mean 0 and standard deviation 5 microns. In case of independent observations the standard deviation of the y-parallax was, therefore,

$$S_{rp} = (25 + 25)^{1/2}u = 7 \text{ microns.}$$

If not stated otherwise, the coordinates were measured with a monocomparator. The relative orientation was computed as has been described by Jerie (1956). An approximate value of the orientation elements of the second photograph was obtained from the parallax equation

$$y_2 - y_1 = x_2 \times \kappa + \frac{x_2 y_2}{f} \times \phi + f \times \left(1 + \frac{y_2^2}{f^2}\right) \times \omega + y_2 \times \left(\frac{x_1 - x_2}{f}\right) \times \frac{bz}{bx} + (x_1 - x_2) \times \frac{by}{bx}$$

where x_1, y_1, x_2, and y_2 refer to the coordinates of photos 1 and 2, respectively, of a stereomodel. The Greek letters κ, ϕ, and ω refer to rotations of the second photograph about the z-, y-, and x-axes, respectively. The symbols bx, by, and bz refer to the x-, y-, and z-distances between the perspective centers of the two photographs. After corrections of the measured picture coordinates, the process was repeated until the changes were negligible.

Mountainous Terrain

Three types of terrain which are often encountered in highway construction were investigated; namely, flat terrain, valley type, and side-sloping terrain. The height differences averaged 10 percent of the flying height for valley-type terrain and 15 percent for side-sloping terrain. The previously mentioned generator was also used to generate valley and sidesloping terrain.

The points at the center of the strip were designated by CC and the points outside the strip axis by UU, UC, LC, and LL. (see Figure 1). The elevations of the terrain points along the lines UU, UC, CC, LC, LL were generated with mean M and standard deviation S. The values for M and S for a flying height of 1500 meters are shown in Table 1.

The choice of these figures was not based on strict logic, but the following criteria served as guidelines:

1. The valley and side-sloping terrains generated were to have height differences

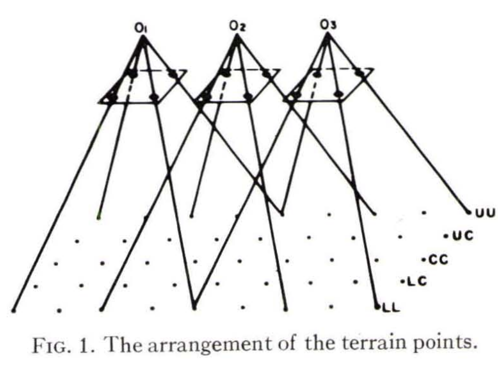

FIG. 1. The arrangement of the terrain points.

TABLE 1. TERRAIN CHARACTERISTICS

	Valley		Slope	
	M	*S*	*M*	*S*
UU	+70 m	40 m	+150 m	40 m
UC	−30	30	+ 50	30
CC	−80	20	− 50	30
LC	−30	30	−100	20
LL	+70	40	− 50	30

averaging 1 percent and 15 percent of the flying height respectively.

2. The variations needed to be large enough to make the results applicable to as many practical cases as possible but, on the other hand, not so large as to make the physical characteristics of the valley and slope terrain unrealistic and the results meaningless.

As can be seen from Table 1, the sideslope was not really a slope but a valley with a high side and a low side. The negative values for the elevations were chosen to get an average photo scale which was equal for the three types of terrain.

MODEL SIMULATION

The simulations were started for one model. To calculate the orientation elements, 6 points as well as 15 points were used. The model was simulated 36 times. Each time *new terrain* was generated, the terrain points projected on the photographs and new errors were assigned to the image points. In Figure 2 are shown the standard deviation of the residual parallaxes in each point for 36 repetitions.

Because of the different elevations of the terrain points, the location of the image points on the photograph vary in a systematic way. Therefore different values for the residual parallaxes can be expected for the three types of terrain. However, it was decided to have the terrain points in a regular pattern on the ground, even though this might cause slight variations in the projected images. As can be seen from Figure 2, the standard deviation of the residual parallaxes was in accordance with the theoretically expected values. Where 15 points were used for relative orientation, the standard deviation of the orientation elements and *BZ* was significally smaller than was expected. Therefore, it was decided to calculate also the standard deviation of the orientation elements of a cantilever extension and this was the next phase carried out.

