Dr. Paul R. Wolf*
*University of Wisconsin*
*Madison, Wis. 53706*

# Independent-Model Triangulation

## The Balplex 760 plotter demonstrated remarkable accuracy capability: H/6,000 horizontally and H/2,700 vertically.

Abstract: *A Balplex 760 plotter was studied as an instrument for independent model aerotriangulation to determine horizontal and vertical accuracy. The test consisted of the triangulation of a strip of nine photographs having a dense network of control. The results were compared to those obtained with a Wild A-5 plotter used both as a strip-bridging instrument and also as an independent-model instrument. The Balplex root-mean-square errors in planimetry and height were 1/6,000 and 1/2,700 of the flying height, respectively. Two different schemes were used to determine the coordinates of the perspective centers: a grid-plate and a two-point method.*

### Introduction

In recent years a considerable gain has been made in the popularity of the Independent-Model† method for photogrammetric control extension. This popularity growth may be attributed primarily to several significant advantages that the method holds over control extension by fully analytic or stereo-bridging techniques.‡ Because of its dependence on the digital computer, the current popularity of the Independent Model method is also a direct result of the development of the computer.

A number of recently published research papers have dealt with the subject of accuracy of control extension by the method of Independent Models (References[1,2,3,4,5]). These studies have generally involved the use of high-order stereoscopic plotters having optical or mechanical projection systems and binocular viewing systems. The results of these studies have indicated that control extension by the method of Independent Models yields accuracy in computed coordinates as good as, or slightly better than, stereo-bridging and it yields accuracy only slightly inferior to fully analytic procedures.

In comparing the two basic analogic control extension procedures, one of the important advantages of the independent-model method is that the base-in base-out capability of the plotting instrument is not required as it is with stereo-bridging. Double-projection, direct-viewing stereoscopic plotters, if equipped for digital read-out of $X$, $Y$ and $Z$ model coordinates, are therefore capable of being employed in Independent Model Aerotriangulation.

Double-projection, direct-viewing stereoscopic plotters, with few exceptions,[6] have been utilized as compilation instruments almost exclusively up to this point and have generally been omitted from consideration in published independent model accuracy studies. These plotters are, however, less expensive and owned in much greater abundance by mapping firms than optical- or mechanical-projection, binocular-viewing instruments. And it seems plausible that by employing independent-model procedures, these compilation instruments can be used on many projects for control extension in addition to map compilation. In view of these considerations, justification seems to exist for a published account of an accuracy study involving the use of a double-projection, direct-viewing stereoscopic plotter.

* Presented at the Annual Convention of the American Society of Photogrammetry, Washington, D. C. March 1970. Formerly at University of California, Berkeley.

† Some authors refer to the method of Independent Models as Semi-Analytical Aerotriangulation.

‡ The term *stereo-bridging* as used herein denotes continuous-strip stereotriangulation by means of a base-in base-out instrument and includes the absolute orientation phase.

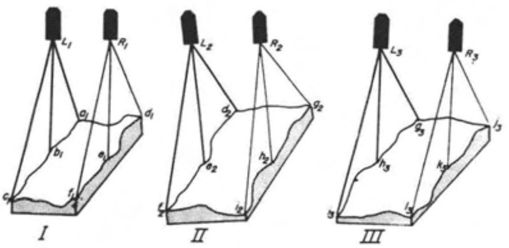


FIG. 1. Independent-model aerotriangulation.

This paper summarizes the findings of an accuracy study in three dimensional control extension utilizing a Balplex 760(ER-55) plotter and the method of independent-model aerotriangulation. Some additional studies in perspective-center coordinate determination were conducted and are also discussed herein.

## Capitulation of the Method of Independent Models

In brief, the method of independent-model aerotriangulation as applied to strips consists of the following basic steps:

- Relative orientation in the stereoscopic plotter of each of the stereo pairs of the strip and readout of the $X$, $Y$ and $Z$ model coordinates of all control points and pass points of each of the models; the coordinate system being arbitrary and unique for each independent model. The first three consecutive independent models of a strip are depicted in Figure 1.
- Determination of the $X$, $Y$ and $Z$ model coordinates of the perspective centers (shown as $L$ and $R$ on Figure 1) of the projectors in the arbitrary model coordinate systems.
- Strip formation consisting of the application of three-dimensional coordinate transformations which relate the coordinate systems of each successive model to the arbitrary coordinate system of the first model.
- Transformation and adjustment of the arbitrary strip coordinates to the ground coordinate system.

