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Lunar Shape and Gravity Field

Photogrammetric missions can be utilized not only to establish control points on the moon but also to obtain gravity information on the far side.

INTRODUCTION

METHODS OF SELECTING orbits to map planetary surfaces in an *optimal* manner have been presented by Stern and Stern [1968]. However, these authors define *optimal* as the condition that a maximum of area of the planet is covered with a minimal amount of overlap. Such a planetary mission, of course, would obtain only pictures of the surface of the planet. If a sufficient overlap of the pictures would be attained and if the pictures would be taken with a metric camera, the photographs of the planet could be used to make a map of the surface of the planet, as applied to the moon by the Lunar Orbiter System [Norman, 1969].

ABSTRACT: *The positions of a metric camera in a spacecraft orbiting the moon can be treated as unknown parameters in a photogrammetric solution for establishing control points on the surface of the moon. According to Newton's law of gravitation, however, the orbit of a spacecraft around the moon, or the positions of the camera in a satellite on a photogrammetric mission, can be expressed by six orbital elements and the parameters of the lunar gravity field. Such a substitution of the camera positions not only reduces the number of unknown parameters but also provides an opportunity to gather information about the gravity field of the farside of the moon where spacecraft cannot be tracked from earth.*

The scale and orientation of such a map can be obtained from photographs taken by a pair of satellite-borne stellar cameras, from altimeter measurements of the distances between the camera and the surface of the planet, and from tracking of the spacecraft from the earth. This approach was proposed by the C&GS [1965] and by the Geodesy and Cartography Working Group in NASA [1967] for a more accurate mapping of the moon than that obtained by the Lunar Orbiters. Brown [1968] showed that without stellar cameras and altimeter measurements the scale and a mass-centered coordinate system can be obtained by short-arc methods.

Precise timing of the photographic exposures is required for these proposals. We will assume that this requirement is fulfilled in a photogrammetric mission for the moon or for another planet. We are then able to determine not only the geometric shape of the moon but also its gravity field. The orbit of a spacecraft around the moon on a photogrammetric mission is highly sensitive to the irregularities of the gravity field because of its low altitude. The positions of the spacecraft at the exposure time are obtained from the photogrammetric solution. These positions can be regarded as observations of the orbit and will allow conclusions about the lunar gravity field not only for the moon's nearside but also for the farside where the occultation of the spacecraft by the moon makes direct measurements from the earth impossible.

One might object that extracting this additional information will weaken the photogrammetric solution. However, in a purely photogrammetric solution the spacecraft position for each photograph enters the solution with three unknown parameters. If, for instance, the entire moon is covered with photographs, this will amount to thousands of unknown parameters for the camera positions. These many unknown parameters can be replaced by the parameters of the gravity field of the moon and six orbital elements for each orbit, because the orbit of the spacecraft around the moon can be expressed by Newton's law of gravitation. Although it is known from the analysis of the tracking data of the Lunar Orbiters that the gravity field of the moon is rather irregular [Muller and Sjogren, 1968], it will be possible to express the gravity field by fewer parameters than there are unknown coordinates for the camera positions, without distorting the results for the camera positions. Such a reduction of the number of unknown parameters, of course, will improve the photogrammetric solution.

A photogrammetric mission which completely covers the moon cannot be expected in the near future. However, photogrammetric pictures taken from an Apollo command module in an orbit moderately inclined to the lunar equator are planned. Even for such a photogrammetric mission one should try to replace the coordinates of the camera positions by the parameters of the lunar gravity field because results for the lunar gravity field of the nearside are available from the analysis of the Lunar Orbiters, in addition to the tracking of the spacecraft from the earth. Such a method will give a unique opportunity to gather information about the gravity field of the moon's farside.

