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Relative Control for Extraterrestrial Work

Information for analytical techniques is available for use either with a sequential or a simultaneous solution.

(*Abstract on next page*)

INTRODUCTION

EXTRATERRESTRIAL APPLICATIONS of analytical photogrammetry can be divided into two broad categories: those dealing with photographs from orbiting vehicles, and those concerned with photographs acquired at or near the surface. Common to both of these categories is the problem that control points, defined in the conventional absolute sense, are simply not available in extraterrestrial work.

In the first category concerning photographs from orbiting vehicles, this problem is alleviated through the use of orbital constraints in the analytical reduction. In the second category, however, the photographs are not generally related to each other by the nice geometric constraint of an orbit. These photographs are normally taken (or to be taken) from soft-landed vehicles, from roving vehicles, or from hand-held cameras used by landed astronauts traversing the surface. Consequently, it cannot be assumed that the geometric relationships between different camera stations are always known. Instead, we must consider the possibility of a variety of relative control information which may exist in the object space. It is the objective of this paper to deal with this type of information in relation to applications within the second category of extraterrestrial photographs.

By relative control information is meant anything other than points with known coordinates in any type of coordinate system. The exception applies to both partially and completely known control points. The types of relative control considered in this paper include distances of any kind (i.e., between camera stations, between object points, etc.); points known to lie on planes of any orientation (i.e., vertical, horizontal, etc.); points which lie on lines of different orientations; known angles; and known geometric shapes.

Considering the fact that there are two distinct procedures of analytical reduction techniques, namely sequential and simultaneous, the use of the above mentioned information will be considered in relation to both techniques.

It is important to note at the outset, that this general subject of use of constraints in analytical photogrammetry has been already treated in the literature.[1,2] However, most of the work published emphasizes regular aerial, and in some cases orbital, photographs. This paper, on the other hand, is concerned with extraterrestrial applications of more or less close-range photography.

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SEQUENTIAL TRIANGULATION SYSTEM

A sequential triangulation system is considered here to be divided into two basic operations, relative and absolute orientation. As we are concerned in this study with extraterrestrial application of photographs taken on the surface, we will confine ourselves to a stereo pair situation. However, all the relations derived here would apply equally well to multiple stereo pairs which have been collected into one coordinate system after we will study each category of relative control and find out which of the *seven* absolute orientation parameters can be determined.

ABSTRACT: *The logical techniques for use in extraterrestrial mapping are probably those of analytical photogrammetry. Such techniques are equally applicable for photographs from descending vehicles, orbiting vehicles, or from the surface of the extraterrestrial body (via soft-landed vehicles or landed human beings). In all cases, lack of known absolute or partial control in the object space is a reality to be reckoned with. A variety of possible* relative control *information is available for use in the reduction of photographs taken from points on or near the surface of the extraterrestrial body. Such relative control may include: known distances, points known to lie on lines or planes of different orientations, and known angles and shapes. The contribution of each of these pieces of information to the solution of the photogrammetric problem is effected through the derivation of corresponding equations. A consideration is given to sequential as well as simultaneous analytical triangulation systems as affected by the incorporation of such relative control data.*

independent relative orientation and assuming no significant distortions between them.

Relative control information is then used in the step of absolute orientation which is conventionally performed using the transformation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = s\boldsymbol{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \tag{1}$$

where:

s	is a scale change
$\boldsymbol{R}$	rotation matrix in terms of three independent parameters
C_x, C_y, C_z	three translations
(x, y, z)	model coordinates
(X, Y, Z)	control coordinates

The application of Equation 1 necessitates that control information be in the form of coordinates with respect to one unified system (X, Y, Z). It is not necessary, however, that all three coordinates be known for a control point, as one can easily utilize *partially* known control points. Nevertheless, Equation 1, in its given form, cannot be used for *relative* control information. In the following section

DISTANCES

The first type of relative control information is known distances which may be between camera stations, between object points, or between a camera station and an object point. Any of these, or combinations thereof, would simply determine the scale factor s.