CANTILEVER EXTENSION

A cantilever extension of 10 models was simulated for the three types of terrain. For these types, the same series of errors were generated and assigned to the photo coordinates. The relative orientation was computer using 6 points as well as 15 points. The model was scaled by equating the elevations of the transfer points (3 points and 5 points respectively). The bridging was repeated 36 times, generating new terrain for each case.

The standard deviation of the *X*-, *Y*-, and *Z*-coordinates is shown in Figure 3.

As can be seen from Table 2, the standard deviation of the orientation elements was in accordance with the theoretically expected values, which are shown in brackets. It indicates that the computing procedure as well as the variable-number generator was reliable for this particular purpose.

STRIP ADJUSTMENT

To investigate the absolute accuracy, 25 independent strips of 10 models were bridged and adjusted to the ground coordinates. To transform the strip coordinates to the ground coordinates, a second degree conformal transformation was applied, as has been described by Schut (1962). This transformation is a sequence of conformal transformations in two dimensions xz, yz, and xy, which is of particular interest in mountainous terrain. The results are shown in Table 3. The accuracy for x, y, and z is shown for points located along the lines *UU*, *UC*, *CC*, *LC*, and *LL*.

As can be seen from Table 3, the accuracy for flat terrain, valley, and slope was the same for height differences of 10 percent and 15 percent of the flying height. Where 15 points were used instead of 6 for relative orientation, a 25 percent gain in accuracy was obtained in planimetry and 10 percent in height.

The accuracy of the x-coordinates is not only uniform but considerably greater. A reduction of the difference in y-accuracy can be attained through an improvement of the accuracy of the orientation elements κ and ω with respect to the other orientation elements. An alternative solution is to force a strong correlation in the κ and ω errors of successive models. The results shown so far are obtained by simulating triangulations for a monocomparator or a stereocomparator with three plate carriers.

STEREOCOMPARATOR

If a two-plate stereocomparator is used, the points located in three photographs are

