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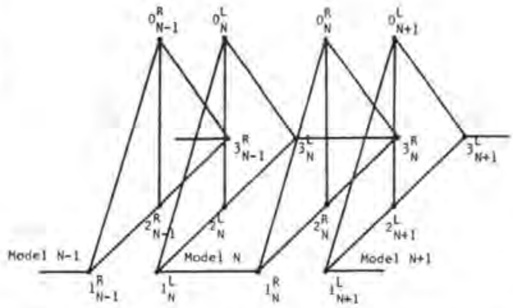

FIG. 1. The layout for the semi-analytical approach. The letters *R* and *L* denote right and left, respectively.

Semi-Analytical Aerotriangulation

Results reveal the merits of this approach both from the economic and accuracy points of view.

(*Abstract on next page*)

INTRODUCTION

ONE APPROPRIATE kind of semi-analytical aerotriangulation involves the independent relative orientation in an analogue type plotter of each model of a strip of photographs, followed by the linking of these models into a strip assembly by mathematical means. The plotter used need not have the Zeiss parallelogram nor the related prisms necessary to change the optical path as required whenever the conventional aeropolygon approach is utilized. Thus, a relatively inexpensive instrument can be used both for the aerotriangulation as well as for the plotting of each individual model. Several methods of linking the models to each other have been previously suggested.[2,4] Most of these methods require the knowledge of the position in space of the camera's perspective centers. This, in turn, enables the superposition of corresponding vertical surfaces ($0_N{}^L, 1_N{}^L, 2_N{}^L, 3_N{}^L$ and $0_{N-1}{}^R$, $1_{N-1}{}^R$, $2_{N-1}{}^R$, $3_{N-1}{}^R$ in Figure 1) which results in the linking of model N to model *N-1*. It is then a matter of successive absolute orientations that finally renders the strip assembly. Once the cantilevered strip is obtained, its transformation and final adjustment may follow exactly the same procedure as that of any analog or purely analytical aerotriangulation technique.

In establishing such a method, most photogrammetrists have concentrated mainly on the mathematical manipulations involved in the absolute orientation process, neglecting somewhat the practical aspects of the execution. For example, Schut's method,[4] widely accepted in the United States, does not provide means of obtaining the coordinates of the perspective centers which are required in the process. The USGS,[1] using this method, has resorted to Inghillerie's method of space resection[2] in order to get the perspective centers' coordinates.

It is the intent of this paper to present a complete semi-analytical aerotriangulation technique based upon the conclusions derived from several extensive fundamental investigations.

NEED FOR FUNDAMENTAL RESEARCH

In order to clarify the concept of *fundamental research* in this context, let us follow the most common operational steps involved in semi-analytical aerotriangulation. A grid plate is first inserted in each of the cameras in the plotter and the coordinates of several grid intersections are recorded. These coordinates serve as input in a space-resection program, resulting in the spatial coordinates of the camera's perspective center. The grid plates are then replaced by the actual photographs, a conventional independent relative orientation is performed using the rotational elements of orientation only (ϕ, ω, κ), and the coordinates of the passpoints ($1_N{}^L$, $2_N{}^L$, $3_N{}^L$, $1_N{}^R$, $2_N{}^R$, $3_N{}^R$ in Figure 1) are then recorded. As the base has not been altered during this process, it is convenient to assume that the coordinates of the perspective centers as determined for the first model do not change, and hence can be used for the second model and indeed for all subsequent models as well. This amounts to a significant saving in operation and computation time, because the next model, replacing the previous one, may be directly oriented and its passpoints recorded without the need to determine the position of its pair of perspective centers. The operational stage terminates when the last individual model in the strip has been set up, relatively oriented and the coordinates of its passpoints recorded. The linking of these individual models into one strip assembly is performed mathematically by rotating and superimposing surface $0_N{}^L$, $1_N{}^L$, $2_N{}^L$, $3_N{}^L$ onto surface $0_{N-1}{}^L$, $1_{N-1}{}^L$, $2_{N-1}{}^L$, $3_{N-1}{}^L$, carrying along the entire model N with its right cross-section $0_N{}^R$, $1_N{}^R$, $2_N{}^R$, $3_N{}^R$. Next, surface $0_{N+1}{}^L$, $1_{N+1}{}^L$, $2_{N+1}{}^L$, $3_{N+1}{}^L$ is rotated to fit the newly obtained position of surface $0_N{}^R$, $1_N{}^R$, $2_N{}^R$, $3_N{}^R$ and this procedure is repeated for all the models in the strip.

