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Strip Adjustment using Harmonic Analysis

The scheme is compared to the graphic method.

ABSTRACT: *A new approach to aerial triangulation adjustment using Harmonic Analysis has been suggested. Using all the available ground control points, a correction surface is generated using numerical integration for Fourier components. Corrections were computed back from the Fourier function so computed. The residuals were determined and compared with those obtained by the graphical (parabolic) method. This new approach has been tested for all three dimensions.*

INTRODUCTION

HARMONIC ANALYSIS has been extensively used in the solution of several engineering and scientific problems, i.e., in wave form analysis and in tidal predictions etc. This is based on Fourier expansion of a given function, subject to satisfying certain conditions, into a large number of simple trigonometric functions of fundamental and higher harmonics. Mathematically this is expressed as

$$F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx + \theta_m) + \sum_{n=1}^{\infty} B_n \sin(nx + d_n) \qquad (1)$$

where the constant and the amplitudes A_0, A_n, B_n are determined from

$$A_0 = \frac{1}{T}\int_0^T F(x)dx$$
$$A_n = \frac{2}{T}\int_0^T F(x)\cdot\cos nxdx \qquad (2)$$
$$B_n = \frac{2}{T}\int_0^T F(x)\cdot\sin nxdx$$

T being the period of the periodic function $F(x)$ and θ_n, α_n being the phase differences. T has limits 0 to 2π in normal cases but can be changed to limit of 0 to $2l$ by simple substitutions

$$y = \frac{xl}{\pi}.$$

Conversely if the trigonometric components along with their amplitudes are known, the function $F(x)$ can be evaluated to a degree depending on the number of harmonic components known. The problem can therefore be looked at in another way by considering it as a curve-fitting problem. That is, if we have a number of observed functional values of $F(x)$ in the range 0 to $2l$, then the unknown $F(x)$ can be determined by means of a best fit. For example the line $y=x$ can be fitted fairly well by four harmonic components, i.e.,

$$y = 2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x)$$

between the limits $-4\pi/5$ to $+4\pi/5$ with immediate divergence thereafter (See Figure 1). Similarly in Figure 2 the periodically discontinuous series of straight lines is wel fitted by

$$y = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x\right)$$

the fit being perfect at $(2n+1)$ points equally or unequally spaced. The actual method of fit involves calculating Fourier components from the known or available $F(x)$ values in the range 0 to $2l$ by numerical integration instead of the continuous integration as given in Equations 2, and then summing up all the components to obtain $F(x)$ as best as possible. Obviously the greater the number of computed components, the greater will be the possibility of obtaining the best fit. Theoretically it means summing an infinite number of components. However, practical solutions have shown that a reasonably good fit can

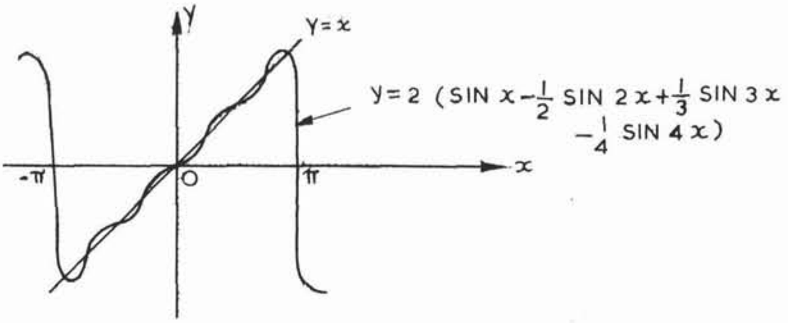

FIG. 1. The line $y=x$ can be fitted fairly well with four harmonic components.

be obtained with 5 to 10 components. The use of an electronic digital computer greatly facilitates the computations of a large number of components and the discrepancy between the known and the computed value of $F(x)$.

Strip Analysis

It is well known that the strip deformation in aerial triangulation is very prominent in the x-direction (direction of flight), the deformation being described mathematically by the equation

$$\Delta x = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + \qquad (3)$$

where Δx is the error in the x coordinate and is given by

$$\Delta x = x_{\text{terrain}} - x_{\text{machine}}. \qquad (4)$$

In graphical strip adjustment the correction $e = -\Delta x$ is obtained from the parabolic deformation curve plotted from known Δx at a number of control points. An attempt has been made in this paper to show that if we know Δx at a number of given points (control points), then we can build up the Fourier components from these known deformations (or errors) and then get the correction function by summing up the components. Once a good fit has been obtained at the known points, the correction at any point within the range can be worked out. This involves first knowing the range. In order to accommodate all points of the strip, a range can be defined by

$$l = X_f - X_i$$

where X_f is the X-coordinate of the last control point in the last model of the strip and X_i that of the first control point of the first model.