If a known distance in the object space is D_i and the corresponding distance in the model is d_i, the scale factor is directly obtained from

$$s = \frac{1}{n}\sum_{i=1}^{n}(D_i/d_i) \tag{2}$$

where n is the total number of known distances. If the distance is between two camera stations, then

$$d_i = [b_x^2 + b_y^2 + b_z^2]_i^{1/2} \tag{3}$$

which is the length of the base at the arbitrary scale of relative orientation.

If the distance is between two points r and t, we have

$$d_i = [(x_r - x_t)^2 + (y_r - y_t)^2 + (z_r - z_t)^2]^{1/2} \tag{4}$$

which is distance in the model between the two points.

Here, the two points r and t may both be object points or one may be an object point and the other a camera station.

The value of s obtained from Equation 2 represents the simple mean of n values. However, if the known information is of varying known weight, then the weighted mean should be used.

PLANES

The second valuable source of relative information is the knowledge that a number of points lie on a plane in the object space. Such a plane would basically assume one of three positions, either vertical, or horizontal, or inclined. We will study each of these cases independently.

Known Vertical Plane. The model coordinates x, y, z are operated on by a rotation matrix such that after rotation the coordinates of a given set of points lie on a vertical plane. The rotation matrix in three dimensional space is a 3×3 orthogonal matrix which would have a maximum of three independent parameters. Schut[3] has shown different methods of constructing such a matrix. One such method, which we shall choose as suitable for this case, is by performing a single rotation α about a directed line. If the direction cosines of that line are λ, μ, γ then the rotation matrix is given by

$$R = \begin{bmatrix} \lambda^2(1-\cos\alpha)+\cos\alpha & \lambda\mu(1-\cos\alpha)-\gamma\sin\alpha & \lambda\gamma(1-\cos\alpha)+\mu\sin\alpha \\ \lambda\mu(1-\cos\alpha)+\gamma\sin\alpha & \mu^2(1-\cos\alpha)+\cos\alpha & \mu\gamma(1-\cos\alpha)-\lambda\sin\alpha \\ \lambda\gamma(1-\cos\alpha)+\mu\sin\alpha & \mu\gamma(1-\cos\alpha)+\lambda\sin\alpha & \gamma^2(1-\cos\alpha)+\cos\alpha \end{bmatrix} \tag{5}$$

Four parameters seem to be involved in $\boldsymbol{R}$ but one of these is a dependent parameter because of the fact that the three directon cosines are related by

$$\lambda^2+\mu^2+\gamma^2 = 1. \tag{6}$$

In our problem, the directed line would be chosen to pass through the origin and parallel to the trace of the given plane. Hence, from Figure 1,

$$\lambda = \cos\beta \qquad \mu = \sin\beta \qquad \gamma = 0 \tag{7}$$

and $\boldsymbol{R}$ reduces to

$$\boldsymbol{R} = \begin{bmatrix} \cos^2\beta(1-\cos\alpha)+\cos\alpha & \sin\beta\cos\beta(1-\cos\alpha) & \sin\alpha\sin\beta \\ \sin\beta\cos\beta(1-\cos\alpha) & \sin^2\beta(1-\cos\alpha)+\cos\alpha & -\sin\alpha\cos\beta \\ -\sin\alpha\sin\beta & \sin\alpha\cos\beta & \cos\alpha \end{bmatrix} \tag{8}$$

The transformation equation would be

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \boldsymbol{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9}$$

where x, y, z are model coordinates and x', y', z' are the transformed coordinates. The transformed coordinates of the points known to lie on a vertical plane must then satisfy the equation

$$y' = x'\tan\beta + C. \tag{10}$$

If the equation for the first of the points is subtracted from those for all other points, C can be eliminated and the equation for the i-th point reduces to

$$(y_i' - y_1') = (x_i' - x_1')\tan\beta. \tag{11}$$

Using Equation 8 into 11 and expanding, the latter simplifies to

$$(x_i - x_1)\cos\alpha\sin\beta - (y_i - y_1)\cos\alpha\cos\beta + (z_i - z_1)\sin\alpha = 0. \tag{12}$$

The unique case should be when three points are known because they a e the minimum that determines a plane. This is true because we would have two of Equations 12 in the two unknowns α and β.

