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Triangulation with Independent Models

Models can be formed either numerically or with an analog instrument. They are then joined together to form a strip which is adjusted to fit control by means of polynomials.

ABSTRACT: *Strip triangulation is regarded as a connection procedure between consecutive overlapping models. The models in turn can be formed either numerically or with an analog instrument. A method for forming a model is demonstrated and the connection routine is described. Finally the triangulation is demonstrated by a computed example. The results show that the final coordinates obtained from comparator measurements have a high standard of accuracy and are undoubtedly superior to those derived from a Wild A8. Nevertheless the results obtained from the A8 strip are sufficiently accurate for mapping purposes, and an instrument like the A8 can be used to advantage for triangulation procedures.*

INTRODUCTION

THIS TRIANGULATION procedure uses independent models for creating a strip. The method accepts models derived from stereoinstruments, such as the Wild A8, as well as models obtained numerically, where the coordinates of the image points are measured with the aid of mono- or stereocomparators.

A strip is said to be formed where all models constituting it are uniformly scaled, identically oriented, and are referred to a common reference frame. All these requirements can be met by performing a spatial linear transformation of coordinates which transforms each consecutive model to the preceding one. The essential transformation elements are derived from coordinates of points lying in the region common to both models. The strip coordinate system is chosen arbitrarily, its origin is usually located in the projection centre of the first photograph.

The triangulation process consists of two major routines:

1. Formation of a model.
2. Connection of two consecutive models to each other.

FORMATION OF A MODEL, NUMERICAL SOLUTION.

Two photographs with an overlap area are regarded to be oriented and to form a model if each pair of corresponding rays intersects if the photographs are projected into a common space. Because of physical restrictions imposed by a variety of factors, the intersection of all pairs of rays never occurs. Thus, from a geometric point of view, the perfect model can never be formed. Hence, we must always accept some approximate model and accomplish an orientation which is, in a certain sense, the best one.

The problem of forming a model is named *relative orientation*. This in turn is performed with or without linear orientation elements. Orientation elements always depend on the coordinate system to which the model is referred. It is always possible to choose a reference frame in which the linear elements, the base components, are equal to zero, and

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solve an orientation which depends on angular elements only.

The formation of a model is done iteratively. Stopping the process is regulated by a criterion which has to indicate that the elements found assume their final values and cannot be improved by further iteration steps. The criterion applied here is the mean square value of the discrepancies which come forth from the intersection condition. After each iteration step this mean is computed and compared with that obtained from the preceding iteration step. As long as this value decreases significantly the process is reiterated. Besides, the criterion is checked against a maximum tolerable value which varies according to the photo-material used.

The solution of the orientation is based upon the coplanarity condition of corresponding rays. This condition is expressed by the known formula:

$$\frac{(X_2')}{(X_3')} = \frac{(X_2'')}{(X_3'')} \cdot \quad (1.1)$$

The coordinates (X_i) are defined according to Figure 1. The notation introduced is convenient for programming purposes and is used throughout the paper. All coordinates and orientation elements are written as subscripted variables. The quantities referred to the left hand photograph are denoted by a prime index, those referred to the right hand photograph by a double-prime index.

The quantities in the brackets are coordinates transformed from the rotated phos tographs to two parallel reference systemwith a common (X_1)-axis which coincides with the projection base. Both of the origins are located in the respective projection centers. The transformation uses a sequence of rotations similar to that used in the Wild Autographs and its matrix consists of the following elements:

$$\begin{aligned}
a_{11} &= \cos\alpha_2 \cos\alpha_1\\
a_{12} &= -\cos\alpha_2 \sin\alpha_1\\
a_{13} &= -\sin\alpha_2\\[6pt]
a_{21} &= \sin\alpha_3 \sin\alpha_2 \cos\alpha_1 + \cos\alpha_3 \sin\alpha_1\\
a_{22} &= -\sin\alpha_3 \sin\alpha_2 \sin\alpha_1 + \cos\alpha_3 \cos\alpha_1 \qquad (1.2)\\
a_{23} &= \sin\alpha_3 \cos\alpha_2\\[6pt]
a_{31} &= \cos\alpha_3 \sin\alpha_2 \cos\alpha_1 - \sin\alpha_3 \sin\alpha_1\\
a_{32} &= -\cos\alpha_3 \sin\alpha_2 \sin\alpha_1 - \sin\alpha_3 \cos\alpha_1\\
a_{33} &= \cos\alpha_3 \cos\alpha_1.
\end{aligned}$$

α_1 denotes the κ-rotation, α_2 denotes the ϕ-rotation and α_3 the ω-rotation.