In order to make the range dimensionless and make possible the evaluation of the cosine/sine terms of the Fourier components a new range of $2l'$ has been chosen given by

$$2l' = \frac{X_f - X_i}{B}$$

where B is the average air base of n models. This is an assumption whose validity has been proved by the results obtained.

Data and Results

The above analysis was used for testing the deformation along X-axis for x-coordinate correction as strip deformation is most significant in the direction of flight. The data used is taken from "Strip Triangulation" with a Wild A-9 at a model scale of 1:80,000, strip No. 135, plates from No. 18 to 29 of the Zurich Area. The aerial triangulation was completed as part of the studies by the au-

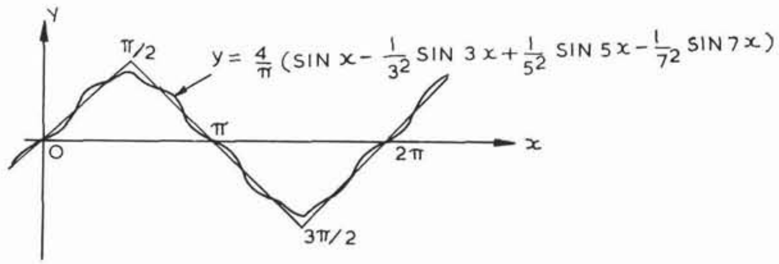

FIG. 2. A periodically discontinuous series of straight lines is well fitted with different harmonic components.

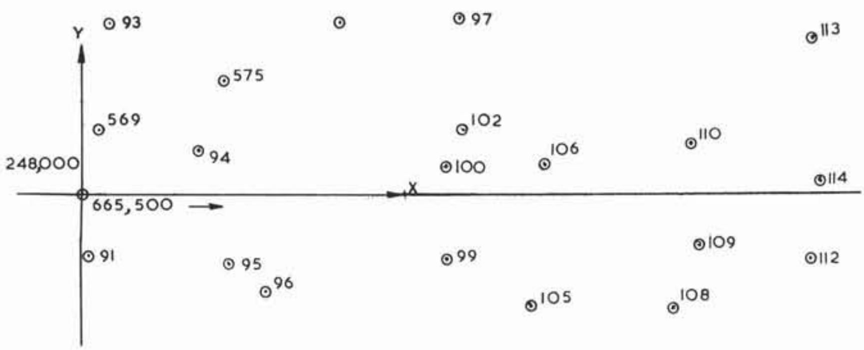

FIG. 3. Distribution of ground control points.

thor at ITC Delft during 1967. For adjustment (graphically) and checking, 20 ground control points were available and were well distributed in the strip (see Figure 3). Model coordinates were transformed to strip coordinates using four ground control points with the help of ITC standard computation forms. During studies at ITC, a graphical adjustment by Zarzycki's method was performed. Table 1 shows the corrections applied and residuals with mean-square errors obtained by the graphical method. Table 2 shows the results obtained by performing the harmonic analysis. The computations for the Fourier Analysis were programmed on I.B.M. 7044 Digital Computer; the program may be obtained from the author. Table 3 shows data for performing the numerical integration for the harmonic components.

The first part of the program uses the known errors to x-coordinates, performs numerical integration and computes the amplitude, period and phase of the Fourier components. The second part then takes these components and calculates the corrections. The discrepancies and also percentage discrepancies are printed out to show the difference between actual and calculated values. On account of 20 ground control points being available, 20 harmonic com-