It is important to emphasize that after the rotation in Equation 8 the given plane would be perpendicular to the $x'y'$-plane and parallel to the z'-axis. However, there is no reason at all that the z'-axis would be parallel to the Z-axis of the control system. Hence, the information given by the knowledge of *one* vertical plane is not sufficient to *level* the model.

If the known vertical plane is to be taken parallel to, for example, the XZ-plane, the result would be that the y'-axis will be parallel to the y-axis. The same applies for the case of the YZ-plane. In this instance a more appropriate method of constructing $\boldsymbol{R}$ would be to choose two sequential rotations. For example, if the vertical plane is to be parallel to the XZ-plane, two sequential rotations ω about x-axis and κ about z-axis will be used. The rotation matrix will then be

$$\boldsymbol{R} = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}$$

or

$$R = \begin{bmatrix} \cos\kappa & \cos\omega \sin\kappa & \sin\omega \sin\kappa \\ -\sin\kappa & \cos\omega \cos\kappa & \sin\omega \cos\kappa \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \tag{13}$$

After transformation, the condition for points in the known plane would be

$$y' = \text{constant}$$

or

$$y_i' - y_1' = 0$$

or

$$-(x_i - x_1)\sin\kappa + (y_i - y_1)\cos\omega\cos\kappa + (z_i - z_1)\sin\omega\cos\kappa = 0 \tag{14}$$

where

$(x, y, z)_1$ are model coordinates for first point in the plane

$(x, y, z)_i$ are model coordinates for other points in the plane.

Similar derivation can be performed for the condition of the known plane being parallel to the YZ-plane.

If we now return to the condition of a general vertical plane we must consider the situation where two or more such planes are available. This is important because it has already been shown that one vertical plane alone does not bring the z'-axis to be parallel to the Z-axis. Two or more, however, would effect such a parallelism provided these planes are not themselves parallel.

To bring the z'-axis to be parallel to the Z-axis the rotation matrix $\boldsymbol{R}$ would be formed of two sequential rotations ω and ϕ about the x- and y-axes, respectively. Hence

$$R = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}$$

or

$$\mathbf{R}\begin{bmatrix} \cos\phi & \sin\omega \sin\phi & -\cos\omega \sin\phi \\ 0 & \cos\omega & \sin\omega \\ \sin\phi & -\sin\omega \cos\phi & \cos\omega \cos\phi \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}. \tag{15}$$

Now suppose that there are known m vertical planes $P_1, P_2 \cdots P_m$ in the object space. Suppose further that there are n_1 points in plane P_1, n_2 points in plane $P_2 \ldots$ and n_m points in plane P_m. The transformed x'- y'-coordinates of all the points in any one of these planes must satisfy the equation

$$(x_i' - x_2')(y_i' - y_1') - (x_i' - x_1')(y_i' - y_2') = 0 \tag{16}$$

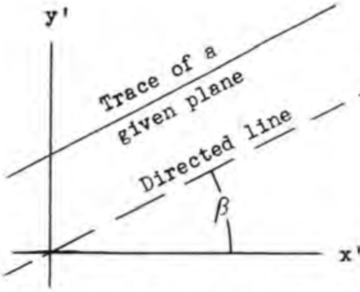

FIG. 1. Geometric situation involved where a plane is known to be vertical.

where

(x_1', y_1') and (x_2', y_2') are transformed coordinates of the first two points in one plane,

(x_i', y_i') $i = 3, 4 \cdots n$ are transformed coordinates of any other point in the same plane,

and

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} [x\ y\ z]^t. \tag{17}$$

There will be $(n_1 - 2)$ Equations 16 for P_1, $(n_2 - 2)$ for $P_2 \cdots$ and $(n_m - 2)$ for P_m, all in terms of only two unknowns ω and ϕ. These can then be solved by the method of least squares, provided m, the number of given vertical planes, is equal to or more than two. When the numerical value of $\boldsymbol{R}$ is evaluated, all other points in the model can be transformed. If the model was originally at the right scale, differences between the transformed z'-coordinates would be true differences in elevation.