## Description of the Accuracy Test

A strip of nine photographs taken in the vicinity of the Sequoia National Park in California served as the basis for this accuracy test. The strip contained 42 panelled control points, 41 of which were controlled vertically and 35 of which were controlled horizontally. Figure 2 indicates the distribution of control in the strip and also shows the control that was used to adjust the strip. The camera used was a Zeiss RMK A 15/23 and the flying height was approximately 6,000 feet above average terrain. The range in relief in the strip was in excess of 500 feet.

In order to provide a basis for comparing the accuracy of independent-model aerotriangulation with the Balplex plotter, the strip was first stereo-bridged using a Wild Autograph A-5. The A-5 was completely rebuilt and calibrated in 1965 and may be considered to be a first-order instrument. Next the strip was triangulated using the A-5 and the method of independent models. Finally the Balplex was utilized in performing the triangulation by the method of independent models. In each of these three instances, model coordinates of all ground control points were read. Using only a nominal amount of ground control to transform and adjust the strip, the balance of control was available for evaluating the accuracies of the computed coordinates.

The Balplex was equipped with a Coradograph coordinatograph which was capable of readings in $X$ and $Y$ to the nearest 0.0005 inch. Vertical model coordinates were read directly from the tracing table to the nearest 0.01 millimeter. The A-5 model coordinates were taken directly from the instrument, the least gradation being 0.01 millimeter.

In this study two different methods were used in determining the coordinates of the perspective centers and these are discussed subsequently. The strip formation was performed using the method of Schut,[7] and the method of Keller and Tewinkel[8] was used to perform the strip transformation and adjustment.

With the abundance of horizontal and vertical control available in the strip, many different control combinations existed with which to transform the strip coordinates to ground. Several control combinations were used in performing the adjustment, and of course as would be expected, the greater the density of control used, the better the accuracy in the computed coordinates of points. The control combination that was selected for purposes of reporting accuracies in this paper, as shown in Figure 2, consisted of nine vertical and seven horizontal points.

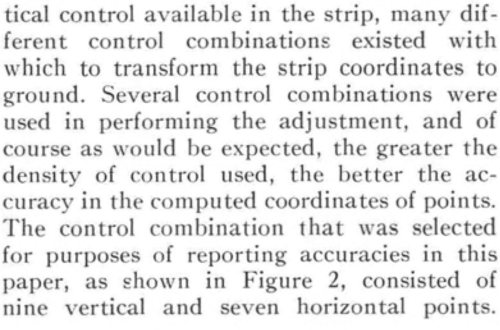


FIG. 2. Control used in strip adjustment.

TABLE 1. AVERAGE ERRORS IN PHOTOGRAMMETRIC CONTROL EXTENSION

| Method | *Average Errors in Feet (H=6,000′)* | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | *In Control Used for Adjustment* | | | *In All Control* | | | *In Control Not Used for Adjustment* | | |
| | X (a) | Y (b) | Z (c) | X (d) | Y (e) | Z (f) | X (g) | Y (h) | Z (i) |
| A-5 Bridging | 0.8 | 0.6 | 0.6 | 0.9 | 0.7 | 0.9 | 0.9 | 0.7 | 1.0 |
| A-5 Ind. Models | 0.6 | 0.6 | 0.9 | 0.7 | 0.6 | 0.9 | 0.7 | 0.6 | 1.0 |
| Balplex Ind. Mod. | 1.2 | 1.0 | 0.9 | 0.9 | 0.9 | 1.7 | 0.9 | 0.9 | 1.9 |

Several different combinations of nine vertical and seven horizontal control points were used and each yielded essentially the same results. Both the horizontal and vertical strip adjustments were performed using the third degree option of the aforementioned strip adjustment method.

## TEST RESULTS

The accuracies of the three separate aerotriangulations are summarized in Tables 1 and 2. In Table 1 average errors in feet in the computed *X*, *Y* and *Z* ground coordinates are listed. The average errors are the means of the absolute values of the discrepancies between control coordinates and computed coordinates. In columns *a*, *b* and *c* are listed average errors for only those control points used to adjust the strip. Columns *d*, *e* and *f* list average errors for all control, including that used to adjust the strip. Finally columns *g*, *h* and *i* list average errors for only the control which is in excess of that used to adjust the strip. This latter average error is most revealing and should be of primary interest, because in aerotriangulation projects these points exclusive of the control used in strip adjustment are the unknowns that are sought.