TABLE 1. ADJUSTMENT BY GRAPHICAL METHOD

Sl. No.	*Point No.*	X_m	*C (Parabolic)*	X'_m	X_t	r	r^2
1	112	685,216.7	+1.4 m	685,218.1	685,218.7	+0.6 m	0.36
2	113	685,123.2	−1.5	685,121.7	685,120.8	+0.9	0.81
3	109	682,075.8	+1.0	682,076.8	682,072.2	−4.6	21.16
4	110	681,871.1	−0.3	681,870.8	681,866.2	−4.6	21.16
5	108	681,393.4	+2.2	681,395.6	681,390.4	−5.2	27.04
6	106	677,825.6	−1.9	677,823.7	677,821.4	−2.4	5.76
7	105	677,709.0	+1.7	677,710.7	677,695.4	−15.3	234.09
8	103	676,227.1	+0.9	676,228.0	676,222.5	−5.5	30.25
9	102	675,570.5	−3.4	675,567.1	675,560.2	−6.9	47.61
10	101	675,366.9	−6.6	675,360.3	675,356.1	−4.2	17.64
11	100	675,040.6	−2.7	675,037.9	675,035.7	−2.2	4.84
12	99	675,018.2	−0.1	675,018.1	675,010.0	−8.1	65.61
13	96	670,076.8	0.0	670,076.8	670,069.8	−7.0	40.96
14	95	669,171.2	−1.0	669,170.2	669,170.8	+0.6	0.36
15	575	668,927.5	−6.0	668,921.5	668,921.6	0.1	0.01
16	94	668,330.1	−4.0	668,326.1	668,324.4	−1.7	2.89
17	93	665,839.6	−8.3	665,831.3	665,836.8	+5.5	30.25
18	569	665,502.0	−5.5	665,496.5	665,493.9	−2.6	6.74
19	91	665,142.4	−2.1	665,140.3	665,138.4	−1.9	3.61
							$\Sigma r^2 = 561.15$

Mean-square error = 5.43 m.
Mean-square error at the scale of negative = 64 micrometers.

TABLE 2. ADJUSTMENT BY HARMONIC ANALYSIS

Sl. No.	*Point No.*	X_m	C	X'_m	X_t	r	r^2
1	112	685,216.7	−3.0 m	685,213.7	685,218.7	+5.0 m	25.00
2	113	685,123.2	−2.7	685,120.5	685,120.8	+0.3	0.09
3	109	682,075.8	−3.9	682,071.9	682,072.2	−0.3	00.09
4	110	681,871.1	−4.3	681,866.8	681,866.2	−0.6	0.36
5	108	681,393.4	−3.2	681,390.2	681,390.4	+0.2	0.04
6	106	677,825.6	−8.0	677,817.0	677,821.3	+4.3	18.49
7	105	677,709.0	−9.9	677,699.1	677,695.4	−3.7	13.69
8	103	676,227.1	−5.3	676,221.8	676,222.5	+0.7	0.49
9	102	675,570.5	−10.0	675,560.5	675,560.2	−0.3	0.09
10	101	675,366.9	−9.7	675,357.2	675,356.1	−1.1	1.21
11	100	675,040.6	−7.5	675,033.1	675,035.7	+2.6	6.76
12	99	675,018.2	−7.4	675,010.8	675,010.0	−0.8	0.64
13	96	670,076.8	−6.9	670,069.9	670,069.8	−0.1	0.01
14	95	669,171.2	−0.8	669,170.4	669,170.8	+0.4	0.16
15	575	668,927.5	−5.7	668,921.8	668,921.6	−0.2	0.04
16	94	668,330.1	−5.5	668,324.6	668,324.4	+0.2	0.04
17	93	665,839.6	−3.2	665,836.4	665,836.8	+0.4	0.16
18	569	665,502.0	−7.2	665,494.8	665,493.9	−0.9	0.81
19	91	665,142.4	−5.3	665,137.1	665,138.5	+1.3	1.69
							$\Sigma r^2 = 69.86$

X_m = machine X-coordinate
(Strip coordinate)
C = correction from Harmonic Analysis
X_t = Terrain coordinate
r = residual
Mean-square error = 1.91 meters
Mean-square error on the scale of negative = 24 micrometers.