Photogrammetric Approach

Using the colinearity equations, the vector $\boldsymbol{o}_i$ of the measurements on the plate i can be represented as a function of the vector $\mathbf{r}_i$ of the three elements of the orientation of the plate i, as a function of the vector $\boldsymbol{x}_i$ of the three coordinates of the camera position of the plate i, and of the vector $\boldsymbol{y}_j$ of the three coordinates of the ground point j:

$$\boldsymbol{o}_i = \boldsymbol{o}_i(\boldsymbol{r}_i, \boldsymbol{x}_i, \boldsymbol{y}_j) \tag{1}$$

where $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ are expressed in the same coordinate system. Approximate values for $\boldsymbol{r}_i$, $\boldsymbol{x}_i$, and $\boldsymbol{y}_j$ can be obtained so that we apply Taylor's theorem and obtain from Equation 1 the observation equations which lead to the normal Equations 2 of the least-squares adjustment for n photographs with m ground points in common [Schmid, 1958]. Here $\Delta\boldsymbol{r}_i$ represents the vector of corrections to the approximate values for the three elements of the orientation of the plate i, $\Delta\boldsymbol{x}_i$ the vector of corrections to the three coordinates of the camera position of the plate i, and $\Delta\boldsymbol{y}_j$ the vector of corrections to the three coordinates of the ground point j. The matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\boldsymbol{C}_i$, $\boldsymbol{D}_i$, $\boldsymbol{E}_{ij}$, and $\boldsymbol{F}_{ij}$ are of dimension 3×3. The vectors $\boldsymbol{l}_i$, $\boldsymbol{l}_i$, and $\boldsymbol{l}_j$ of the absolute values of the normal equations each contain three elements.

$$\left|\begin{array}{cccccccccccc}
A_1 & C_1 & 0 & 0 & \cdots & 0 & 0 & E_{11} & E_{12} & \cdots & E_{1m} \\
C_1^T & B_1 & 0 & 0 & \cdots & 0 & 0 & F_{11} & F_{12} & \cdots & F_{1m} \\
0 & 0 & A_2 & C_2 & \cdots & 0 & 0 & E_{21} & E_{22} & \cdots & E_{2m} \\
0 & 0 & C_2^T & B_2 & \cdots & 0 & 0 & F_{21} & F_{22} & \cdots & F_{2m} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & \cdots & A_n & C_n & E_{n1} & E_{n2} & \cdots & E_{nm} \\
0 & 0 & 0 & 0 & \cdots & C_n^T & B_n & F_{n1} & F_{n2} & \cdots & F_{nm} \\
E_{11}^T & F_{11}^T & E_{21}^T & F_{21}^T & \cdots & E_{n1}^T & F_{n1}^T & D_1 & 0 & \cdots & 0 \\
E_{12}^T & F_{12}^T & E_{22}^T & F_{22}^T & \cdots & E_{n2}^T & F_{n2}^T & 0 & D_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
E_{1m}^T & F_{1m}^T & E_{2m}^T & F_{2m}^T & \cdots & E_{nm}^T & F_{nm}^T & 0 & 0 & \cdots & D_m
\end{array}\right|
\left|\begin{array}{c}
\Delta r_1 \\ \Delta x_1 \\ \Delta r_2 \\ \Delta x_2 \\ \vdots \\ \Delta r_n \\ \Delta x_n \\ \Delta y_1 \\ \Delta y_2 \\ \vdots \\ \Delta y_m
\end{array}\right|
=
\left|\begin{array}{c}
l_1 \\ l_1' \\ l_2 \\ l_2' \\ \vdots \\ l_n \\ l_n' \\ \bar{l}_1 \\ \bar{l}_2 \\ \vdots \\ \bar{l}_m
\end{array}\right| \tag{2}$$

As is known, the coefficient matrix of Equation 2 is singular. In order to obtain a unique solution, adjacent photographs must have at least five ground points in common. In addition to this, seven coordinates of ground points have to be known for orientation and scaling. These values can be easily introduced into Equation 2 as observations with zero variances [Koch and Pope, 1969].