The coordinates to be inserted in the coplanarity condition (1.1) are found from the transformations:

$$\begin{aligned}
(X_i') &= A_{ij}'X_j'\\
(X_i'') &= A_{ij}''X_j''.
\end{aligned} \quad (1.3)$$

A_{ij} are the matrixes defined by (1.2) and X_j', X_j'' are the measured image coordinates. For the left hand photograph the matrix A_{ij} is obtained from Equations 1.2 by assuming $\alpha_3' = 0$.

Expressions 1.3 can be expanded in a series, linear with respect to the orientation angles:

$$\begin{aligned}
(X_i') &= \overline{A}_{ij}'X_j' + dA_{ij}'X_j'\\
&= (\overline{X}_i') + dA_{ij}'X_j'\\
(X_i'') &= \overline{A}_{ij}''X_j'' + dA_{ij}''X_j''\\
&= (\overline{X}_i'') + dA_{ij}''X_j''.
\end{aligned} \quad (1.4)$$

Substituting Expressions 1.4 into the Condition 1.1 results in a system of linear equations:

$$P_i = b_{ij}x_j \qquad (i = 1, n \quad j = 1, 5). \quad (1.5)$$

P_i is the discrepancy in the intersection condition, b_{ij} is a coefficient matrix with dimensions $n \times 5$, and x_j is the vector of the unknowns to be solved. The coefficients b_{ij} are functions of the measured coordinates, approximately transformed coordinates $(\overline{X}_i')$, $(\overline{X}_i'')$, and the initial angular orientation elements around which the matrixes A_{ij}' and A_{ij}'' are expanded in series:

$$\begin{aligned}
b_{i1} &= V' + \frac{(\overline{X}_2')(\overline{X}_2'')}{(\overline{X}_3'')} \sin\alpha_2'\\
b_{i2} &= -\frac{(\overline{X}_2'')(\overline{X}_1')}{(\overline{X}_3'')}\\
b_{i3} &= -\frac{(\overline{X}_2')}{(\overline{X}_3'')}(V'' \sin\alpha_1'' + W'' \cos\alpha_3'' \sin\alpha_2'')\\
&\quad - CV'' \cos\alpha_3'' \qquad (1.6)\\
b_{i4} &= \frac{(\overline{X}_2')(\overline{X}_1'')}{(\overline{X}_3'')} \cos\alpha_3'' - C(\overline{X}_1'') \sin\alpha_3''\\
b_{i5} &= -\frac{(\overline{X}_2')(\overline{X}_2'')}{\overline{X}_3''} - (\overline{X}_3').
\end{aligned}$$

The auxiliary quantities V', V'', W'' and C are defined by the expressions:

$$\begin{aligned}
V' &= X_1' \cos\alpha_1' - X_2' \sin\alpha_1'\\
V'' &= X_1'' \cos\alpha_1'' - X_2'' \sin\alpha_1''\\
W'' &= X_1'' \sin\alpha_1'' + X_2'' \cos\alpha_1'' \qquad (1.7)\\
C &= \frac{(\overline{X}_3')}{(X_3'')} \cdot
\end{aligned}$$

All coordinates on the right hand side of Expressions 1.6 and 1.7 refer to an orientation point i, and should have been subscripted by the index i. In order to simplify the notation this index has been omitted.

The discrepancies P_i are defined by:

$$P_i = (\overline{X}_2')_i - C(\overline{X}_2'')_i. \qquad (1.8)$$

The evaluation of the coefficients b_{ij} and discrepancies P_i requires a set of values for the orientation elements. The first iteration assumes all elements to be equal to zero. Each following step uses the elements obtained from the preceding one.

