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Trilaterated Photo Coordinates

Image coordinates of 5-micron standard deviation have been obtained with a glass scale and with the aid of a small computer.

INTRODUCTION

THE ADVENT OF THE electronic computer has made possible many new and useful applications in photogrammetry which would not have otherwise been feasible. One such application is the unusual method of obtaining trilaterated photo coordinates presented herein. This method, which can be performed using simple and inexpensive apparatus, deviates from the conventional approach of measuring photo coordinates with respect to reference comparator axes or a reference fiducial axis system scratched on the emulsion. Instead the method, which has been alluded to by Brown,[1] involves measurement of the distances from each of the fiducial marks to any point *P* whose photo coordinates are to be determined. Distance measurements from two fiducial marks, whose coordinates have been determined by least squares quadrilateral trilateration, establish the coordinates of *P*. Four or eight fiducial mark formats provide for two or six redundant measurements respectively. By taking advantage of this redundancy, the most probable values of X_p and Y_p are obtained through a least squares trilateration solution.

ABSTRACT: *A method is presented whereby photo coordinates may be obtained using inexpensive apparatus in conjunction with trilateration computational techniques and an electronic computer. The method takes advantage of redundant measurements and consequently it makes possible the calculation of the most probable values of the photo coordinates as well as their estimated standard deviations. The method has been tested using a simple glass scale and the results indicate average estimated standard deviations in* x *and* y *of approximately five microns. These results are not as accurate as those possible with a precise comparator. The method does, however, provide unusually good results in relation to cost and it also provides several significant advantages over other lower order procedures.*

This trilateration method produces the following distinct advantages: (a) the most probable X and Y coordinates of the point P are obtained, i.e., those which minimize the sum of the squares of the measurement residuals; (b) the estimated standard deviations of X_p and Y_p are obtained which provides a basis for rejection of sub-standard measurements; (c) the covariance matrix is obtained as a by-product which may be useful for weighting photo coordinates in subsequent analytical calculations, (d) fiducial lines scratched on the emulsion, which deface the imagery, are not needed; and (e) inexpensive equipment is utilized.

THEORETICAL DEVELOPMENT

The following three basic steps are involved in obtaining photo coordinates with this trilateration method: (1) the least squares solution for the most probable coordinates of the fiducial marks in an arbitrary rectangular system; (2) the least squares solution for the most probable photo coordinates X_p and Y_p in the same arbitrary rectangular system as that of the fiducial coordinates; and (3) a transformation which shifts the origin and orientation of the coordinate axes to the conventional position and corrects for shrinkage and expansion. In the theoretical development herein, the four corner fiducial format is depicted although the method is equally adaptable to the side fiducial format.

STEP ONE

On Figure 1, points A, B, C and D are the imaged positions of the fiducials. The six distances AB, BC, CD, DA, AC and BD are observed. Note that these are analogous to measurements taken in ordinary quadrilateral trilateration. An arbitrary rectangular coordinate system is initially assigned which places the origin at A, with the X-axis positive to the right and passing through B. With this arbitrary choice, $X_a = Y_a = Y_b = 0$. The remaining five unknown fiducial coordinates can be calculated by using any five of the observed distances. The sixth observation therefore is redundant. Using all six observations in an ordinary observation equation least-squares trilateration adjustment,[2] the most probable values for the five unknown fiducial coordinates can be computed in the aforementioned arbitrary rectangular coordinate system.

The observation equations may be written, one for each observed distance, by using the following prototype equation:

$$L_{ij} + V_{ij} = ((X_j - X_i)^2 + (Y_j - Y_i)^2)^{1/2}. \quad (1)$$

In Equation 1, L_{ij} is the observed length of the line I–J, V_{ij} is the residual error in the observation, and X_i, Y_i, X_j and Y_j are the most probable coordinates of the points I and J. This non-linear equation is not used in this form but rather it is linearized by using a Taylor series expansion and dropping as negligible all terms of order two or higher. Evaluating the partial derivatives of the function and substituting them into the Taylor series, there results the following prototype of the linearized observation equation:

$$\begin{aligned} K_{ij} + V_{ij} &= \left[\frac{Xi_0 - Xj_0}{(IJ_0)}\right](dXi) + \left[\frac{Yi_0 - Yj_0}{(IJ_0)}\right](dYi) \\ &\quad + \left[\frac{Xj_0 - Xi_0}{(IJ_0)}\right](dXj) + \left[\frac{Yj_0 - Yi_0}{(IJ_0)}\right](dYj). \end{aligned} \quad (2)$$