Table 2 lists root-mean-square errors in feet in the computed *X*, *Y* and *Z* ground coordinates. In columns *a*, *b* and *c* are the *rms* errors for only that control used to adjust the strip. Columns *d*, *e* and *f* list the *rms* errors for all control and columns *g*, *h* and *i* contain the *rms* errors for only the excess control.

The results of this one test, of course, could not be considered conclusive enough to enable the drawing of firm conclusions. However, this test does indicate the following:

* The methods of stereo-bridging and independent models yield approximately the same accuracy which is in agreement with published findings of others.
* Both methods yield higher accuracy in planimetric coordinates than in the *Z*-coordinate which of course is expected.
* Although the first-order plotter yields the higher accuracy of the two instruments, the Balplex shows a capability for quite remarkable accuracy.

Considering columns *g*, *h* and *i* of Table 1, it is seen that the average error with the Balplex was approximately 1/6,700 of the flying height for computed planimetric coordinates and approximately 1/3,100 of the flying height for computed vertical coordinates. In Table 2, columns *g*, *h* and *i* indicate that with the Balplex, root-mean-square errors in computed planimetric coordinates were approximately 1/6,000 and 1/2,700

TABLE 2. ROOT-MEAN-SQUARE ERRORS IN PHOTOGRAMMETRIC CONTROL EXTENSION

| Method | *Root Mean Square Errors in Feet (H=6,000′)* | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | *In Control Used for Adjustment* | | | *In All Control* | | | *In Control Not Used for Adjustment* | | |
| | X (a) | Y (b) | Z (c) | X (d) | Y (e) | Z (f) | X (g) | Y (h) | Z (i) |
| A-5 Bridging | 1.1 | 0.8 | 0.8 | 1.1 | 0.8 | 1.1 | 1.1 | 0.8 | 1.2 |
| A-5 Ind. Models | 0.7 | 0.7 | 1.2 | 0.9 | 0.8 | 1.2 | 0.9 | 0.8 | 1.2 |
| Balplex Ind. Mod. | 1.0 | 1.0 | 1.0 | 1.1 | 1.1 | 2.0 | 1.0 | 1.0 | 2.2 |

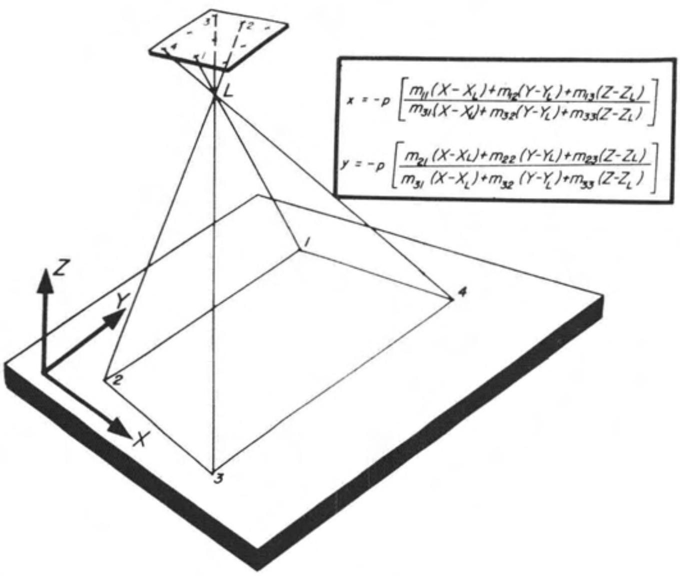


FIG. 3. Grid-plate method for determining the coordinates of the perspective center.

of the flying height, respectively. From these results it seems evident that double-projection, direct-viewing plotters are capable of satisfactory accuracy in control extension for many mapping projects. For certain compilation projects where the full *C*-factor capability of the Balplex is to be utilized, this vertical control accuracy must be considered borderline.

## DETERMINATION OF PERSPECTIVE CENTER COORDINATES

As a part of this study, two different methods for determining perspective center coordinates were investigated. The first method, illustrated in Figure 3, was used on both the A-5 and Balplex and consisted of using a precise grid plate.[1] In this method, the grid plate was inserted into each projector and the $X$, $Y$, and $Z$ model coordinates were monocularily read for a number of grid intersections. Treating the projector principal distance as analogous to the camera focal length, and considering the precise coordinates of the grid intersections and their respective model coordinates as analogous to photo coordinates and ground control coordinates respectively, the space resection problem was solved. The basic and well known collinearity equations which are solved in the resection are given in Figure 3. The solution provided the $X$, $Y$ and $Z$ coordinates of the perspective centers in the model system and in addition it yielded the orientation angles. A minimum of three intersections must be read for a solution, but the four corner points are recommended as a practical minimum. A greater number of redundant readings provides statistically improved results.