Equations 1.5 provide a normal equation system from which corrections to the assumed orientation are solved. These added to the previous angles establish a new set of orientation elements and a new iteration step is taken if the criterion so requires. It should be noted that the transformation matrixes formed by a set of orientation elements are always applied to the originally measured image coordinates X_i', X_i''.

The orientation elements found from the final iteration are used now to compute model coordinates. First the measured coordinates are transformed by the Matrix 1.2, thereafter model coordinates are computed from the formulas:

$$\begin{aligned} XM_3 &= \frac{(X_3')(X_3'')}{(X_1')(X_3'') - (X_1'')(X_3')}\, B \\ XM_1 &= \frac{(X_1')}{(X_3')}\, XM_3 \\ XM_2 &= 0.5\left(\frac{(X_2')}{(X_3')} + \frac{(X_2'')}{(X_3'')}\right) XM_3 \end{aligned} \qquad (1.9)$$

As the discrepancies due to the intersection condition are unavoidable, the XM_2 coordinate is determined as a mean of two values, one resulting from the left hand photograph and the other from the right hand photograph. The base B is chosen arbitrarily. It can assume a constant value for all models.

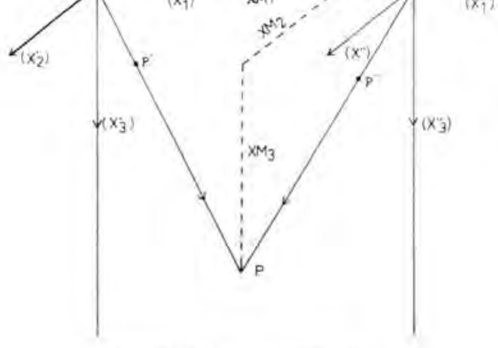

FIG. 1. The definition of the model coordinate system.

CONNECTION PROCEDURE AND FORMATION OF A STRIP

The transformation procedure by means of which two consecutive models are connected to each other is based on points common to both models. In what follows, the model to be connected is denoted the new model, and that to which it is connected is called the preceding model. Each common point has two sets of coordinates, one referred to the system of the new model and the other related to the strip system. The origin of the new model is located in the left projection center, the projection center of the common photograph. To facilitate the solution of the transformation sought, the strip coordinates of the transfer points have to be translated to the same origin. For this reason the strip coordinates of the above mentioned projection center have to be known. In the first model these coordinates are equal to B, 0, 0. The strip coordinates of other projection centers are determined during the connection procedure.

The transformation by means of which the new model is transformed can be written as:

$$XG_i = XT_i + SA_{ij}XM_j \quad (i = 1,3 \;\; j = 1,3) \qquad (2.1)$$

where XG_i are the temporarily translated coordinates of the preceding model, XM_j denote the coordinates of the new model, XT_i is an additional translation and S is a scale factor. The system of equations resulting from Expression 2.1 is not linear with respect to the unknown elements of A_{ij}. Supposing that an approximate solution exists for the unknowns, Expression 2.1 can be linearized as follows:

$$\begin{aligned} XG_i = \overline{S}\overline{A}_{ij}XM_j + XT_i + dS\overline{A}_{ij}XM_j \\ + \overline{S}d\overline{A}_{ij}XM_j. \end{aligned} \qquad (2.2)$$

All quantities resulting from the approximate solution are denoted with a bar. The sequence of rotations applied here is identical with that used in the orientation process, hence the matrix A_{ij} in Equations 2.1 and 2.2 is identical with the matrix given by Equations 1.2. Indroducing an auxiliary vector $XA_i = \overline{A}_{ij}XM_j$ into Formula 2.2 yields:

$$XG_i - \overline{S}XA_i = XT_i + dSXA_i + \overline{S}d\overline{A}_{ij}XM_j. \qquad (2.3)$$