In Equation 2, X_{i_0}, Y_{i_0}, X_{j_0} and Y_{j_0} are initial approximations of the unknown coordinates X_i, Y_i, X_j and Y_j; and dX_i, dY_i, dX_j and dY_j are corrections to be applied to the initial approximations such that:

$$X_i = X_{i_0} + dX_i$$
$$Y_i = Y_{i_0} + dY_i$$
$$X_j = X_{j_0} + dX_j$$
$$Y_j = Y_{j_0} + dY_j$$

Also:

$$K_{ij} = L_{ij} - IJ_0$$
$$IJ_0 = ((X_{j_0} - X_{i_0})^2 + (Y_{j_0} - Y_{i_0})^2)^{1/2}$$

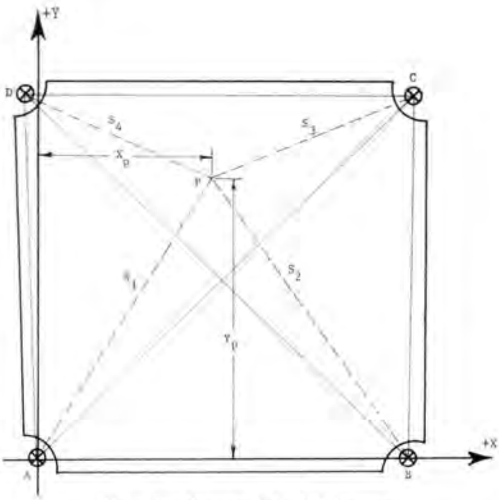

FIG. 1. Corner fiducial format.

The only unknowns in Equation 2 are the residual and the corrections to the coordinates. The initial approximations of the X and Y coordinates of the fiducials may be calculated by simple intersection from five selected observations. The Taylor series method of solution is iterative but the initial approximations, so calculated, will be sufficiently close to the most probable values so that only one iteration of the computations will be required.

Prototype Equation 2 is used to formulate the six observation equations for calculating the most probable fiducial coordinates. For illustrative purposes, the following equation for observation BC is presented:

$$\begin{aligned} K_{bc} + V_{bc} &= \left[\frac{X_{b_0} - X_{c_0}}{(BC_0)}\right] dX_b + \left[\frac{X_{c_0} - X_{b_0}}{(BC_0)}\right] dX_c \\ &\quad + \left[\frac{Y_{c_0} - Y_{b_0}}{(BC_0)}\right] dY_c. \end{aligned} \quad (3)$$

Note that because Y_b is fixed, there is no dY_b term. With the six observation equations formulated, the most probable values of the unknown dX's and dY's are calculated using the following equal-weight least-squares equation:

$$\underset{n\ 1}{X} = (\underset{n\ m}{A^T}\ \underset{m\ n}{A})^{-1}\ \underset{n\ m}{A^T}\ \underset{m\ 1}{K} \quad (4)$$

where

- m is the number of equations and n is the number of unknowns
- A is the matrix of coefficients of the unknown dX's and dY's
- X is the matrix of unknowns (dX's and dY's)
- K is the matrix of constant terms (K_{ij}).

Upon calculating the unknowns, they are applied to the initial approximations to obtain most probable values for the fiducial coordinates. Also the estimated standard deviations of the most probable fiducial coordinates may be calculated, and it is noteworthy that the accuracy of the fiducial coordinates is important since the subsequent calculation of photo coordinates depends on them.