The second method for the coordinates of the perspective center, depicted in Figure 4, did not require a grid plate but rather it utilized the diapositives of the models themselves.[9] The method consisted of monocularily reading the $X$ and $Y$ model coordinates of random image points at two or more different $Z$ settings which were as widely spaced as possible. The two-point form of the equation for a straight line, given in Figure 4, was then applied to calculate the perspective center coordinates.

In the two-point equation, $X_L$, $Y_L$ and $Z_L$ are the model coordinates of the perspective center, $X_1$, $Y_1$ and $Z_1$ are the model coordinates at the first $Z$ setting, and $X_2$, $Y_2$ and $Z_2$ are the model coordinates at the second $Z$ setting. A minimum of two different points each read at two $Z$ settings is required for a solution; however, four corner points are again recommended as a practical minimum and more redundancy improves the results.

Both methods have advantages and disadvantages. In the two-point method an expensive precise grid plate is not needed and furthermore the added operation of placing the grid plate in the projectors is eliminated. The data processing with the two-point method is somewhat simplified also, as grid-plate coordinates need not be introduced into

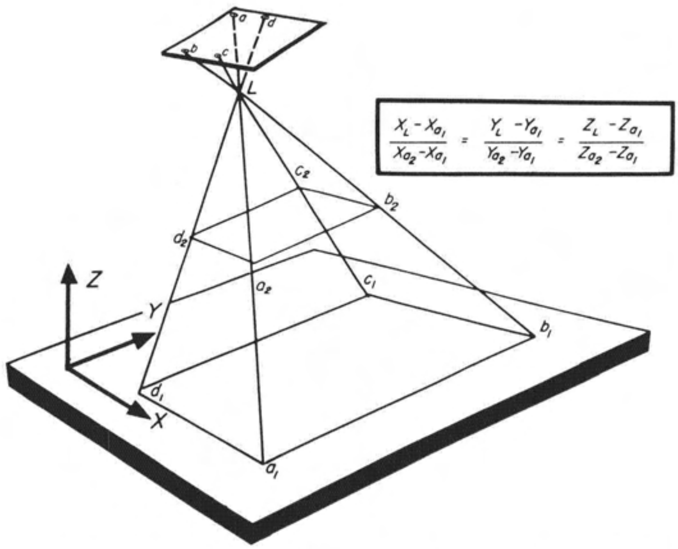


FIG. 4. Two-point method for determining the coordinates of the perspective center.

the system. In addition, errors due to faulty centering of the grid plate are not introduced. With the A-5 plotter, the standard deviations in $X$ and $Y$ for ten grid-plate centering tests was about $\pm 10$ microns at the nine-inch photo scale. A similar test with the Balplex yielded standard deviations of about $\pm 50$ microns at the nine-inch photo scale.

In the grid-plate method, all model coordinates of grid intersections may be read at one $Z$-setting, thereby reducing the reading time and effort considerably. In addition, fewer readings are needed to obtain standard deviations in the perspective center coordinates that are equivalent to the standard deviations obtained by the two-point method. Also, a large range in $Z$ of the plotter is not necessary for the grid plate method as is the case with the two-point method.

Both of these methods were utilized with the A-5 plotter and both yielded satisfactory results. Only the grid-plate method was used with the Balplex, however, because its $Z$ range, limited mechanically by the screw on the tracing table, is only six inches, or about 1/5th of the optimum projection distance. The two-point method of perspective center determination was used in obtaining the A-5 independent-model results which are included within Tables 1 and 2.

Tests were performed to ascertain the relative precisions of the two methods of perspective-center coordinate computation. In these tests, many different perspective-center determinations were made using both methods, with variations in the number of points read. Care was exercised to retain geometically strong point configurations as the number of points used was reduced. From these determinations, the curves of Figure 5 were prepared. Figure 5(A) indicates the standard deviations obtained through many grid-plate determinations of perspective center coordinates using the A-5 plotter. The curve is a plot of the number of grid-plate intersections read on the abscissa, versus standard deviations in the computed perspective-center coordinates on the ordinate. The standard deviations are given in microns at the nine-inch photo size. The errors of centering the grid plate is not included in the $X$ and $Y$ values.