The method described may give rise to several fundamental problems. The grid-plate measurement technique not only requires additional and time-consuming work, but it is also questionable whether or not the resulting coordinates of the perspective center (O) can be treated together with the passpoints ($1, 2, 3$) as if both were measured homogenously, sharing the same coordinate system. Obviously, a slight displacement in the centering of the grid plate, compared with the centering of the photograph, will result in two different origins for the coordinate systems. Further, weighting problems arise in the mathematical manipulation of directly measured coordinates (of points $1, 2, 3$) with computed coordinates (of point O).

Questions such as, "What is the optimal density and distribution of the intersections on the grid plates to be measured for space resection?" and "At what datum should the grid plates be measured?" also have their bearing on the economy and the resulting accuracy of the suggested approach.

One should be skeptical with regard to the assumption that once the coordinates of the perspective centers are determined, they can be used for all the models in the strip. If the cameras in the instrument do not rotate precisely about their corresponding perspective centers during the relative orientation phase, or if the instrument settles a certain amount over extended periods of time, this assumption obviously cannot hold. However, even if the coordinates of the perspective centers can mechanically be kept constant, it is sometimes necessary to change the base intentionally, as for example in the case where the Z-range of the instrument is insufficient to accommodate great height differences.

The bridging microscopes on the Kern PG-2 enable a direct and instant *measurement* of the position of the perspective centers in each model; it does not require a space resection such as the one described here and hence it does away with most of the associated operational problems. However, this elegant solution is currently feasible only in the Kern instruments, and the semi-analytical approach is too attractive to be restricted to only one type of instrument. Thus, efforts should be made to investigate the above mentioned problems and to incorporate the conclusions in a universal procedure which can be applied to any instrument.

So far we have raised mainly technical and operational questions; however we are concerned also with the accuracy resulting from the particular method that is being used. For example: Will a mathematical refinement of the relative orientation significantly affect the values of the passpoint coordinates and thus improve the final results? Which mathematical approach for linking one model to the previous one is more efficient and more rigorous? Should the program provide an

ABSTRACT: *A semi-analytical aerotriangulation technique was based on conclusions derived from extensive fundamental investigations. The technique is universal in that it can be performed on any stereoplotter which enables the recording of coordinates. The simple operation is followed by a computation of space resection, orientation refinement and strip assembly, and the results obtained compare favorably with other approaches.*

exact fit at the common projection centers (O_L^N and O_{N-1}^R) and a least-squares fit at the passpoints in the triple overlap, or should all four points be subject to least-squares adjustment? The answer to these and other related questions are instrumental in adopting the semi-analytical approach as the conventional means of aerotriangulation.

The Suggested Universal System

Research along the outlines discussed in the previous section has been conducted at the University of Illinois and the conclusions of the various investigations have been incorporated into a universal method whose description is given below. The operation calls for centering the diapositives on the plate holders that are actually framed grid plates. It is then possible to record the coordinates of the grid intersections of the grid plate and the passpoints on the photographs simultaneously. This rather unique device eliminates the need to replace alternately the photographs with the grid plates, which is time consuming and an inconvenient procedure. It also eliminates the danger of operating with two different coordinate systems, one for the passpoints and one for the perspective center. It enables re-determination of the position of the perspective centers at any desired model in a very quick and efficient way, and it eliminates the need to assume or forcefully maintain a constant base.

It was found that the distribution of the measured grid intersections on the grid plate do affect the accuracy of the resulting perspective centers. The larger the spread of the points, the more accurate the result is. However, it is not necessary for the points to be on the outside edge of the grid plate. For example, if nine points distributed symmetrically about the principal point and 40 mm apart from each other are used for space resection on the Wild A-9, the accuracy of the results will not be significantly higher than if the points are 20 mm apart.

It was further found that the number of grid intersections used for the space resection determination has little effect on the resulting accuracy, provided that this number is not below 5. Practically the same results will be obtained from 25 points and 9 points. One may conclude that wherever the determination of the perspective centers is necessary, 9 points should be used, whereas in checking for changes in the location of a certain perspective center, 5 points are sufficient.