If the distances from the camera to the ground are measured (for instance, by a laser-altimeter) as proposed by the C&GS [1965] and by the Geodesy and Cartography Working Group in NASA [1967], these measurements will introduce the scale into the Equation 2. The laser measurements have to be synchronized with the opening of the camera shutter. From a calibration, the angle between the principal axis of the photogrammetric camera and the laser beam will be known. Hence, the spot on the photograph can be marked where the laser beam hit the ground and an observation equation can be formed according to Equation 1. If $\boldsymbol{s}_i$ is the laser measurement at the time of the ith photograph and $\boldsymbol{Y}_{js}$ the vector of the coordinates of the ground point which reflected the laser beam, we have the relationship

$$\boldsymbol{s}_i = \boldsymbol{s}_i(\boldsymbol{x}_i, \boldsymbol{y}_{js}) \tag{3}$$

Stellar cameras combined with the photogrammetric camera will give the camera orientation in the astronomic right ascension and declination system which by means of the lunar ephemerides can be transformed into a moon-fixed system, hence

$$\boldsymbol{p}_i = \boldsymbol{p}_i(\boldsymbol{r}_i) \tag{4}$$

where $\boldsymbol{p}_i$ are the observations of the stellar camera at the time of the exposure of the plate i. The Observations 3 and 4 in addition to Equation 1 yield the normal Equations 5.

(*See page 378 for Equation 5*)

The matrices $\boldsymbol{G}$ denote the contribution of the stellar camera Observations 4 to the normal equations, and $\boldsymbol{H}$, $\boldsymbol{I}$, and $\boldsymbol{J}$ the contribution of the altimeter Observations 3 which is shown only for plate 2 and ground point js. The solution of Equation 5 gives the coordinates of surface points of the moon in a moon-fixed coordinate system whose origin is in an arbitrary position.

Dynamical Approach

So far the camera positions $\boldsymbol{x}_i$ have been treated as independent parameters. However, the orbit of the spacecraft around the moon can be expressed as a function of the orbital elements $\boldsymbol{e}_o$ at an epoch t_o and of the parameters χ of the lunar gravity field

$$\boldsymbol{x}_i = \boldsymbol{x}_i(\boldsymbol{e}_0, \chi). \tag{6}$$

While flying over the moon's nearside the spacecraft will be tracked from the earth so that

$$\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{e}_0, \chi) \tag{7}$$

where $\boldsymbol{t}$ denotes the tracking data. Finally, from previous tracking of lunar satellites information $\boldsymbol{g}$ about the gravity field of the moon has been gathered which can be introduced by

$$\boldsymbol{g} = \boldsymbol{g}(\chi). \tag{8}$$

If the entire surface of the moon would be covered with photographs from heights, for instance, less than 100 km., the elimination of the camera positions according to Equation 6 could considerably reduce the number of unknown paraeters in the normal Equations 5. But even if only an equatorial belt of the moon is covered by photographs, it might be possible to reduce the number of unknown

$$
\begin{vmatrix}
A_1+G_1 & C_1 & 0 & 0 & \cdots & 0 & 0 & E_{11} & \cdots & E_{1j} & \cdots & E_{1m} \\
C_1^T & B_1 & 0 & 0 & \cdots & 0 & 0 & F_{11} & \cdots & F_{1j} & \cdots & F_{1m} \\
0 & 0 & A_2+G_2 & C_2 & \cdots & 0 & 0 & E_{21} & \cdots & E_{2j} & \cdots & E_{2m} \\
0 & 0 & C_2^T & B_2+H_2 & \cdots & 0 & 0 & F_{21} & \cdots & F_{2j}+I_{2j} & \cdots & F_{2m} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & 0 & 0 & \cdots & A_n+G_n & C_n & E_{n1} & \cdots & E_{nj} & \cdots & E_{nm} \\
0 & 0 & 0 & 0 & \cdots & C_n^T & B_n & F_{n1} & \cdots & F_{nj} & \cdots & F_{nm} \\
E_{11}^T & F_{11}^T & E_{21}^T & F_{21}^T & \cdots & E_{n1}^T & F_{n1}^T & D_1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
E_{1j}^T & F_{1j}^T & E_{2j}^T & F_{2j}^T+I_{2j}^T & \cdots & E_{nj}^T & F_{nj}^T & 0 & \cdots & D_j+J_j & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
E_{1m}^T & F_{1m}^T & E_{2m}^T & F_{2m}^T & \cdots & E_{nm}^T & F_{nm}^T & 0 & \cdots & 0 & \cdots & D_m
\end{vmatrix}
\begin{vmatrix}
\Delta r_1 \\ \Delta x_1 \\ \Delta r_2 \\ \Delta x_2 \\ \vdots \\ \Delta r_n \\ \Delta x_n \\ \Delta y_1 \\ \vdots \\ \Delta y_{jk} \\ \vdots \\ \Delta y_m
\end{vmatrix}
=
\begin{vmatrix}
m_1 \\ m_1' \\ m_2 \\ m_2' \\ \vdots \\ m_n \\ m_n' \\ \overline{m}_1 \\ \vdots \\ \overline{m}_j \\ \vdots \\ \overline{m}_m
\end{vmatrix}
\qquad (5)
$$