The equations derived in this section regarding one vertical plane, as well as two or more such planes, have been programmed and tested. Simulated data was generated and used to verify the more important situation of several vertical planes. The results obtained were exactly as expected, thus indicating correct mathematical formulations and geometric concepts.

Known Horizontal Plane. With a known horizontal plane it should be obvious that the transformed z'-axis would be parallel to the Z-axis. This is the same as the preceding instance of two or more vertical planes, and therefore the rotation matrix $\boldsymbol{R}$ of Equation 15 would be applicable here. The only difference is that after rotation the z'-coordinates of all points in the known plane must be equal. Hence the condition equation

$$z_i' - z_1' = 0 \tag{18}$$

where

$$z' = \mathbf{r}_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{19}$$

For a given plane with n points, $(n-1)$ Equations 18 can be written and solved by least squares for ω and ϕ. If there is more than one horizontal plane, appropriate equations may be written making sure that z_1' in Equation 18 is the first point in *each plane*. Therefore, for first plane $(z_1')_1$ will be used for all points in that plane; a $(z_1')_2$ will be used for all points in the second plane; and so on. For a scaled model, differences in elevation can be directly obtained after transformation.

Known Inclined Plane. If all that is known is that a set of points lie on a plane in the object space with no information regarding the plane, no parameters for absolute orientation can be determined. This should be readily obvious since the the same set of points would lie on a plane in the model assuming that model distortions after relative orientation are negligible.

LINES

Similar to the instance of a plane, a given line can be vertical, horizontal, or inclined in the object space. We will likewise treat each of these situations individually.

Known Vertical Lines. In this instance, the model coordinates are transformed such that the z'-axis will become parallel to the object Z-axis. The transformation matrix $\boldsymbol{R}$ would be constructed by two sequential rotations and would take the form given by Equation 15. After transformation, coordinates of points along the given line must satisfy the relations

$$\begin{aligned} x' &= k_1 \\ y' &= k_2 \end{aligned} \tag{20}$$

where k_1 and k_2 are constants and x', y' as defined by Equation 17.

If the contribution of the first point is subtracted from those for all other points, the relations for any point i would be

$$\begin{aligned} x_i' - x_1' &= 0 \\ y_i' - y_1' &= 0. \end{aligned} \tag{21}$$

Obviously, for the unique case of two points, one pair of Equations 21 would be solved for the two unknowns ω and ϕ. For more than two points the method of least squares would be applied. It is, furthermore, a straightforward matter to extend the treatment to the condition of several vertical lines.

It should be noted here that the situation of one given vertical line is equivalent to two given vertical planes that are not parallel. This is easily ascertained by the fact that two non-parallel vertical planes would intersect in a vertical line. This situation of vertical lines has also been programmed and tested with satisfactory results.

Known Horizontal Line. After model transformation the known line would be parallel to the $x'y'$-plane and hence each point on that line will have the same z'-coordinate. As a stright line is defined by two points in space, only one equation of the following type would be obtained:

$$z_i' - z_1' = 0. \tag{22}$$

Consequently the rotation matrix $\boldsymbol{R}$ must include no more than one independent parameter. This parameter would be a single rotation about a line through the origin. Such a line would be constructed perpendicular to the projection of the line on the xy-plane. If the line is originally parallel to the model z-axis, the solution is obviously a rotation of 90° about either the model x- or y-axis, without the need for Equation 22. It is important to note that the condition of a horizontal line is equivalent to that of a vertical plane. Both yield no valuable information unless more than one (line or plane) are given. Hence we consider next the case of two or more horizontal lines that are not parallel.

With two or more non-parallel horizontal lines the transformed z'-axis becomes parallel to the Z-axis. This implies that the transformation matrix $\boldsymbol{R}$ can be constructed by two sequential rotations ω and ϕ about x- and y-axes, respectively. Hence the same matrix given in Equation 15. Equation 22 may then be written for all points on the same line with the same value for z_1'. For the second and other lines, each would have a different value for z_1. All the equations are then collected and solved by least squares for ω and ϕ.

Known Inclined Line. Similar to the corresponding case of a plane, none of the absolute orientation parameters can be determined from the mere knowledge that a set of points lie on a straight line.