According to Equation 2.3 each common point provides three equations:

$$\begin{aligned} XG_1 - \overline{S}XA_1 &= XT_1 + XA_1dS \\ &\quad + c_{11}y_1 + c_{12}y_2 + c_{13}y_3 \\ XG_2 - \overline{S}XA_2 &= XT_2 + XA_2dS \\ &\quad + c_{21}y_1 + c_{22}y_2 + c_{23}y_3 \\ XG_3 - \overline{S}XA_3 &= XT_3 + XA_3dS \\ &\quad + c_{31}y_1 + c_{32}y_2 + c_{33}y_3. \end{aligned} \qquad (2.4)$$

The coefficients c_{ij} are obtained from the new model coordinates and approximate connection elements:

$$
\begin{aligned}
c_{11} &= -\bar{S}W \cos\bar{\beta}_2 \\
c_{12} &= -\bar{S}(V \sin\bar{\beta}_2 + XM_3 \cos\bar{\beta}_2) \\
c_{13} &= 0 \\
c_{21} &= \bar{S}(V \cos\bar{\beta}_3 - W \sin\beta_3 \sin\bar{\beta}_2) \\
c_{22} &= \bar{S}XA_1 \sin\bar{\beta}_3 \\
c_{23} &= \bar{S}XA_2 \\
c_{31} &= -\bar{S}(V \sin\bar{\beta}_3 + W \cos\bar{\beta}_3 \sin\bar{\beta}_2) \\
c_{32} &= \bar{S}XA_1 \cos\bar{\beta}_3 \\
c_{33} &= -\bar{S}XA_2.
\end{aligned}
\tag{2.5}
$$

V and W are auxiliary quantities defined by:

$$
\begin{aligned}
V &= XM_1 \cos\bar{\beta}_1 - XM_2 \sin\bar{\beta}_1 \\
W &= XM_1 \sin\bar{\beta}_1 + XM_2 \cos\bar{\beta}_1.
\end{aligned}
$$

The angles β_i denote the three rotations of the model: $\beta_1 \equiv \kappa$, $\beta_2 \equiv \phi$, $\beta_3 \equiv \Omega$, and y_i the anglular corrections to be solved for. Executing the iteration process for solving the transformation unknowns is similar to performing relative orientation and need not be described again.

With the aid of the solved unknowns, the coordinates of the new model are transformed to the strip system. The transformation includes also the vector that represents the base of the new model. Its components in the new model are: *B, 0, 0* and its transformed components are given by:

$$
\begin{aligned}
XO_1 &= SB \cos\beta_1 \cos\beta_2 + XT_1 \\
XO_2 &= SB(\cos\beta_1 \sin\beta_2 \sin\beta_3 - \sin\beta_1 \cos\beta_3) \\
&\quad + XT_2 \\
XO_3 &= SB(\cos\beta_1 \sin\beta_2 \cos\beta_3 - \sin\beta_1 \sin\beta_3) \\
&\quad + XT_3.
\end{aligned}
\tag{2.6}
$$

Adding the values XO_i to the known strip coordinates of the left-hand side projection center provides strip coordinates for the right-hand side projection center of the connected model.

Proceeding in the above manner, a chain of transformations can be carried out along the strip successively until all models are connected into one system.

Iterating the connection of two models i-1, i is speeded up if appropriate initial values are assumed for the unknowns. It is obvious that the photograph common to both models must assume a spatial position in the transformed model, which differs only slightly from that it had in the preceding model. Therefore suitable approximate values to start the iteration process can be obtained from the relations:

$$
\begin{aligned}
\bar{\beta}_{i,3} &= \beta_{i-1,3} \\
\bar{\beta}_{i,2} &= \beta_{i-1,2} + \alpha_{i-1,2}'' - \alpha_{i,2}'.
\end{aligned}
\tag{2.7}
$$

The index i refers to the new model, and α represents the relative orientation angles in question. Approximate values for β_1 and for the scale factor S are easily derived from a linear transformation of plane coordinates of two points lying in the common region.

The connection procedure described regards the coordinates of all points as equally weighted. It is a known fact that the error propagation due to relative orientation causes different errors at various model points. Nevertheless these weight variations have been neglected. It can be shown that neglecting these variations is farily justified.