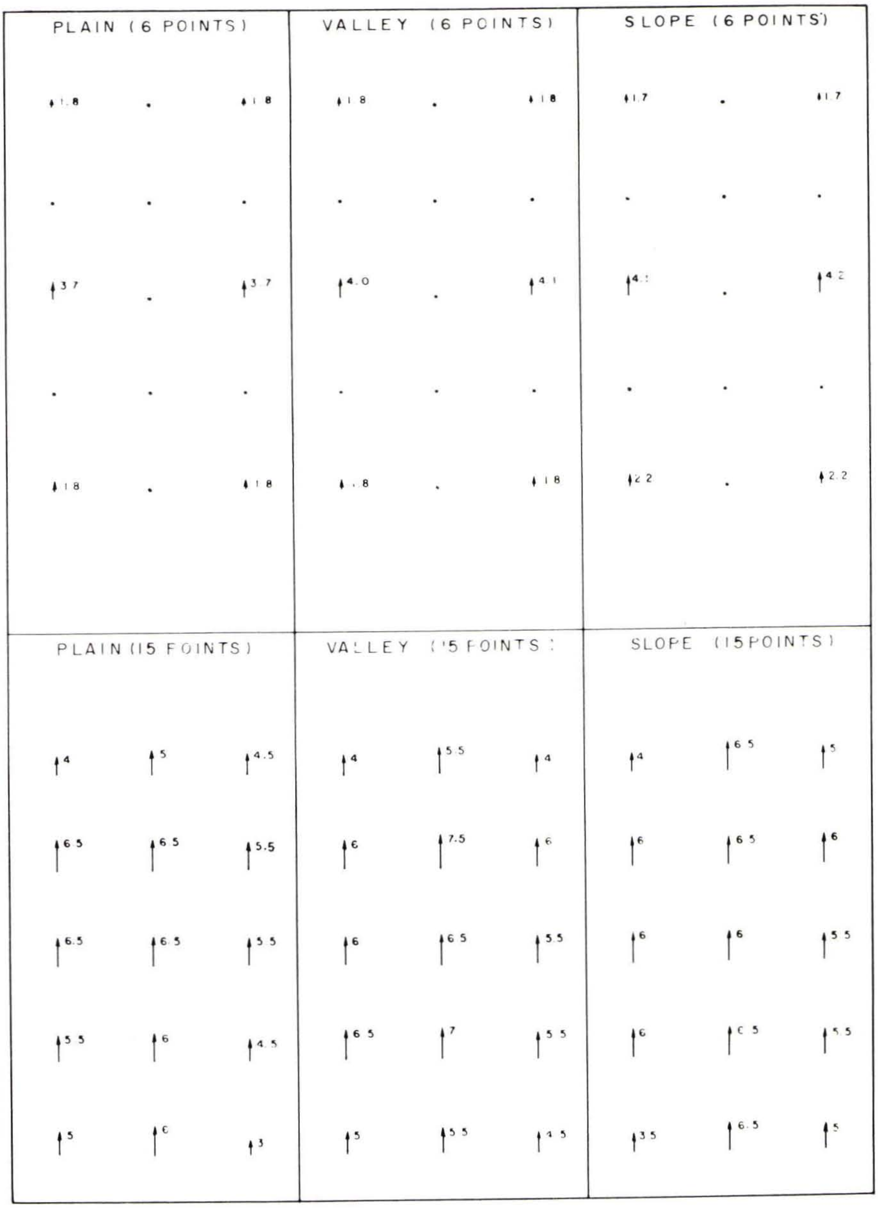

FIG. 2. Standard deviation of residual y-parallaxes.

measured twice. It was assumed that the two measurements were independent. It is very unlikely, however, that two measurements of the same point or different points located in the same area are independent because of film shrinkage, etc. The stereocomparator results as shown in Table 4 can, therefore, be considered to represent an extreme case. As can be seen from Table 4, the overall accuracy dropped by about 10 percent, and also the variation in accuracy for points along the strip axis and the edges became larger. This result is almost contrary to the outcome of the theoretical investigations by v. d. Weele (1964), which may be partly accidental and partly due to the transformation of the strip coordinates to the ground coordinates.

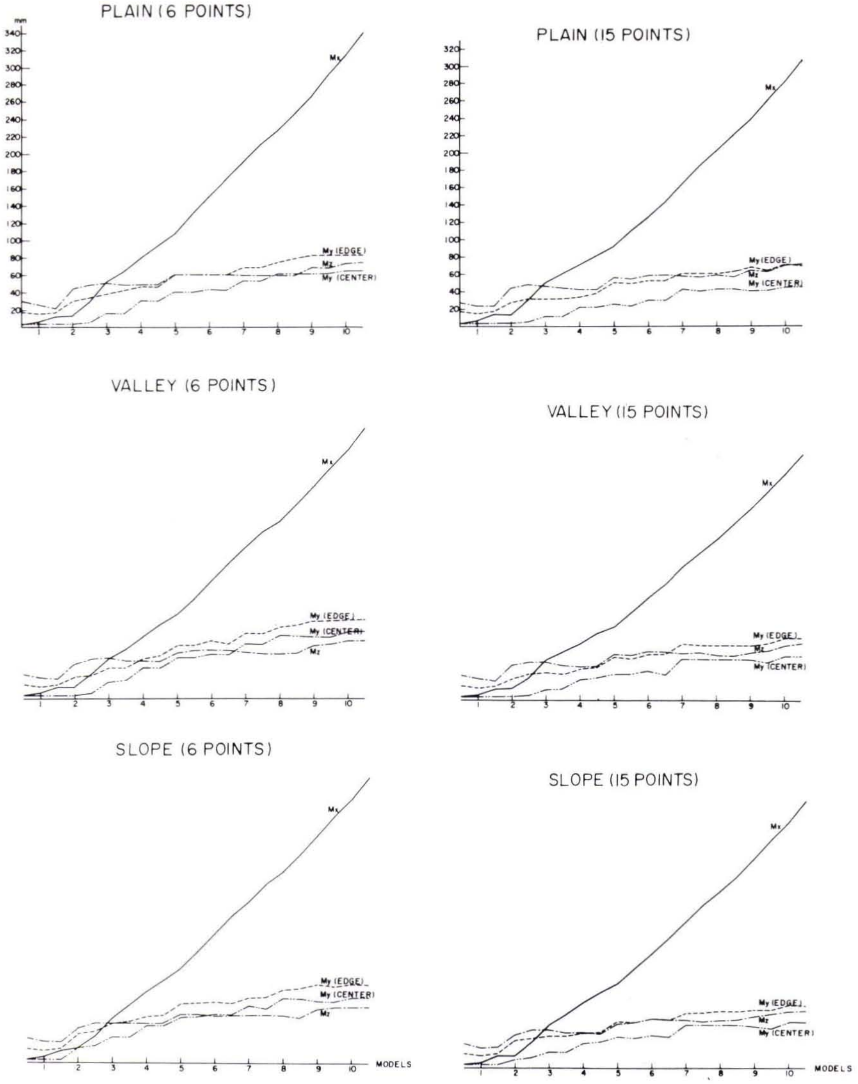

FIG. 3. Cantilever extension of 10 models. (Vertical scales of upper diagrams apply to all of them).