ANGLES

The seven-parameter transformation of absolute orientation (Equation 1) is conformal and does not cause any deformations. Consequently, the internal geometry of a model after relative orientation must be assumed to be correct within the tolerances of the work performed. Angles, naturally, are reconstructed correctly after relative orientation and undergo no change in value during absolute orientation. All that changes is the orientation of the *plane* of the angle. This obviously leads to the fact that a known vertical angle is equivalent to a known vertical plane, a known horizontal angle to a known horizontal plane, and so on.

A known horizontal angle is of particular interest because it allows the leveling of the model. If one of the lines forming the angle is chosen also to be parallel to either the object X- or Y-axis, the third parameter (azimuth) of the rotation matrix can be determined. Consequently, the system may first be transformed such that the plane of the angle is horizontal using the procedure given under *Known Horizontal Plane*. Next, the coordinates x', y', z' may be transformed by

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}. \qquad (23)$$

If one line is now chosen to be parallel to the Y-axis, the coordinates X' of points on that line must satisfy the relation

$$X_i' - X_1' = 0 \qquad (24)$$

where X_1' is for the first point and X_i' for any other. If the minimum condition of two points is considered, the angle κ can be readily determined from

$$\tan\kappa = -(x_2' - x_1')/(y_2' - y_1'). \qquad (25)$$

Otherwise the method of least squares is used.

At this point, if the model was originally scaled, the system of coordinates would be parallel to the object-space coordinate system. Absolute orientation would further be completely determined if a point, such as the apex of the angle, is chosen as the origin of the coordinate system. This would account for three translations and make one of the lines of the angle the Y-axis (or the X-axis).

GEOMETRIC SHAPES

An object or objects of known geometric shape may supply some information useful for control purpose. Such shapes may include squares, rectangles, circles, spheres, etc. If the shapes of these objects are the only known information, no parameter of absolute orientation can be determined. This is, as pointed out earlier, because after relative orientation the shapes of all objects would be correctly reconstructed. However, if these geometric objects are also of known dimensions, then the scale of the relatively oriented model can be determined. Of course if some geometric shape is broken down to known planes and/or lines, then more parameters can be determined. But then we would be concerned with situations already covered in the preceding sections.

SIMULTANEOUS TRIANGULATION SYSTEM

Basically two types of simultaneous triangulation systems are available, one built on the collinearity equation, and the other utilizing the coplanarity-scale-restraint combination of equations. A mathematical model built on the collinearity condition is considered by this author easier to augment by the conditions arising from relative control information. Consequently, it is within the bounds of such a simultaneous triangulation system that we consider the use of relative information.

The linearized form of collinearity equation takes the form

$$v + \dot{B}\dot{\Delta} + \ddot{B}\ddot{\Delta} = F^0 \qquad (26)$$

where

- v is the vector of residuals in the observed photo-coordinates
- $\dot{B}$ coefficient matrix composed of the partial derivatives of the equations with respect to the photo parameters
- $\dot{\Delta}$ corrections to approximate values of photo parameters
- $\ddot{B}$ matrix of partial derivatives with respect to object point coordinates
- $\ddot{\Delta}$ corrections to approximate values of object point coordinates
- F^0 values of the equations evaluated at the given observations and assumed approximations to the parameters.

The set of Equations 26 are those in the photogrammetric problem arising from the geometry of projectivity between object- and photo-spaces. Any additional information can be incorporated by augmenting Equation 26 with the corresponding condition equations. In the following sections we shall derive these additional condition equations and study the means of incorporating them into the mathematical model.

DISTANCES

As before three types of known distances will be considered.

Distance Between Two Camera Stations. Let the distance L_{ir} be known between the two camera stations i and r. The condition equation is

$$F_{ir} = L_{ir}^2 - (X_{ci} - X_{cr})^2 - (Y_{ci} - Y_{cr})^2 - (Z_{ci} - Z_{cr})^2 = 0 \tag{27}$$

where (X_c, Y_c, Z_c) refers to the coordinates of camera station.