Models Evaluated in the Autograph A8

Model coordinates determined in the Autograph A8 relate to a reference system which differs from the model system defined previously. Firstly, the origin of the recorded coordinates does not lie in the left projection center of the model, and secondly, the positive direction of the observed height differences is upwards, whereas the positive direction of the calculated XM_3 coordinate in the model system used before, is downwards. To allow for these differences two measures have to be taken: (1) the coordinates of the left projection center must be determined in the reference system of the Autograph projection plane, and (2) the direction of the measured heights must be reversed.

The primary axis of the instrument that coincides with the projection base is defined in the projection plane by the equation $XM_2 = 200.00$, where XM_2 is read on the appropriate scale (according to the notation used, XM_2 is equivalent to Y). As the primary axis passes through the projection center, the XM_2-coordinate of this point is already known. The XM_1-coordinate of the left projection center is defined by: $XM_1 = 200.00 - B/4$. B is the length of the base as read on the base scale. It is divided by 4 because the scales in the projection plane are subdivided into two-centimeter intervals. All plane coordinates read can be translated now to the left center of projection:

$$
\begin{aligned}
XM_1 &= (XM_{r1} - 200.00 + B/4)\cdot 2 \\
XM_2 &= (XM_{r2} - 200.00)\cdot 2.
\end{aligned}
\tag{3.1}
$$

The expressions on the right-hand side of Equation 3.1 are multiplied by 2 in order to obtain the coordinates in the correct units. The subscript r stands for the recorded value.

The XM_3-coordinate is handled similarly. To read it in millimeters one should use the height scale 1:10,000. After inserting this glass scale a reference reading is taken, so that a simple integral number on the scale coincides with a number on the scale attached to the vertical column. It may be convenient to choose the reading $Z=300$. Thereafter the XM_3-coordinate in the model, which increases from the base downwards, is determined as follows:

$$XM_3 = Z_{in} - \Delta r. \qquad (3.2)$$

Z_{in} denotes the chosen reading on the vertical column, and Δr is the difference between the two readings on the height scale, the difference between the height reading of the point in question and the reference reading which conforms to Z_{in}.

The procedure to relate the observed coordinates of model points to a system with the origin in the left projection center uses data supplied by the Autograph itself. If one does not rely on these data he can check them by a proper calibration of the instrument.

Each model should provide the following information: spatial coordinates of at least six points; the length of the base and the orientation elements α_2', α_4'', α_3'' (equivalent to ϕ', ϕ'', ω''). The orientation elements are needed to establish a set of initial values for the connection of two consecutive models.

Further processing of the data and the strip formation is identical with that described in the previous section.

Results of Tests

The preceding sections give a general outline of the computation method and demonstrate the geometrical model on which the triangulation is based. But the actual photograph deviates from its mathematical description and for this reason it becomes necessary to allow for such deviations as far as possible. Consequently, the triangulation procedure contains additional routines which correct the measured image coordinates of each point before the orientation procedure is brought into action. The routines mentioned are corrections for radial distortion and film shrinkage. Film shrinkage is eliminated to some extent by a linear transformation of the measured coordinates to a set of given data of fiducial marks.

The orientation method used does not make any assumptions as to the number and location of orientation points, size of orientation angles and character of terrain. Convergence of the iteration process is rapid, models with moderate tilts up to 5 degrees are oriented after three iterations, for models with tilts of the order of magnitude of 20 degrees, five iterations are required.

Connecting the models uses all common points located in the overlap area; however three common points are sufficient. Practice shows that a larger number of points does not necessarily improve the solution. The connection procedure uses redundant observations. This in turn entails two sets of coordinates for each common point, one in the preceding model the other in the new model. For strip triangulation the final coordinates of each connection point are obtained by averaging the two sets. In case of block triangulation another approach may be necessary. The accuracy of the connection procedure can be estimated from the coordinate differences and expressed by the mean-square difference. Such an estimate has been computed from differences in 24 connected models. The mean-square difference in plane coordinates was 0.01 mm in photography scale, the respective figure for heights was 0.02 mm. Figure 2 is a histogram that shows the distribution of the differences. It is worth noting that the distribution of the height differences tends to be rectangular.