Figure 5(B) is a plot of standard deviations obtained with the two-point method and the A-5. In this figure the number of images read at two $Z$ settings is on the abscissa and the ordinate is the same as that of Figure 5(A). Figure 5(C) is a plot of the same information as Figure 5(A), except that it applies to the Balplex. Ordinate values are again given at nine-inch photo size for comparative purposes.

From a study of the grid-plate curves of Figure 5(A) and 5(C), one may conclude that very little precision is gained by reading more than about ten grid intersections. Similarily from the two-point curves of Figure 5(B), ten image points also seem to be about optimum; however, it is to be noted that this represents a total of 20 readings. A further study of these curves indicates that the grid-plate method yields smaller standard devia-

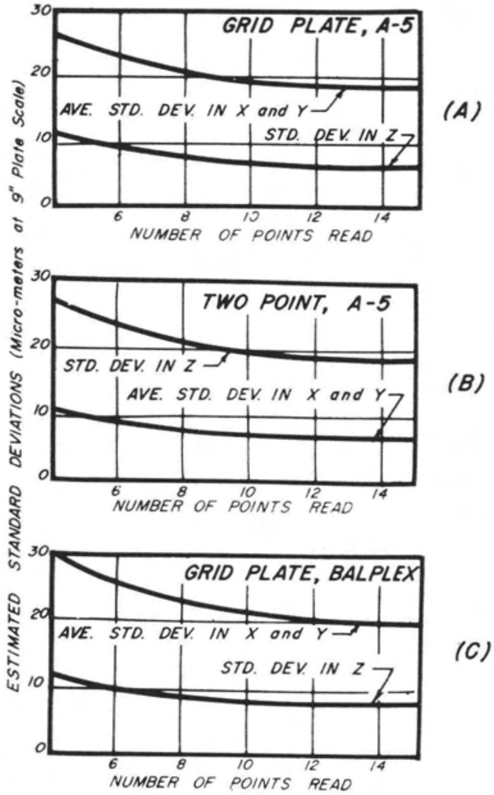


FIG. 5. Standard deviations versus number of points observed.

tions in the $Z$ coordinate than in $X$ and $Y$. Interestingly, the reverse is true with the two-point method. In addition, it is noted that the standard deviations of the $Z$ coordinate, as determined by the grid plate method, are smaller than those obtained for $Z$ by the two-point method. Again the reverse is true for the $X$ and $Y$ coordinates.

This latter observation lead to further study into the sensitivity of the computed pass point coordinates to each of the $X$, $Y$ and $Z$ coordinates of the perspective centers. Although perfect, fictitious, independent models would seem to be the preferred approach in this investigation, this study was performed using the Wild A-5 data from the same strip of photos. Perturbations were deliberately introduced into each of the $X$, $Y$ and $Z$ perspective-center coordinates by amounts approximately equal to the standard deviations of those coordinates. The degradation in computed coordinates as a result of these perturbations were then observed. On the basis of these tests the following qualitative conclusions could be drawn.

- $X$-perturbation caused practically no noticeable degradation in computed $X$ or $Y$ ground coordinates. There was, however, a slight degradation in the computed $Z$ coordinates of points.
- $Y$-perturbation caused practically no noticeable deterioration in any computed ground coordinates.
- $Z$-perturbation caused no noticeable deterioration in the computed $Z$ coordinates of points. There was a slight deterioration in $X$, and a slightly more noticeable degradation in the computed $Y$, ground coordinates.

On the basis of these tests alone it is difficult to reach a decision as to the preferable method of perspective-center determination. It seems that the grid-plate method, with its higher standard deviation in $X$, will cause a slightly higher degradation in computed $Z$ ground coordinates. On the other hand, it seems that the two-point method, with its higher standard deviation in $Z$, will cause a slightly higher degradation in computed $X$ and $Y$ ground coordinates. It is this author's opinion that either method, if carefully performed using approximately ten points, will yield highly satisfactory results in independent-model aerotriangulation. Further research in this area would be valuable, however.