TABLE 3. INPUT DATA

$X_f = 685{,}377.8$ m
Base = 3200 m

Sl. No.	*Point No.*	X_m (*machine*)	X_t (*terrain*)	$e = X_m - X_t$	$X_t - X_m$	$\frac{X_f - X_m}{Base} = T_i$
1	112	685,216.7 m	685,218.7	−2.0 m	161.1 m	0.050319
2	113	685,123.2	685,120.8	+ 2.4	254.6	0.079370
3	109	682,075.8	682,072.2	+ 3.6	3302.0	1.031900
4	110	681,871.1	681,866.2	+ 4.9	3506.7	1.095800
5	108	681,393.4	681,390.4	+ 3.0	3984.4	1.245500
6	106	677,825.6	677,921.3	+ 4.3	7552.2	2.360000
7	105	677,709.0	677,695.4	+13.6	7668.8	2.396500
8	103	676,227.1	676,222.5	+ 4.6	9150.7	2.859700
9	102	675,570.5	675,560.2	+10.3	9807.3	3.064900
10	101	675,366.9	675,356.1	+10.8	10,010.9	3.124800
11	100	675,040.6	675,035.7	+ 4.9	10,337.2	3.230300
12	99	675,018.2	675,010.0	+ 8.2	10,359.6	3.237500
13	96	670,076.8	670,069.8	+ 7.0	15,301.0	4.781600
14	95	669,171.2	669,170.8	+ 0.4	16,206.6	5.064700
15	575	668,927.5	668,921.6	+ 5.9	16,450.3	5.140700
16	94	668,330.1	668,324.4	+ 5.7	17,047.7	5.327500
17	93	665,839.6	665,836.8	+ 2.8	19,538.2	6.105600
18	569	665,502.0	665,493.9	+ 8.1	19,575.8	6.211000
19	91	665,142.4	665,138.4	+ 4.0	20,235.4	6.323600

TABLE 4. Y-COORDINATES (IN METERS)

Sl. No.	Point No.	Y_m	C	Y'_m	Y_t	r	r^2	r'
1	112	246,402.6	−1.7	246,400.9	246,405.6	−4.7	22.09	+0.7
2	113	252,334.0	+0.2	252,334.2	252,338.0	−3.8	14.44	+0.9
3	109	246,689.6	+0.8	246,690.4	246,689.7	+0.7	0.49	−0.3
4	110	249,514.5	+0.9	249,515.4	249,516.8	−1.4	1.96	+2.0
5	108	244,804.9	−0.8	244,804.1	244,803.2	+0.9	0.81	−1.9
6	106	248,911.4	−5.0	248,906.4	248,911.2	−4.8	23.04	+2.5
7	105	245,036.8	−7.4	245,029.4	245,023.8	+5.6	31.36	−11.6
8	103	245,557.7	−5.4	245,552.3	245,553.8	−1.5	2.25	−1.8
9	102	249,825.4	−0.3	249,825.1	249,820.6	+4.5	20.25	−0.5
10	101	252,921.4	+1.4	252,922.8	252,930.0	−7.2	51.84	+14.4
11	100	248,869.9	+0.2	248,870.1	248,866.1	+4.0	16.00	0.3
12	99	246,299.6	+0.1	246,299.7	246,297.2	+2.5	6.25	0.4
13	96	245,353.2	−8.5	245,344.7	245,344.7	0.0	0.00	−5.3
14	95	246,059.3	−5.1	246,054.2	246,055.5	−1.3	1.69	−0.1
15	575	251,096.4	−6.1	251,090.3	251,089.3	+1.0	1.00	−0.1
16	94	249,075.3	−5.8	249,069.5	249,069.8	−0.3	0.09	+0.3
17	93	252,551.0	−12.0	252,539.0	252,538.6	+0.4	0.16	−4.3
18	569	249,706.2	−9.4	249,696.9	249,698.4	−1.6	2.56	−7.7
19	91	246,311.6	−1.7	246,309.9	246,303.8	+6.1	37.21	−4.2
							$\Sigma r^2 = 233.49$	

Y_m = transferred machine Y-coordinate.
C = correction from harmonic analysis adjustment.
r = residual from harmonic analysis adjustment.
r' = residual from graphical adjustment.
Mean-square error in meters = 3.5 m.
Mean-square error on the scale of the negative = 43 micrometers.
Mean-square in meters by graphical method = 5.1 m.
Mean-square at the scale of negative = 63 micrometers.

ponents were computed. It is seen that this analysis gives better values of corrections and the resulting residuals, and mean-square error are lower compared to those obtained by the graphical method.

CONCLUSIONS

i. Tests have been conducted only for the x-deformation to see the validity of the assumptions made in the program.

ii. The residual and the mean-square error is much smaller than for the graphical method. This may be due to the deformation surface being generated from actual errors obtained rather than from expected or theoretical errors.

iii. Analysis has been done to a strip of 11 models. Tests are needed for a larger number of models. Corrections for y- and z-deformations are under study.

iv. This method has advantage over the graphical method that all the available ground control points are used in the computation of the corrections whereas the graphical method takes only those points which are suitably situated along the x-axis, the rest not being used at all.