Whether or not the cameras in the instrument are actually rotating about their corresponding perspective centers, thus permitting the assumption that the coordinates of the perspective center remain constant regardless of the orientation, should be investigated for each case separately. In the experiments conducted on the Wild A-9, which was available for this investigation, a decrease in the Z-coordinate of the perspective center seemed to occur as the amount of tilt increased; however this phenomenon was not observed in X and Y. Moreover, the range of differences between the various determinations of the perspective center coordinates at various tilts did not exceed the standard deviation within which any other point in the model was determined. Therefore, in our particular case, it was concluded that unless extreme values of tilt are applied, it may indeed be assumed that the rotation of the cameras is in fact about the perspective centers. In case of extreme values of tilt, new determination of the perspective centers' coordinates, or at least a check on them, is mandatory.

In order to satisfy the requirements and implement the recommendations stated above, a flexible computer program has been designed. This program consists of three main sections; space resection, relative orientation refinement, and model linking. Following the outlines formulated by Inghilleri,[2] the three main sections were extensively modified and arranged as subroutines in a master program whose flow chart is presented in Figure 3. The program is so designed as to offer a high degree of flexibility to the user. For example, it is not necessary that all three subroutines be utilized for each model in the strip. If it is desired that all models utilize the perspective centers as defined in the first model, then the space resection subroutine will be called only once. However, should the the need arise to incorporate other values of perspective centers' coordinates, the subroutine can be called upon to compute the new values. Similarly, if the parallaxes were completely taken care of during the operation, there is no need to utilize the relative orientation refinement subroutine (see also the next section), in which case the linking of the models to each other will take place directly.

The equations involved in each of the three main subroutines have one common origin, namely the well known relationship:

$$\begin{bmatrix} X_m - X^0 \\ Y_m - Y^0 \\ Z_m - Z^0 \end{bmatrix} = \lambda \cdot \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} X_G \\ Y_G \\ P \end{bmatrix}, \quad (1)$$

where:

X_G, Y_G = Given grid plate coordinates
X_m, Y_m, Z_m = corresponding measured model coordinates
X^0, Y^0, Z^0 = coordinates of the perspective center
P = principal distance
λ = scale factor, and
A_{ij} = coefficients of the orthogonal rotation matrix.

The equations utilized for the space resection program are the collinearity condition equations, derived from Equation 1;

$$\frac{X_m - X^0}{Z_m - Z^0} = \frac{A_{11}X_G + A_{12}Y_G + A_{13}P}{A_{31}X_G + A_{32}Y_G + A_{33}P},$$

and (2)

$$\frac{Y_m - Y^0}{Z_m - Z^0} = \frac{A_{21}X_G + A_{22}Y_G + A_{23}P}{A_{31}X_G + A_{32}Y_G + A_{33}P}.$$

Denoting

$$\frac{X_m - X^0}{Z_m - Z^0} = T_x; \qquad \frac{Y_m - Y^0}{Z_m - Z^0} = T_y; \qquad \frac{X_G}{P} = t_x \quad \text{and} \quad \frac{Y_G}{P} = t_y \tag{3}$$

Equations 2 become:

$$T_x = \frac{A_{11}t_x + A_{12}t_y + A_{13}}{A_{31}t_x + A_{32}t_y + A_{33}}; \qquad T_y = \frac{A_{21}t_x + A_{22}t_y + A_{23}}{A_{31}t_x + A_{32}t_y + A_{33}}. \tag{4}$$

Expansion by Taylor series and evaluating at $\phi = \omega = \kappa = 0$, will render the following linearized observation equations:

$$(1 + t_x^2)\Delta\phi + (t_xt_y)\Delta\omega - (t_y)\Delta\kappa - \left(\frac{1}{Z_m - Z^0}\right)\Delta X^0 + \left(\frac{X_m - X^0}{(Z_m - Z^0)^2}\right)\Delta Z^0 + F_1^0 = 0 \tag{5}$$

and

$$(t_xt_y)\Delta\phi + (1 + t_x^2)\Delta\omega + (t_x)\Delta\kappa - \left(\frac{1}{Z_m - Z^0}\right)\Delta y^0 + \left(\frac{Y_m - Y^0}{(Z_m - Z^0)^2}\right)\Delta Z^0 + F_2^0 = 0$$

where the constant vectors:

$$F_1^0 = T_x - t_x, \qquad F_2^0 = T_y - t_y. \tag{6}$$

The unknowns are solved by the least-squares technique and the approximate perspective center coordinates are corrected accordingly:

$$X^0 = \bar{X}^0 + \Delta X^0, \qquad Y^0 = \bar{Y}^0 + \Delta Y^0, \qquad Z^0 = \bar{Z}^0 + \Delta Z^0 \tag{7}$$