## SUMMARY

An investigation was made to determine the accuracy of horizontal and vertical control extension employing a double-projection direct-viewing stereoscopic plotter and the methods of independent-model aerotriangulation. In particular, the study featured a Balplex 760 plotter whose optimum model scale is equivalent to 5× enlargement from the 9-inch photo scale. The test consisted of the triangulation of a strip of nine photos dense in ground control. To form a basis for accuracy comparison, the strip was also triangulated using a first-order Wild A-5 plotter. The results indicated that, although the first-order instrument yielded superior accuracy, the Balplex demonstrated quite remarkable accuracy capability. The Balplex root-mean-square errors in the planimetric and vertical coordinates of the extended control were approximately 1/6,000 and 1/2,700 of the flying height, respectively. Although a single test is not sufficient to enable drawing firm conclusions, it does indicate that control extension to satisfactory accuracy using double-projection direct-viewing plotters may be feasible for many mapping projects.

A further phase of this study has indicated that with a plotter having a substantial range in $Z$, perspective-center coordinates may be

determined to satisfactory accuracy by employing either a *grid-plate* or a *two-point* method. No definite preference in methods could be established, although both methods possess certain advantages and disadvantages. With either method, the optimum number of points that should be read seemed to be approximately ten; the practical minimum is four corner points. If care is exercised, either method should yield highly satisfactory results in independent-model aerotriangulation.

Although this project was conducted without the aid of encoders for automatic, direct-recording of model coordinates, such equipment is definitely recommended for utmost efficiency in independent-model aerotriangulation.

ACKNOWLEDGMENTS

The author gratefully acknowledges the cooperation on this project of the photogrammetry section of the California Division of Highways for providing the photography and ground control data for this study. Grateful acknowledgment is also due the photogrammetry section of the Pacific Region of the U. S. Geological Survey for preparation of the Balplex diapositives and the use of other of their equipment. Special thanks are also due to Prof. Francis H. Moffitt of the University of California for his assistance on this project and to Mr. Michael Thomas, photogrammetry student at the University of California, who assisted with much of the data processing.

1. Inghilleri, G. and Galetto, R., "Further Developments of the Method of Aerotriangulation by Independent Models," *Photogrammetria*, Vol. 22, No. 1, pp. 13–28, January 1967.
2. Shmutter, B., "Triangulation with Independent Models," *Photogrammetric Engineering*, Vol. XXXV, No. 6, pp. 548–553, June 1969.
3. Weissman, S., "Semi-Analytical Aerotriangulation," *Photogrammetric Engineering*, Vol. XXXV, No. 8, pp. 789–795, August 1969.
4. Wong, K., "Computer Programs for Strip Aerotriangulation," *Journal of the Surveying and Mapping Division, Proceedings of the American Society of Civil Engineers*, Vol. 95, No. SU1, pp. 71–80, October 1969.
5. Thompson, E. H., "Aerial Triangulation and the Thompson Watts Plotter," A paper published at the University College London, London, England, July 1964.
6. Altenhofen, R. E., "Semianalytical Aerotriangulation in Quadrangle Mapping," *Proceedings of the 34th Annual Meeting*, the American Society of Photogrammetry, pp. 5–27, Washington, D. C., March 1968.
7. Schut, G. H., "Formation of Strips from Independent Models," *Report AP-PR 36*, National Research Council of Canada, Ottawa, July 1967.
8. Keller, M. and Tewinkel, G. C., "Aerotriangulation Strip Adjustment," *Technical Bulletin No. 23*, U. S. Dept. of Commerce, Coast and Geodetic Survey, Washington, D. C., August 1964.
9. Ligterink, G. H., "Aerial Triangulation by Independent Models, the Coordinates of the Perspective Center and Their Accuracy," A paper published at the Technological University, Delft, Netherlands, February 1969.

## Notice to Contributors

1. Manuscripts should be typed, double-spaced on $8\frac{1}{2}\times 11$ or $8\times 10\frac{1}{2}$ white bond, on *one* side only. References, footnotes, captions—everything should be double-spaced. Margins should be $1\frac{1}{2}$ inches.
2. *Two* copies (the original and first carbon) of the complete manuscript and two sets of illustrations should be submitted. The second set of illustrations need not be prime quality.
3. Each article should include an abstract, which is a *digest* of the article. An abstract should be 100 to 150 words in length.
4. Tables should be designed to fit into a width no more than five inches.
5. Illustrations should not be more than twice the final print size: *glossy* prints of photos should be submitted. Lettering should be neat, and designed for the reduction anticipated. Please include a separate list of captions.
6. Formulas should be expressed as simply as possible, keeping in mind the difficulties and limitations encountered in setting type.