STEP TWO

Referring again to Figure 1, P is a point whose coordinates X_p and Y_p are to be determined. The distances from the fiducial marks to the point P of S_1, S_2, S_3 and S_4 are observed. With the coordinates of the fiducials known from the calculations of Step One, X_p and Y_p may be calculated explicitly by using only the S_1 and S_2 measurements. The following equations apply:

$$\left.\begin{aligned} X_p &= \frac{X_a + X_b}{2} + \left[\frac{S_1^2 - S_2^2}{2AB^2}\right](X_b - X_a) \\ Y_p &= \frac{2Z}{AB^2}(X_b - X_a) \end{aligned}\right\} \quad (5)$$

where

$$S = \tfrac{1}{2}(S_1 + S_2 + AB)$$
$$Z = (S(S - S_1)(S - S_2)(S - AB))^{1/2}.$$

As only two observations explicitly provide X_p and Y_p, two redundant observations remain and consequently the most probable values of X_p and Y_p may be calculated through a least-squares solution. Equations 5 provide X_{p_0} and Y_{p_0} values to be used as initial approximations in the solution. Four observation equations are formulated by again using prototype Equation 2. For illustrative purposes, the observation equation for S_1 is:

$$K_{ap} + V_{ap} = \left[\frac{X_{p_0} - X_a}{(AP_0)}\right]dX_p + \left[\frac{Y_{p_0} - Y_a}{(AP_0)}\right]dY_p. \quad (6)$$

Note that in these observation equations, the fiducial coordinates are held fixed to those calculated in Step One. The only unknowns therefore are the residuals and dX_p and dY_p. The unknowns dX_p and dY_p are computed using Equation 4, followed by the calculations of the most probable coordinates X_p and Y_p and their estimated standard deviations.

With a small computer Steps One and Two are performed separately as just described. If a larger computer is available, however, statistically better results are obtained by combining the two steps, whereupon the fiducial coordinates and all photo coordinates are obtained simultaneously in a single adjustment.

STEP THREE

In this study the transformation presented by Keller and Tewinkel[3] was used to shift the origin of coordinates from fiducial A to the principal point, to rotate the arbitrary coordinate axes into the fiducial axis system, and to correct for any shrinkage and expansion. The following two equations apply:

$$\begin{aligned} X' &= X + a_1 + b_1X + c_1Y + d_1XY \\ Y' &= Y + a_2 + b_2X + c_2Y + d_2XY \end{aligned} \quad (7)$$

where:

X' and Y' are calibrated fiducial coordinates,

X and Y are computed fiducial coordinates from Step One, the a's, b's, c's and d's are coefficients to be determined.

A pair of these equations are written for each fiducial mark, yielding four X' equations and four Y' equations. These equations are solved for the coefficients which are subsequently used to calculate corrected and transformed X_p' and Y_p' coordinates.

TEST RESULTS

This method of obtaining trilaterated photo coordinates has been tested by using a glass

TABLE I. CALCULATION OF FIDUCIAL COORDINATES IN ARBITRARY SYSTEM

Line	Measured Length (mm.)	Fiducial Mark	Arbitrary Coordinates (mm.)		Estimated Standard Deviations (mm.)	
			X	Y	X	Y
AB	159.850	A	0.000	0.000	fixed	fixed
BC	159.815	B	159.848	0.000	.005	fixed
CD	159.880	C	159.966	159.813	.008	.005
DA	159.880	D	0.088	159.878	.008	.005
AC	226.015					
BD	226.115					

TABLE II. TRILATERATED PHOTO COORDINATES

Point	Measured Distances (mm.)				Final Transformed Coordinates (mm.)		Estimated Standard Deviations (mm.)		Remarks
	S_1	S_2	S_3	S_4	X	Y	X	Y	
1	189.89	100.47	59.36	171.65	71.892	−42.818	.004	.003	PUG Mark
2	95.25	81.99	141.70	149.72	−24.336	−34.699	.006	.007	Panelled Point
3	57.71	105.49	177.96	154.47	−62.644	−28.149	.004	.005	Panelled Point
4	82.62	103.48	145.48	131.40	−31.698	−14.491	.007	.007	Panelled Point
5	114.93	97.34	113.33	128.68	0.799	−15.652	.004	.004	PUG Mark
6	76.59	121.87	149.62	115.70	−36.521	3.257	.008	.008	Panelled Point
7	119.71	114.23	106.41	112.17	6.641	1.049	.004	.004	PUG Mark
8	156.08	134.70	72.30	106.85	42.283	14.904	.005	.005	Panelled Point
9	96.44	157.08	145.40	75.82	−26.173	41.868	.007	.006	Panelled Point

scale and found to consistently yield very favorable results. Tables I and II list actual sample results for one of the tests where Steps One and Two were performed separately. This test was performed on a diapositive having a side fiducial format. Figure 2 illustrates the arbitrary axis system for side fiducials. The final transformation step rotates this arbitrary system into correct orientation.