Likewise, the condition imposed upon any two corresponding rays for the relative orientation refinement is (see also Equation 4):

$$\frac{b_{21}t_{x_N}^R + b_{22}t_{y_N}^R + b_{23}}{b_{31}t_{x_N}^R + b_{32}t_{y_N}^R + b_{33}} - \frac{A_{21}t_{x_N}^L + A_{22}t_{y_N}^L + A_{23}}{A_{31}t_{x_N}^L + A_{32}t_{y_N}^L + A_{33}} = 0. \tag{8}$$

where A_{ij} are the coefficients of the rotation matrix associated with the left photograph and b_{ij} the coefficients associated with the right photograph of model N. Linearization of Equation 8, in the same way as Equation 4 was treated, yields:

$$t_{x_N}^L t_{y_N}^L \Delta\phi_N^L + t_{x_N}^L \Delta k_N^L - t_{x_N}^R t_{y_N}^R \Delta\phi_N^R - (1 + t_{y_N}^{2R})\Delta\omega_N^R - t_{x_N}^R \Delta\kappa_N^R + F^0 = 0, \tag{9}$$

where:

$$F^0 = t_{y_N}^R - t_{y_N}^L.$$

Equation 1 is also utilized for the model-linking program, except that here X^0, Y^0, Z^0 are the coordinates of the center of gravity associated with points *0, 1, 2, 3* (see Figure 1). Inasmuch as the coordinates are reduced to this center, Equation 1 can be rewritten in the following form:

$$\begin{bmatrix} \bar{X}_{i_{N-1}}^R \\ \bar{Y}_{i_{N-1}}^R \\ \bar{Z}_{i_{N-1}}^R \end{bmatrix} = \lambda \cdot \boldsymbol{m} \cdot \begin{bmatrix} \bar{X}_{i_N}^L \\ \bar{Y}_{i_N}^L \\ \bar{Z}_{i_N}^L \end{bmatrix} \quad (i = 0, 1, 2, 3), \tag{10}$$

where λ is the scale factor between models N and N-*1*, the bars above X, Y, Z indicate reduced coordinates to the center of gravity, and $\boldsymbol{m}$ is the rotation matrix for transferring model N into the system of model N-*1*. Linearizing and evaluating at $\phi = \omega = \kappa = 0$ the equations become:

$$\bar{\bar{Z}}_{i_N}^L \Delta\phi_N - \bar{\bar{Y}}_{i_N}^L \Delta\kappa_N + F_x^0 = 0, \qquad \bar{\bar{Z}}_{i_N}^L \Delta\omega_N + \bar{\bar{X}}_{i_N}^L \Delta\kappa_N + F_y^0 = 0, \quad \text{and} \qquad -\bar{\bar{X}}_{i_N}^L \Delta\phi_N - \bar{\bar{Y}}_{i_N}^L \Delta\omega_N + F_z^0 = 0, \tag{11}$$

where the double bars indicate reduced and scaled coordinates.

Solving for the unknowns, corrections are found to the approximate values of the orientation matrix m. With the aid of this orientation matrix, the entire model is transformed and linked to the previous one.

All three subprograms were designed to share the same basic contruction which, in a general form, is represented in the flow chart of Figure 2, and consists of the following steps:

1. Initialization of the orientation elements, i.e., $\varphi=\omega=\kappa=0$.
2. Construction of the observation equation coefficients' matrix $\mathbf{A}$, the normal matrix $\mathbf{A}^T\mathbf{A}$ and the vector $\mathbf{A}^T\mathbf{L}$. (Equations 5 and 6 for space resection; Equation 11 for the linking of the models).
3. Inversion of the normal matrix and solving for the corrections to the unknowns.
4. Construction of a partial rotation matrix using the results of "SOLVE" ($=\mathbf{M}_p$).
5. Transformation of the coordinates by means of $\mathbf{M}_p$ to assume a new position which will serve as initial position for the next iteration.
6. Updating the total rotation matrix by computing the product of the partial matrices $\mathbf{M}_p$ as obtained in each iteration.
7. Checking whether another iteration is needed.