Equation 27 when linearized becomes

$$\begin{aligned} dL_{ir} - 1/L_{ir}[&(X_{ci} - X_{cr})(\delta X_{ci} - \delta X_{cr}) \\ &+ (Y_{ci} - Y_{cr})(\delta Y_{ci} - \delta Y_{cr}) \\ &+ (Z_{ci} - Z_{cr})(\delta Z_{ci} - \delta Z_{cr})] = -(F_{ir}^0/2L_{ir}) \end{aligned} \tag{28}$$

where F_{ir}^0 is the value for Equation 27 if approximations are used for the unknowns. Equation 28 can be written in more concise form, with obvious correspondence in terms, as

$$\underset{1,1}{\boldsymbol{V}_L} + \underset{1,6}{\dot{\boldsymbol{B}}_L}\, \underset{6,1}{\dot{\Delta}} = \underset{1,1}{\boldsymbol{F}_L^0} \tag{29}$$

which represents the condition equation for one known distance.

Distance Between Two Object Points. The known distance between two object points j and s is denoted by S_{js} such that

$$S_{js}^2 - (X_j - X_s)^2 - (Y_j - Y_s)^2 - (Z_j - Z_s)^2 = 0. \tag{30}$$

The linearization of Equation 30 is identical to that for 28, except for using the appropriate subscripts. In matrix form it is

$$\underset{1,1}{\boldsymbol{V}_s} + \underset{1,6}{\ddot{\boldsymbol{B}}_s}\, \underset{6,1}{\ddot{\Delta}} = \underset{1,1}{\boldsymbol{F}_s^0}. \tag{31}$$

Distance Between a Camera Station and an Object Point. Designating the distance between camera station i and object point j by D_{ij}, the matrix form of the linearized equation is

$$\underset{1,1}{\boldsymbol{V}_D} + \underset{1,3}{\dot{\boldsymbol{B}}_D}\underset{3,1}{\dot{\boldsymbol{\Delta}}_i} + \underset{1,3}{\ddot{\boldsymbol{B}}_D}\underset{3,1}{\ddot{\boldsymbol{\Delta}}_j} = \underset{1,1}{F_s^0}. \tag{32}$$

Any or all of the three condition Equations 29, 31, or 32 can easily be combined with the set of equations represented by Equation 26. The merged condition equations may then be reduced by least squares using well-established procedures.

PLANES

As we have done before, we shall consider three types of known planes.

Known Vertical Plane. A vertical plane can be determined by two points provided they do not lie on a vertical line. Therefore, in order that a condition may be furnished and enforced, at least three points must be designated as lying on the vertical plane. If n points are given, where $n \geq 3$, then $(n-2)$ conditions can be written.

The general equation of a vertical plane through points 1, 2, and i are the same as Equation 16, or

$$(Y_i - Y_1)(X_i - X_2) - (X_i - X_1)(Y_i - Y_2) = 0. \tag{33}$$

The linearized form of this equation is

$$\underset{1,4}{\boldsymbol{C}_{vp}}\, \underset{4,1}{\ddot{\boldsymbol{\Delta}}} = \underset{1,1}{G_{vp}^0}. \tag{34}$$

Equation 34 does not include any observational errors, and its incorporation in the least square adjustment is not as straight forward as those obtained in the preceding section. A brief description of a procedure for dealing with this type of constrained least squares is given in an Appendix.*

Known Horizontal Plane. Points in the object space known to lie on a horizontal plane would evidently have equal elevation, although the value of such elevation may not be known. Hence, if two such points are 1 and i, the general condition equation is

$$Z_i - Z_1 = 0 \tag{35}$$

or, in matrix form,

$$\underset{1,2}{C_{hp}}\, \underset{2,1}{\ddot{\boldsymbol{\Delta}}} = \underset{1,1}{G_{hp}^0} \tag{36}$$

It is then obvious that for n given points, $(n-1)$ condition equations can be written.

Known General Plane. As three points define a plane, four or more points are required to effect additional conditions. If the first three points are 1, 2, and 3 the general condition equation for a point i would be given by:

$$\begin{vmatrix} (X_i - X_1) & (X_i - Y_1) & (Z_i - Z_1) \\ (X_2 - X_1) & (Y_2 - Y_1) & (Z_2 - Z_1) \\ (X_3 - X_1) & (Y_3 - Y_1) & (Z_3 - Z_1) \end{vmatrix} = 0. \tag{37}$$

It is interesting to note that the preceding two condition Equations 33 and 35 for the vertical and horizontal planes can be obtained from Equation 37 if the appropriate restrictions are imposed.