RESULTS OF Y AND Z ADJUSTMENTS

Subsequent to the study of the X-adjustment, those for Y and Z were completed and furnished to the Editor, with the following conclusions.

v. The adjustment of the Y-coordinates were performed in the same way as for the X-coordinates. The results are shown in Table 4. The mean square error is lower than that obtained by the graphical method.

vi. For the Z-coordinate (height), the machine heights are corrected first by adding the Linear Correction in the same way as for the graphical method. The discrepancies between the terrain heights and the computed heights are then used for harmonic analysis by numerical integration with the same computer program as used for the X-deformation. The corrections so obtained are then applied and the residuals computed. (Table 5).

vii. The Fourier components summation

TABLE 5. Z-COORDINATE (HEIGHT) IN METERS

Sl. No.	Point No.	Zm (machine)	Linear correction ΔZ	$Z_m+\Delta Z$	C	Z'	Z_t	r	r^2	r'
1	112	459.0	+ 3.2	462.2	−14.0	448.2	462.1	−13.9	193.21	+ 0.5
2	113	416.0	+ 4.0	420.0	− 8.5	411.5	425.4	−13.9	193.21	+ 0.6
3	109	404.0	+ 28.7	332.7	−13.8	418.9	418.9	0.0	0.0	− 2.6
4	110	388.0	+ 30.4	418.4	−14.6	403.8	405.0	−1.2	1.44	− 4.1
5	108	408.0	+ 30.4	438.4	−14.8	423.6	426.6	− 3.0	9.00	+ 0.1
6	106	412.0	+ 63.3	475.3	−30.9	444.4	445.8	− 1.4	1.96	− 9.5
7	105	470.0	+ 64.4	534.4	−32.4	502.0	500.6	+ 1.4	1.96	− 7.4
8	103	436.0	+ 76.4	512.4	−38.2	474.5	474.5	− 0.3	0.09	− 8.9
9	102	359.0	+ 81.7	440.7	−36.5	404.2	403.0	+ 1.2	1.44	−14.6
10	101	390.0	+ 83.4	473.4	−36.2	439.2	440.5	− 1.3	1.69	−14.9
11	100	425.0	+ 86.1	511.1	−35.9	475.2	472.1	+ 3.1	9.61	−13.3
12	99	436.0	+ 86.2	522.2	−35.9	486.3	486.1	+ 0.2	0.04	− 5.9
13	96	461.0	+126.5	587.5	−37.5	550.0	550.0	0.0	0.00	0.2
14	95	385.0	+133.8	518.8	−32.6	486.2	483.2	+ 3.0	9.00	0.9
15	575	577.0	+135.8	712.8	−29.7	683.1	687.4	− 4.3	18.49	0.9
16	94	478.0	+140.7	618.7	−29.4	589.3	587.0	+ 2.3	5.29	− 1.2
17	93	322.0	+160.9	482.9	−20.0	462.9	468.4	− 5.5	30.25	+ 8.4
18	569	236.0	+163.7	399.7	−26.8	372.9	371.3	+ 1.6	2.56	+ 0.2
19	91	283.0	+166.6	449.6	−14.1	435.5	417.5	+18.0	324.00	+3.7
									$\Sigma r^2=803.14$	

C = correction from harmonic analysis adjustment.
r = residual from harmonic analysis.
r' = residual from graphical adjustment.
Mean-square error in meters from harmonic adjustment = 6.5 m.
Mean-square error in meters by graphical adjustment = 7.2 m.

seems to diverge as evidenced by the large residuals at the beginning and the range. A good fit seems to occur in between.

viii. On the whole, even taking into account the large residuals, the mean-square error as obtained with the harmonic analysis adjustment is lower than that by means of the graphical method.

REFERENCES

1. K. Kubik, I.T.C. DELFT Lecture Notes on *Aerial Triangulation and Adjustments*, 1967–69.
2. Bachman George, *Elements of Harmonic Analysis*, N. Y. Academic Press.
3. Katznelson Yitzhak, *An Introduction to Harmonic Analysis*, Wiley, New York 1968.
4. Loomis Lynnh, *An Introduction to Abstract Harmonic Analysis*, N. J. Van Nostrand.

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