Once this iterative process is terminated, each sub-program proceeds to accomplish its task. Thus in the space resection program the corrections are added to the approximate values of the perspective centers' coordinates

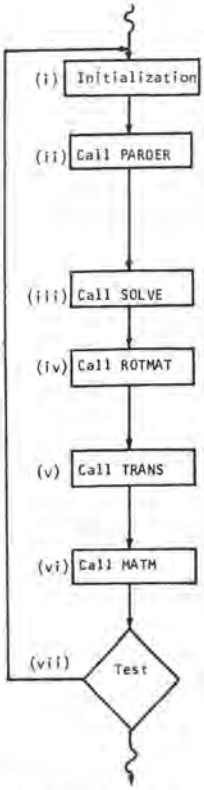

FIG. 2. The basic loop and its subroutines.

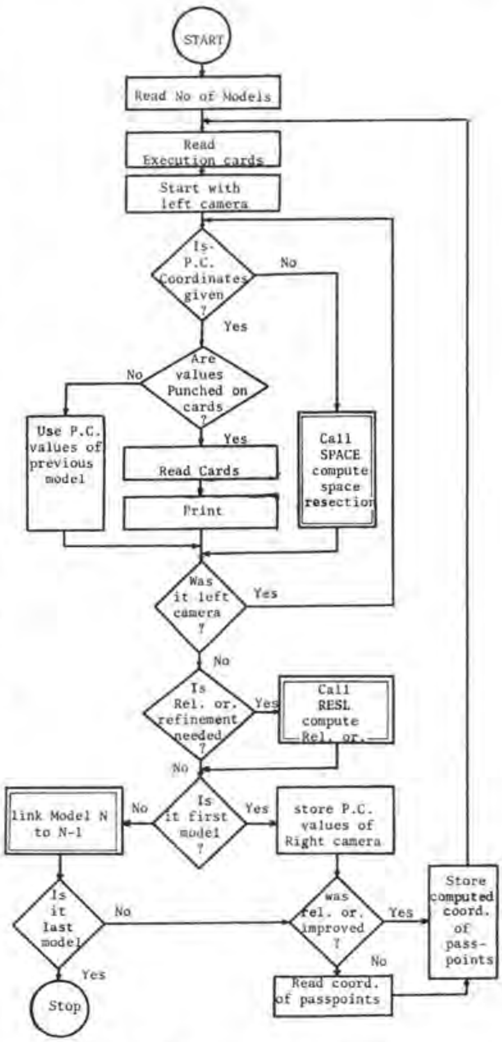

FIG. 3. The general flow chart of the main program.

(Equation 7); similarly in the relative orientation program the improved machine coordinates will be computed, and in the model linking program the transformation of the rest of the points in the model is accomplished, following the transformation of crossection ($_N{}^L$) to crossection ($_{N-1}{}^R$), (Equation 10). The basic loop (Figure 2) and its subroutines are shared by all three sub-programs which together constitute the main program as outlined in Figure 3.

A Note on the Relative Orientation Refinement

It has previously been determined[2] that relative orientation refinement and con-

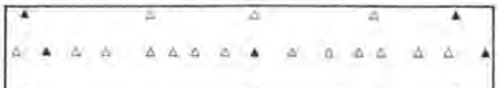

FIG. 4. The nine-model strip. Triangles denote ground control points, and black triangles denote those control points that were used in the adjustment.

sequently the refinement of the measured coordinates indeed improves the results of the semi-analytical aerotriangulation. Another advantage is that an operator with minimal training can be employed and that the time and effort required to clean entirely the parallaxes from the model can be shortened considerably.

However, the need to measure the parallax as input for the relative orientation refinement is a drawback in the system. The parallax is measured by means of the element *by*, if the instrument possesses such an element, and with the element ω if it does not. Element *by* is the most convenient and precise means of parallax measurements, yet its use disturbs the position of the perspective center which, in the case of semi-analytical aerotriangulation, is generally considered as fixed. Although *by* can be brought back to its original position after the measurements of the parallax, it is doubtful whether this can be achieved with sufficient accuracy. This in turn might necessitate the re-determination of the perspective center position for each photograph in the strip. Moreover, one of the main advantages of the semi-analytical approach is that even second-order instruments, having only the rotational orientation elements, can be used for the triangulation. In this case the element *by* does not exist at all.