$$
\begin{vmatrix}
A_1+G_1 & 0 & \cdots & 0 & L_1 & N_1 & E_{11} & \cdots & E_{1j} & \cdots & E_{1m} \\
0 & A_2+G_2 & \cdots & 0 & L_2 & N_2 & E_{21} & \cdots & E_{2j} & \cdots & E_{2m} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & \cdots & A_n+G_n & L_n & N_n & E_{n1} & \cdots & E_{nj} & \cdots & E_{nm} \\
L_1^T & L_2^T & \cdots & L_n^T & K & O & Q_1 & \cdots & Q_j & \cdots & Q_m \\
N_1^T & N_2^T & \cdots & N_n^T & O^T & M & R_1 & \cdots & R_j & \cdots & R_m \\
E_{11}^T & E_{21}^T & \cdots & E_{n1}^T & Q_1^T & R_1^T & D_1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
E_{1j}^T & E_{2j}^T & \cdots & E_{nj}^T & Q_j^T & R_j^T & 0 & \cdots & D_j+J_j & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
E_{1m}^T & E_{2m}^T & \cdots & E_{nm}^T & Q_m^T & R_m^T & 0 & \cdots & 0 & \cdots & D_m
\end{vmatrix}
\begin{vmatrix}
\Delta r_1 \\ \Delta r_2 \\ \vdots \\ \Delta r_n \\ \Delta e \\ \Delta \chi \\ \Delta y_1 \\ \vdots \\ \Delta y_{jk} \\ \vdots \\ \Delta y_m
\end{vmatrix}
=
\begin{vmatrix}
m_1 \\ m_2 \\ \vdots \\ m_n \\ m' \\ m'' \\ \overline{m}_1 \\ \vdots \\ \overline{m}_j \\ \vdots \\ \overline{m}_m
\end{vmatrix}
\qquad (9)
$$

parameters by the substitution Equation 6 if we take Equations 7 and 8 into account.

From the tracking of the Lunar Orbiters a good knowledge has been obtained for the gravity field of the nearside of the moon (Muller and Sjogren [1968], Blackshear [1969]). Satellites on the lunar farside, however, cannot be reached by earth tracking. If no synchronous satellites are sent up for the moon from which moon satellites are tracked, a photogrammetric mission will provide an excellent possibility of obtaining information about the gravity field of the farside. But in contrast to tracking from synchronous satellites, a photogrammetric mission will also give the geometric features of the surface of the moon.