Until now triangulation is regarded as a procedure which generates spatial coordinates for a number of points related to an arbitrary coordinate system. If these points are to be used for mapping purposes as a basis for orienting the individual models the manifold of strip coordinates has to be transformed to the system to which the mapped object is referred.

Due to unavoidable error accumulation the strip becomes deformed. Various methods are applied to allow for such deformations, most of which utilize control points lying in the middle and the edges of the strip. The triangulation reported here uses a second degree correction polynomial constructed on the basis of nine points uniformly distributed along the strip.

In order to compare the validity of the connection procedure the same strip has been measured both in the Wild stereocomparator and in the Autograph A8. The scale of photography was 1:17,000 and the strip consisted of eight models. In addition to those used for establishing the correction polynomial, 36 control points were available. After transforming the strip to the ground system and correcting the coordinates, residual errors were computed for all points. The residuals enable one to estimate the accuracy of the

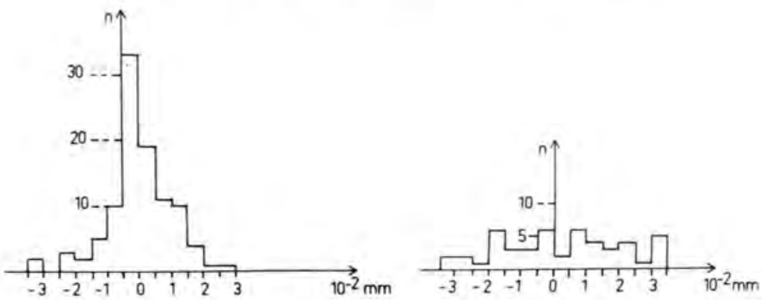

FIG. 2. Histograms showing the distribution of the differences. Plane (horizontal) coordinates are shown on the left, and heights on the right.

final coordinates. Table 1 summarizes the results. Columns r_x, r_y, r_z indicate the number of residuals in each class.

It is also of interest to compare the final coordinates of the new points evaluated in the comparator strip and in the A8 strip. As one and the same strip was triangulated in both instruments the transfer points used were identical in both cases. The differences between the coordinates of these points can be represented by the following figures: mean square difference in x-coordinates 17 cm., in y-coordinates 19 cm. and in heights 25 cm. The largest difference was 29 cm in x, 34 cm. in y, and 45 cm. in height.

TABLE 1. RESIDUAL ERRORS IN CONTROL POINTS

Class of residual, cm.		*Residuals*					
		Comparator			*A8*		
		v_x	v_y	v_z	v_x	v_y	v_z
−45	−35					1	2
−35	−25		1	2	3		1
−25	−15		4	5	3	3	4
−15	− 5	7	7	7	8	2	7
− 5	5	21	19	9	10	9	9
5	15	8	1	6	5	8	3
15	25		3	5	5	8	6
25	35		1	2	2	5	2
35	45						2
Mean square residuals		$s_x = 7$ cm. $s_y = 11$ cm. $s_z = 15$ cm.			$s_x = 15$ cm. $s_y = 19$ cm. $s_z = 22$ cm.		
Extreme residuals		$e_x = -15$ cm. $e_y = -28$ cm. $e_z = -31$ cm.			$e_x = 33$ cm. $e_y = -43$ cm. $e_z = -44$ cm.		

SUMMARY

Strip triangulation is conceived as a connection procedure between consecutive models with an overlap area. The models in turn can be formed numerically or mechanically, with the aid of analog instruments. The paper demonstrates a method to form a model and describes the connection routine between models. Finally the triangulation is demonstrated by a computed example. The results show that the final coordinates obtained from comparator measurements have a high standard of accuracy and are undoubtedly superior to those derived from Autograph A8 models. Nevertheless the coordinates obtained from the Autograph strip are sufficiently accurate for mapping purposes, and an instrument like the A8 can be used to advantage for triangulation procedures.

LITERATURE

Hallert B., *Fotogrammetri*, Stockholm 1964.

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