The linearized form of Equation 37 is

$$\underset{1,12}{C_p}\, \underset{12,1}{\ddot{\boldsymbol{\Delta}}} = \underset{1,1}{G_p^0}. \tag{38}$$

LINES

We shall again consider three types of lines.

Known Vertical Line. If the first point is designated by 1, the condition equation for another point i is

$$\begin{aligned} X_i - X_1 &= 0 \\ Y_i - Y_1 &= 0 \end{aligned} \tag{39}$$

which, if linearized, becomes

* Due to space limitations, the Appendix is not reproduced here but is contained in Reference 4.—*Editor.*

$$\underset{2,4}{C_{vl}}\,\underset{4,1}{\ddot{\mathbf{\Delta}}} = \underset{2,1}{G_{vl}^{0}} \tag{40}$$

Known Horizontal Line. If the first two points are designated 1 and 2, and any additional point i the following conditions hold:

$$Z_i - Z_1 = 0 \qquad i = 2, 3, \cdots, n \tag{41}$$

and

$$(Y_i - Y_1)(X_2 - X_1) - (Y_2 - Y_1)(X_i - X_1) = 0. \tag{42}$$

Equation 41 is applicable if two or more points appear on the horizontal line, whereas Equation 42 is used only if more than two points are involved. The reason for this is that Equation 41 expresses the fact that the points lie on a *horizontal* line, thus are of equal elevation. However, to distinguish this from the condition of a horizontal plane, any more points than the minimum of two (which define a line) must satisfy Equation 42, which is the equation of a line in two dimensions. The general linearized form is (assuming three points):

$$\underset{3,9}{C_{hl}}\;\underset{9,1}{} = \underset{3,1}{G_{hl}^{0}} \tag{43}$$

Known General Line. As two points, 1 and 2, would uniquely define a line, additional conditions would be written only if three or more points are given. For one such a point i the following condition must be fulfilled:

$$\frac{(X_i - X_1)}{(X_2 - X_1)} = \frac{(Y_i - Y_1)}{(Y_2 - Y_1)} = \frac{(Z_i - Z_1)}{(Z_2 - Z_1)}\,. \tag{44}$$

This in fact represents two equations,

$$\begin{aligned} &(X_i - X_1)(Y_2 - Y_1) - (X_2 - X_1)(Y_i - Y_1) = 0 \\ &(X_i - X_1)(Z_2 - Z_1) - (X_2 - X_1)(Z_i - Z_1) = 0 \end{aligned} \tag{45}$$

which, in linearized form, are

$$\underset{2,9}{C_{L}}\,\underset{9,1}{\ddot{\mathbf{\Delta}}} = \underset{2,1}{G_{L}^{0}}. \tag{46}$$

ANGLES

An angle measured at point 1 between the two points 2 and 3 (Figure 2) would yield the following condition from the cosine law

$$L_{23}^2 - L_{13}^2 - L_{12}^2 + 2L_{13}L_{12}\cos\theta = 0 \tag{47}$$

where

$$\begin{aligned} L_{12}^2 &= (X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 \\ L_{23}^2 &= (X_2 - X_3)^2 + (Y_2 - Y_3)^2 + (Z_2 - Z_3)^2 \\ L_{13}^2 &= (X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2 \end{aligned} \tag{48}$$

and θ is the known angle.