TABLE 2. STANDARD DEVIATION OF ORIENTATION ELEMENTS
FIGURES IN BRACKETS ARE THE THEORETICAL VALUES

	Flat Terrain		*Valley*		*Slope*	
	6-point	*15-point*	*6-point*	*15-point*	*6-point*	*15-point*
$\kappa \cdot 10^{-6}$ rad.	55(57)	46(44)	56(56)	47(44)	56(57)	46(44)
$\phi \cdot 10^{-6}$ rad.	107(105)	97(94)	97(95)	91(87)	98(97)	91(89)
$\omega \cdot 10^{-6}$ rad.	90(91)	66(64)	111(113)	77(76)	102(103)	71(70)
bz_1 (micron)	7(7)	6(6)	6(7)	5(6)	6(7)	5(6)
by_2 (micron)	18(18)	12(12)	22(23)	14(13)	20(21)	13(13)

TABLE 3. STANDARD DEVIATION OF x, y, z-COORDINATES

6-point Method

	Flat			*Valley*			*Slope*		
	Mx	My	Mz	Mx	My	Mz	Mx	My	Mz
UU	19	21	22	20	22	20	19	21	19
UC	17	14	21	16	16	20	16	15	19
CC	15	14	20	14	16	19	14	15	19
LC	16	17	20	16	19	19	16	17	20
LL	20	23	21	20	24	19	20	23	21
Av.	17	18	21	16	20	20	17	18	20

15-point method

	Flat			*Valley*			*Slope*		
	Mx	My	Mz	Mx	My	Mz	Mx	My	Mz
UU	15	18	19	15	18	18	15	18	17
UC	13	13	19	13	14	18	13	13	17
CC	13	11	18	12	12	18	12	11	17
LC	13	13	18	13	14	17	13	14	18
LL	15	18	18	15	18	17	15	18	18
Av.	14	15	19	13	15	18	14	15	18

CORRELATED OBSERVATIONS

As can be expected, the settings of the measuring mark on a point in two photographs are not independent but correlated. Investigations by Visser (1964) show a correlation of 70 percent. Although this may differ from one case to another, it is useful to investigate its influence on the accuracy of a strip triangulation. In the following a correlation of 70 percent was assumed. In order to be able to compare results it was decided to keep the standard deviation of the y-parallaxes at 7 microns. In assigning errors to the photo coordinates the following conditions were imposed. If N and M are two random normal variables with mean 0 and standard deviation 9 microns, then the variables N and $(N+M)/2$ are normally distributed with mean 0 and standard deviation 9 microns and a correlation factor 0.7. This procedure was followed in assigning errors to the x, y-photo coordinates of corresponding points. If y_1 and y_2 are two normally distributed variables with the same mean and standard deviation 9 microns and correlation factor 0.7, then it can easily be shown that the standard deviation of the parallax measurement is $S_p=7$ microns. The results of the triangulations are shown in Table 4. As can be seen, the overall accuracy dropped by 25 percent. The variation in accuracy for points along the strip axis and the edges did not change.

VARYING ACCURACY OF PHOTO COORDINATES

As has been shown by experiments in Sweden and mentioned by Hallert (1964), the accuracy varies greatly from the nadir point towards the edges. As this is very difficult to differentiate from the residual parallaxes, and impossible to find it, only six points are used for relative orientation, it is interesting to investigate this effect on the accuracy of a strip triangulation. It was assumed that the accuracy decreases proportionally to the square of the distance from the nadir point:

$$S_x = 5[(25 + L^2)/138\mu]^{1/2}$$
$$S_y = 5[(25 + L^2)/138\mu]^{1/2}$$

where L is the distance from the nadir point in centimeters. The accuracy of x and y at the nadir point is therefore 2.2, and in the corners 6.2, which is a moderate increase as compared with the variations mentioned by Hallert (1964). If we restricted ourselves to the 9 points on the photo plate used for relative orientation, the overall accuracy of the x and y photo coordinates is 5.3. If all 25 points are included, the overall accuracy is 4.7. In this way the results of the strip triangulation can be compared with results obtained previously, where the accuracy of the photo coordinates is uniform with a standard deviation of 5 microns. As can be seen from Table 4, the overall accuracy did not change significantly. However, the difference in accuracy for points along the strip axis and at the edges did increase 100 percent for y.