Table I lists the data of Step One, that of calculating the fiducial coordinates. In Table II data of Steps Two and Three are tabulated, those of calculating and transforming the photo coordinates. As is seen from the tables, the average estimated standard deviations of the X and Y fiducial coordinates was about six microns and the average estimated standard deviations in X_p and Y_p was about five microns.

The glass scale used to test the method was purchased for slightly less than $200. It is 320 millimeters long and precisely graduated to the tenth of a millimeter. The scale is viewed under magnification by means of movable lenses, one for each eye. This permits readings to be accurately estimated to the nearest hundredth of a millimeter. Glass scales of this type are currently available from at least one European and one American manufacturer.

The measurements were taken with the glass plate lying on a light table. This procedure enhances the fidelity of the imagery and enables more reliable measurements. The measured lengths listed in Table I were taken as the mean of two observations; those of Table II were each observed once. The total measuring time for the number of points presented in the sample test was approximately a half hour. Computer time required was very minimal.

The data of Table I and II was also used in an adjustment which simultaneously performed steps one and two. For this run the average estimated standard deviation in X_p and Y_p was just 5 microns. To further verify the accuracy of the method, a second run for the same points of Table II was made with a second set of independent measurements. The second run yielded photo coordinates which agreed within an average of 5 microns with those of Table II. It is to be noted that for this utmost accuracy the image of points must be clear and discrete; panelled points, PUG or snap marks are excellent.

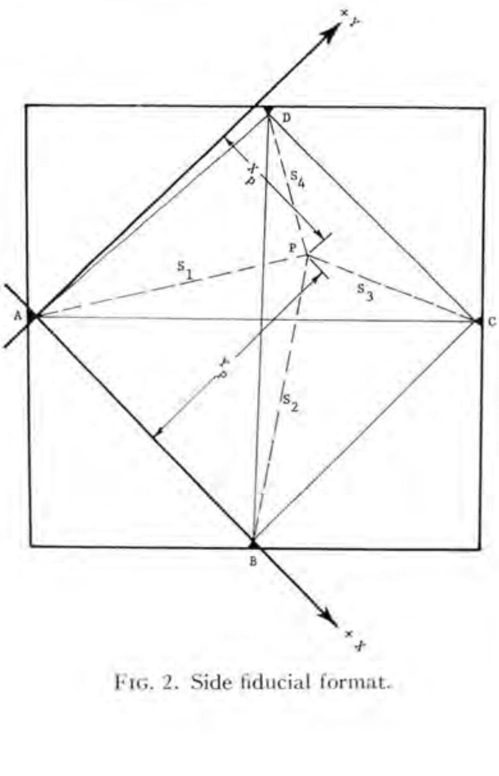

FIG. 2. Side fiducial format.

SUMMARY

A method of obtaining photo coordinates through least squares trilateration has been presented. The method is only feasible when used in conjunction with an electronic computer, but otherwise requires only simple and inexpensive apparatus. The method has been tested using a simple glass scale and has yielded average estimated standard deviations in the X and Y photo coordinates of about five microns. The results of this method do not equal those obtainable with precise comparators, but the method does yield unusually good results in relation to cost if an electronic computer is available. Some significant advantages are held by this method over other conventional low order procedures. It is expected to be of interest to organizations whose accuracy standards can be met with this approach and/or whose budgets may not permit purchase of a precise comparator.

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3. Keller, M. and Tewinkel, G. C., "Aerotriangulation: Image Coordinate Refinement," *Technical Bulletin* No. 25, U. S. Dept. of Commerce, Coast and Geodetic Survey, U. S. Government Printing Office, Washington, D. C., 1965.

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