Equation 47 in linearized form is

$$\underset{1,1}{\mathbf{V}_{\theta}} + \underset{1,9}{\mathbf{B}_{\theta}}\,\underset{9,1}{\ddot{\mathbf{\Delta}}} = \underset{1,1}{F_{\theta}^{0}}. \tag{49}$$

This equation applies when, in general, an angle is of known value in the object space. If

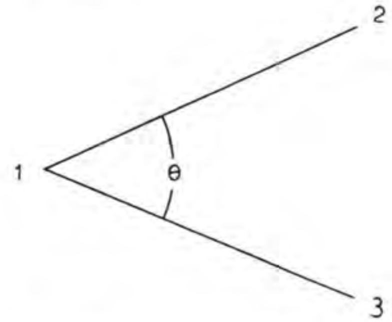

FIG. 2. A general known angle in space.

in addition, the angle is known to lie in a vertical or a horizontal plane, the three points 1, 2, 3, must further be used in the appropriate condition equations (34) or (36), respectively.

GEOMETRIC SHAPES

Although geometric shapes do not influence the operation of absolute orientation, as pointed out earlier in the section on *Geometric Shapes* under the sequential system, they nevertheless contribute constraints on the triangulation process. However, it must be clear that conditions can only be written if the number of points is greater than the minimum number required to define a unique geometric shape. We will discuss in the following only a few examples as it is unrealistic to attempt covering all possible shapes.

Regular geometric shapes which are formed by a discrete number of points, such as corners, vertices, squares, rectangles, cubes, etc., can all be treated as one group. Such figures or shapes can easily be thought of as a defined set of known lengths, lines, angles, and planes. Consequently, these types of information can be readily handled by the techniques developed in the preceding sections. As an example, a known rectangular shape would contribute line, angle, and plane conditions. If the size of the rectangle is also known, then distance conditions may be added to provide a scale.

A second group of geometric shapes include those without any distinct points, such as circles, spheres, and the like. The treatment of these is different inasmuch as they are normally expressed by analytic equations. Let us first consider a given sphere. The general equation of a sphere can be written as

$$X^2 + Y^2 + Z^2 - 2aX - 2bY - 2cZ - d = 0. \tag{50}$$

It is obvious from Equation 50 that a sphere is uniquely defined if four points are given (provided they do not lie on a plane). Hence, a condition will be given for every point in addition to the first four points. Designating the first four points by 1, 2, 3, 4 and the additional points by i, the condition that such five points lie on the same sphere is:

$$[X_i Y_i Z_i 1]\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \\ X_4 & Y_4 & Z_4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} X_1^2 + Y_1^2 + Z_1^2 \\ X_2^2 + Y_2^2 + Z_2^2 \\ X_3^2 + Y_3^2 + Z_3^2 \\ X_4^2 + Y_4^2 + Z_4^2 \end{bmatrix} = [X_i^2 + Y_i^2 + Z_i^2] \qquad (51)$$

This equation can, in principle, be linearized to take the form $\boldsymbol{C\Delta} = \boldsymbol{G}^0$. However, it must be obvious that the linearization in this case is quite a complex operation. Consequently, it might be more realistic to use Equation 50 five times (for the five points) in place of Equation 51. This would lead to the linearized form

$$\underset{5,15}{C_1}\ \underset{15,1}{\Delta} + \underset{5,4}{C_2}\ \underset{4,1}{\Delta'} = \underset{5,1}{G^0} \qquad (52)$$

where Δ' represents the added four parameters a, b, c, and d of Equation 50.

The condition of points known to lie on a circle may be considered as a special case of a sphere, because it is obtained from the intersection of a plane and a sphere. We will choose the general equation to be

$$(X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2 - R^2 = 0 \quad (53)$$

where X_o, Y_o, Z_o are the coordinates of the center of the circle. If the first three points, which are the minimum needed to define a circle, are designated 1, 2, 3 the following must hold to constrain these three points and the center to lie on a plane:

$$\begin{vmatrix} (X_1 - X_0) & (Y_1 - Y_0) & (Z_1 - Z_0) \\ (X_2 - X_0) & (Y_2 - Y_0) & (Z_2 - Z_0) \\ (X_3 - Y_0) & (Y_3 - Y_0) & (Z_3 - Z_0) \end{vmatrix} = 0. \quad (54)$$

In addition to Equation 54, three of Equation 53 may be written making four equations in four unknowns (X_o, Y_o, Z_o, R), or the unique case. For every other point an additional Equation 53 may be written allowing for the corresponding condition that such a point lie on the given circle. The linearized form of Equation 53 is similar to Equation 52.

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