Triplet

Although the total computation time required for a strip triangulation was doubled when using the triplet method, the absolute accuracy as well as the relative accuracy of the strip would be expected to improve. Since the computation time is usually not the main burden of a photogrammetric project, this method may prove to be more economical considering the overall cost. Besides, this method provides more possibilities of detecting gross errors, which may save time on the remeasurements.

The increase in computation time was mainly due to the extra time required to form the normal equations and to solve the 11 unknowns. Using the three-photograph method, 15 condition equations with 11 unknowns were to be solved. There were 6 y-parallax equations for the left pair of photographs and 6 y-parallax equations for the right pair. In addition there were 3 x-parallax equations for the 3 points in the overlapping area of the 3 photographs.

If we designate the coordinates of the three subsequent photographs by x_1, y_1, x_2, y_2, x_3, and y_3, then the three types of equations can be written as follows:

$$y_1 - y_2 = x_1\kappa_1 + \frac{x_1 y_1}{F}\phi + F\left(1 + \frac{y^2}{F^2}\right)\omega_1 + y_1\left(\frac{x_2 - x_1}{F}\right) \times \frac{bz_1}{bx_1} + (x_2 - x_1)\frac{by_1}{bx_1}$$

$$y_3 - y_2 = x_3\kappa_3 + \frac{x_3 y_3}{F}\phi_3 + F\left(1 + \frac{y^2}{F^2}\right)\omega_3 + y_3\left(\frac{x_2 - x_3}{F}\right)\frac{bz_3}{bx_2} + (x_2 - x_3)\frac{by_3}{bx_2}$$

$$\begin{aligned} x_1 + x_3 - 2x_2 &= y_1\kappa_1 + F\left(1 + \frac{x_2}{F^2}\right)\phi_1 + \frac{x_1 y_1}{F} \times \omega_1 \\ &\quad + \frac{x_1}{F}(x_2 - x_1)\frac{by_1}{bx_1} - y_3\kappa_3 \\ &\quad + F\left(1 + \frac{x_3^2}{F^2}\right)\phi_3 + \frac{x_3 y_3}{F^2} \times \omega_3 \\ &\quad + \frac{x_3}{F}(x_2 - x_3)\frac{bz_3}{bx_2} + (x_2 - x_1)\left(\frac{bx_2 - bx_1}{bx_1}\right) \end{aligned}$$

With these condition equations, the normal equations are formed and the 11 unknowns solved. As the unknowns represent approximate values, the process is repeated in an iterative fashion. Successive triplets were oriented until the strip was completed. Points in the beginning and end of the strip were located in one triplet only. Normal points

Table 4. Standard Deviation of x, y, z-Coordinates for Flat Terrain Using 6-Point Method

	x, y-Adjustment, Conformal			*x, y-Adjustment, Independent*			*Stereocomparator*		
	Mx	My	Mz	Mx	My	Mz	Mx	My	Mz
UU	19	21	22	16	20	22	23	25	26
UC	17	14	21	16	14	21	19	17	24
CC	15	14	20	15	13	20	17	15	22
LC	16	17	20	16	16	20	19	18	21
LL	20	23	21	16	22	21	24	24	23
Av.	17	18	21	16	17	21	21	20	23

	Correlation 70% percent			*Varying Accuracy* $S=5[(25+L^2)/138]^{1/2}$			*Triplet*		
	Mx	My	Mz	Mx	My	Mz	Mx	My	Mz
UU	25	27	28	21	23	25	16	17	18
UC	21	19	26	18	14	23	14	10	18
CC	20	17	24	18	12	21	13	9	17
LC	20	20	25	18	15	22	14	12	17
LL	25	27	26	21	23	24	16	18	17
Av.	22	22	26	19	18	23	14	14	17

were covered by two triplets and points located in three photographs by three triplets.