Combination of the Photogrammetric and Dynamical Approach

One way of determining the parameters χ of the lunar gravity field is to adjust the photogrammetric and dynamical data in two steps. In such a solution, Eq. 6 yields the observation equations whose observed quantities (i.e., the coordinates $\boldsymbol{x}_i$ of the camera stations) have been obtained together with their covariance matrix from the solution of the normal Equations 5. These observation equations together with Equations 7 and 8 give the normal equations for the parameters χ of the gravity field. Their solution yields, in addition to the parameters χ, new adjusted camera positions with their covariance matrix. The comparison of this covariance matrix with the one obtained for the camera positions by the inversion of Matrix 5 from the photogrammetric data alone will show the improvement due to the inclusion of the dynamical constraints and orbital data. In addition, at this stage the change in the magnitude of the squares of residuals can be obtained and used to show whether the chosen model for the lunar gravity field is adequate. If indeed this is the situation, the new adjusted camera positions are introduced into the System 5 to obtain a corresponding new set of ground coordinates.

The disadvantage of such an approach is the fact that one has to cope with a great number of camera positions. By simulation studies it should be decided beforehand how many parameters of the gravity field of the moon are necessary to describe the orbits of a planned photogrammetric mission with an accuracy compatible with the one obtained for the camera positions from the photogrammetric solution. Then, the camera positions $\boldsymbol{x}_i$ in Equations 1, 3, and 4 can be substituted according to Equation 6 by the orbital elements $\boldsymbol{e}_o$ and the parameters χ of the gravity field. Hence, we obtain the normal Equations 9 for the combination of the photogrammetric and dynamical approach. Here $\Delta\boldsymbol{e}$ denotes the corrections to the approximate orbital elements and $\Delta\chi$ the corrections to the parameters of the gravity field. The matrices $\boldsymbol{K}, \boldsymbol{L}_i, \boldsymbol{M}, \boldsymbol{N}_i, \boldsymbol{O}, \boldsymbol{Q}_j$, and $\boldsymbol{R}_j$ are of dimensions 6×6, 3×6, $k\times k$, $3\times k$, $6\times k$, 6×3, and $k\times3$, respectively, if k is the number of parameters χ of the gravity field.

(See page 378 for Equation 9)

The normal Equations 9 contain a considerable number of zero elements because no observation equations contain coefficients for more than one set of three orientation parameters. The same holds for the orbital elements for each orbit and for the coordinates of the ground points. In addition, coefficients for ground points appear only in the observation equations for the plates which contain their images. Because of these many zero-elements it will be possible to solve the huge System 9. As in photogrammetry, the orientation parameters or the ground stations could be eliminated analytically before the system goes into the computer for the numerical solution. However, System 9 is much more irregular than, for instance, the normal equations of the adjustment for a photogrammetric block. The compacting of normal Equations 9 should therefore be accomplished by the computer program so that only non-zero elements of the normal equation are stored and used in the solution of Equations 9 [Krakiwsky and Pope, 1967].

Orbit Integration

The equations of motion of a moon satellite are expressed in a coordinate system whose center is the mass center of the moon. The solution of Equation 9 will therefore give the coordinates of the points at the surface of the moon with respect to the mass center. If the photogrammetric mission would be flown without the laser altimeter it would still be possible to scale the results by introducing the product of the lunar mass and the gravitational constant as a fixed quantity into the orbital analysis. If there would be no stellar cameras, the moon-fixed coordinate system could be oriented by earth tracking.

For only half of the lunar surface (or even less) will the lighting conditions permit photography [Konecny, 1968], and radar tracking from the earth is possible only for the lunar nearside. Nevertheless, it will be advisable to use a few long arcs instead of many short ones to reduce the number of unknown orbital elements, although this procedure may not harmonize with the optimal selection of photographs for the photogrammetric approach. The orbits will be integrated numerically, along with the variational equations for the unknown parameters. If necessary, the equations for powered flight will be used.

Harmonic coefficients of the expansion of the lunar gravity potential in spherical harmonics have been determined by Blackshear [1969]. If only parts of the lunar sruface are covei ed by photographs, the gravity field of the moon is better expressed by a known expansion of spherical harmonics up to a low degree and an unknown potential of a simple layer distributed over the surface of the moon [Koch, 1968]. The unknown parameters of the gravity field in this representation are connected with specific surface areas so one can easily account for a great number of observations in certain areas in contrast to few observations in other areas [Koch, 1970].

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