On the other hand, use of ω for parallax measurement is less convenient, especially along the flight axis, and less accurate, mainly because it involves an X-component in eliminating the Y-parallax. Moreover, use of ω as a parallax measuring tool requires a different and more elaborate set of equations for the relative orientation refinement than the one presented here. This problem reveals a gap which sometimes exists between theoretical and practical considerations and demonstrates the need for further fundamental research along these lines.

Evaluation of the Approach

Several experiments have already been conducted to investigate the merits of the semi-analytical approach with respect to other known approaches, yet very few results have been published so far, and it is safe to state that many more experiments are desired in order to arrive statistically at sound conclusions.

Having this in mind, we have conducted several experiments, comparing the semi-analytical method with other methods, and, more specifically, our suggested approach with that of others. Although these experiments are not enough to permit drawing definite conclusions, they nevertheless contribute to the common pool from which such conclusions may be drawn in the future.

One nine-model strip was used for all the experiments (see Figure 4). The same points were observed in each of these experiments by the same operator. The semi-analytical procedure was executed on the Wild A-7, and so was the aeropolygon. The purely analytical aerotriangulation was performed on the Wild STK-1. Once the strip coordinates were obtained, the same adjustment program, based on the same ground control points, was applied to each experiment. The resulting coordinates were compared with the given ground coordinates and the estimate of the various accuracies are presented in Table 1.

These results indicate that although the conventional aeropolygon method yields slightly better results than those obtained by the semi-analytical approach, it is nevertheless questionable whether this slight raise in accuracy justifies the use of expensive first-order instruments in most practical applications. The proximity of the results obtained by the three different approaches indicates that whereas there is not much loss in accuracy when the semi-analytical approach is used, there is definitely a gain from the point of view of ease in operation and handling of the data, and from the point of view of economy in time. Above all, it may encourage users of conventional analog procedures to use a progressive method without the need to transfer completely to purely analytical techniques.

TABLE 1. COMPARISON OF THE RESULTS FROM THREE DIFFERENT APPROACHES

	Average errors			Standard errors		
	x	y	z	x	y	z
Aeropolygon	0.2	0.2	0.2	0.2	0.2	0.3
Purely Analytical	0.1	0.1	0.2	0.2	0.2	0.3
Semi-analytical	0.1	0.2	0.3	0.2	0.3	0.5

In an attempt to evaluate further the suggested method, the resulting unadjusted strip coordinates were compared with those obtained by Schut's method. It was found that the differences in planimetry between the corresponding values increased systematically with the length of the strip. This can easily be taken care of, as it actually was, in the polynomial adjustment which followed. However, the differences in Z between corresponding values in the two groups of data showed a different pattern; although the differences were constant and close to zero along the flight (x) axis, the differences at the upper wing points were significant and equal in magnitude but opposite in sign to the differences at the lower wing points, as the effect upon elevation of an ω rotation about the x-axis. This may indicate a less effective fit across the linked models (i.e. in y-direction) on the part of one of the methods. A second-degree polynomial adjustment was then applied to both groups of data. An examination of the results shows that whereas the suggested method yields standard errors which are only slightly better than those resulting from Schut's data [see Table 2], the residual Z-errors are far better distributed along and across the strip than the results obtained from Schut's data which yields bigger residual errors in the wing points than along the flight axis. These experiments, although by no means conclusive, do indicate a trend and emphasize the potential of the approach.

TABLE 2. COMPARISON OF THE RESULTS FROM THE SUGGESTED METHOD WITH THAT OF MR. SCHUT

	Average errors			Standard errors		
	x	y	z	x	y	z
Suggested method	0.1	0.2	0.3	0.2	0.3	0.5
Shut's method	0.1	0.3	0.4	0.2	0.3	0.6

SUMMARY

The research on semi-analytical aerotriangulation presented here is by no means conclusive. Only a few experiments have been conducted and all phases of the approach have not been thoroughly investigated. However, the results do reveal the merits of this approach both from the economical and accuracy points of view. Economically it enables the performance of aerotriangulation with instruments already available. "The imbalance between plotters and triangulation instruments that can arise at a time can be easily modified since each plotter can be used as a triangulator".[7] From the accuracy point of view, "Data derived from a good plotter is likely to give results only slightly less accurate than those of fully analytical methods".[1]

We have purposely quoted here from articles written by heads of large operational institutions who naturally are concerned with practical approaches to photogrammetry. This enthusiasm, which has also been voiced by others, should encourage researchers to develop and investigate this topic further. The suggested universal system presented here is a contribution to this effort.

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