Flat Terrain, Six-Point Method

In the following, only flat terrain was considered using six points to compute the orientation elements. The results are shown once more in Table 4. It is interesting to note the sharp drop in accuracy for planimetry from the center toward the edge of the strip. The decrease in accuracy was 50 percent for the y-coordinates and 25 percent in x, whereas the accuracy in elevation was almost uniform.

To investigate the flexibility of a second-degree conformal transformation, 10 strips were adjusted with different weights for the ground coordinates. The weight assigned to the planimetric position of the ground control points was in the first case $(1+ay^2)$ and $(2-ay^2)$, respectively, the accuracy in x, y, and z was the same (at least within microns). Apparently a second degree adjustment cannot force a uniform accuracy in strips of 10 models.

The difference in accuracy in y was caused by errors in azimuth κ and transversal tilt ω which could not be eliminated by scaling, because these errors acted with opposite signs in the wing points and cancelled each other. The effect on the y-coordinates was more pronounced than on the z- or x-coordinates. The accuracy of the x-coordinates varied more than the z-coordinates due to the conformal transformation in xy.

In Table 4 the results are shown of strip triangulations, where corrections to x, y, and z were computed independently according to the second-degree polynomial:

$$A_0 + A_1x + A_2y + A_3x^2 + A_4xy + A_5y^2.$$

The coefficients were determined independently for the x and y adjustment contrary to the case where the transformation is conformal in x, y, and with a relationship existing between the coefficients of the x and y transformation. Accordingly, 1, 2, or 3 strip coordinates were determined for the same point. The final strip coordinate was obtained by taking the average.

Although this is a rather arbitrary solution, it is the most obvious one. The strip coordinates were transformed to the ground coordinates with a second degree conformal transformation in xy and independently for z, as had been done before. For flat terrain the conformal transformations in xz and yz resolved automatically in an independent transformation for z. As can be seen from Table 4, the overall accuracy improved by 25 percent. The *variation* in accuracy however was still considerable and no improvement was gained.

As has been stated before, an increase in the accuracy of the elements κ and ω will decrease the variation of the accuracy within the strip. If we consider a triplet, the accuracy of the orientation elements κ, ω, and by improves by 10 percent, but the results are also influenced by the changed correlation factors of the orientation elements. After bridging, the points had 1, 2, or 3 strip coordinates, and the final coordinate was determined by taking the average. The whole process was rather obscure and it is more practical to rely on the results of the simulation.

Conclusions

In interpreting the results, the following facts should be taken into consideration. It has been assumed that the errors in the photo coordinates are normally distributed, the standard deviation of the y-parallaxes being seven microns, a regular pattern of errorless ground control points was available, and second-degree transformation were used to transform the strip coordinates to the ground coordinates. The results, given in microns at photoscale were obtained by analyzing 25 independent strips of 10 models each. When the height differences of valley and side-sloping terrain average 10 percent and 15 percent of the flying height respectively, the accuracy is the same as can be expected for flat terrain. It has also been shown that by using 15 points instead of 6 points for relative orientation, a 25 percent gain in accuracy can be expected in position and 10 percent in height.

The variation in accuracy of points within a strip of 10 models was still considerable. The difference in accuracy for points along the strip axis and along the edges amounts to 50 percent in y and 25 percent in x. Apparently the location of the ground control does not reduce this effect. When the corrections for the x and y strip coordinates are determined independently, the difference in x accuracy is eliminated. The simulated triangulations on a stereocomparator gave accuracies which were 10 percent worse than for a monocomparator. If we assume that the observations of corresponding points were correlated with a correlation factor of 0.7, then a 25 percent drop in accuracy can be expected for strips of 10 models. When the accuracy of the photo coordinates varies in proportion to the square of the distance from the nadir point and the standard deviation is three times as large in the corners than in the nadir point, the overall accuracy was the same. However, the difference in accuracy for points along the strip axis and along the edges increased to 100 percent for y.

In computing the triangulation with the triplet method an increase of 25 percent in overall accuracy can be expected. The use of this method in practice is therefore recommended.

Although some of the results shown are well known and obtained by other means, it shows that this approach is not only very quick but also very effective and has very few limitations.

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The Science and Technology Agency program, under whose auspices the author came to Japan, in addition to its academic objectives, must surely count its contributions to international understanding as an objective, if not the most important goal, for in this scientific age it is all too often a lack of international cooperation and spirit of goodwill rather than a lack of technological knowledge that causes mankind's most bitter hardships.

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